- Beliaev - Dyson - equation:

$$\boxed{G_{\alpha\beta}(\vec{k}, i\omega_n) = G^{(0)}_{\alpha\beta}(\vec{k}, i\omega_n) + \sum_{\gamma,\delta} G^{(0)}_{\alpha\gamma}(\vec{k}, i\omega_n)\, \Sigma_{\gamma\delta}(\vec{k}, i\omega_n)\, G_{\delta\beta}(\vec{k}, i\omega_n)}$$

$$G^{(0)}_{\alpha\beta}(\vec{k}, i\omega_n) = \begin{pmatrix} \frac{1}{i\omega_n - \frac{e_k}{\hbar}} & 0 \\ 0 & \frac{1}{-i\omega_n - \frac{e_k}{\hbar}} \end{pmatrix}$$

$$\Sigma_{\alpha\beta}(\vec{k}, i\omega_n) = \begin{pmatrix} \Sigma_{11}(\vec{k}, i\omega_n) & \Sigma_{12}(\vec{k}, i\omega_n) \\ \Sigma_{21}(\vec{k}, i\omega_n) & \Sigma_{22}(\vec{k}, i\omega_n) \end{pmatrix}$$

[2019.05.14.]

$$(\underline{\underline{G}}^{0})^{-1} \cdot \Big/ \quad \underline{\underline{G}} = \underline{\underline{G}}^{(0)} + \underline{\underline{G}}^{(0)} \underline{\underline{\Sigma}}\, \underline{\underline{G}} \quad \Big/ \cdot \underline{\underline{G}}^{-1}$$

$$\left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{\begin{pmatrix} D & -B \\ -C & A \end{pmatrix}}{AD - BC} \right]$$

- we want to express the elements of $\underline{\underline{G}}$

$$(\underline{\underline{G}}^{(0)})^{-1} = \underline{\underline{G}}^{-1} + \underline{\underline{\Sigma}}$$

$$\boxed{\underline{\underline{G}}^{-1} = (\underline{\underline{G}}^{(0)})^{-1} - \underline{\underline{\Sigma}}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

- Let $D(k, i\omega_n) = -(AD - BC)$

$$D(k, i\omega_n) = -\left[\left(i\omega_n - \frac{e_k}{\hbar} - \Sigma_{11}\right)\left(-i\omega_n - \frac{e_k}{\hbar} - \Sigma_{22}\right) - (-\Sigma_{12})(-\Sigma_{21})\right]$$

$$= \left(i\omega_n - \frac{e_k}{\hbar} - \Sigma_{11}\right)\left(i\omega_n + \frac{e_k}{\hbar} + \Sigma_{22}\right) + \Sigma_{12}\Sigma_{21}$$

$$G_{11}(k, i\omega_n) = \frac{i\omega_n + \frac{e_k}{\hbar} + \Sigma_{22}}{D(k, i\omega_n)}$$

$$G_{12}(k, i\omega_n) = -\frac{\Sigma_{12}(k, i\omega_n)}{D(k, i\omega_n)}$$

$$G_{21}(\ell, i\omega_n) = -\frac{\Sigma_{21}(\ell, i\omega_n)}{D(\ell, i\omega_n)}$$

$$G_{22}(\ell, i\omega_n) = \frac{-i\omega_n + \frac{e_\ell}{\hbar} + \Sigma_{11}}{D(\ell, i\omega_n)}$$

- with Σ_i known the interacting G.f. can be expressed.

Bogoliubov - aproximation

- Same as in quantum gases course
- For weakly interacting bose-gases.

$\downarrow$

typical interaction length scale $\ll$ avarage distance of the atom

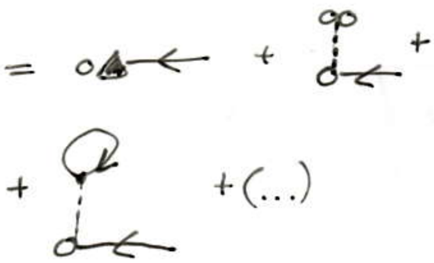
$\rightsquigarrow$ there is a length scale separation $\longrightarrow$ weak interaction: the atoms only hit each other with low energy

$\downarrow$

only s-wave scattering happens. $\rightarrow$ (isotropic in space.)

- which are the relevant graphs in such situations?

$$\boxed{-\Sigma_{10}(\ell, i\omega_n)} \xleftarrow[(0,0)]{} =$$



$\|$

$\langle b_0 \rangle$

$\rightsquigarrow$ for the self energy the incoming link matters not.

$$= \frac{1}{\hbar}\left[N_0^{1/2}(-\mu) + \frac{N_0^{3/2}}{V} V(0) \right]$$

$\rightsquigarrow$ we can neglect other terms far away from T_c, when $N_0 \gg N$

$\downarrow$

this brings particle number outside of condensate (N)

$\downarrow$

$\boxed{T = 0}$

- density must be small, too $\rightsquigarrow$ length separation.

- this is a bad approx for ^4He liquid, due to the strong interaction there.

$$0 = \frac{1}{\hbar}\left[\sqrt{N_0}(-\mu) + \frac{N_0^{3/2}}{V} v(0)\right] = \langle b_0 \rangle = -\Sigma_{10}$$

$$\boxed{\mu^B = v(0)\frac{N_0}{V}}$$

- this approx is equivalent of the G-P-eq. of homogenious sys...
- this is the Bogoliubov-chemical potential.

- condensate wave func.: $\int |\Psi_0|^2 d^3r = N_0$ (normalization)

$$\left[-\frac{\hbar^2}{2m}\Delta + V(r) + g|\Psi_0|^2\right]\Psi_0 = \mu\Psi_0 \qquad \text{(G-P-eq.)}$$

$V_{ext} = 0 \rightsquigarrow \Psi_0(r) = \text{const.} \rightsquigarrow$ condensate is homogenous.

$\hookrightarrow \Delta\Psi_0 = 0$

$$\rightsquigarrow g|\Psi_0|^2 = \mu$$

$$g\underbrace{n_0}_{\frac{N_0}{V}} = \mu \qquad \text{with } g = \frac{4\pi\hbar^2 a}{m}$$

and $v(r) = g\,\delta(r-r')$

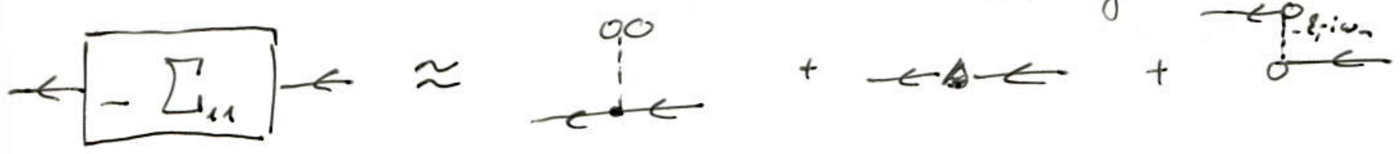
$\downarrow$

In the other formula we have FT of this

$\downarrow$

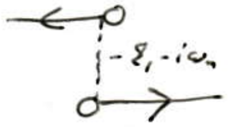
$$v(0) = g$$

- Now for $\underline{\underline{\Sigma}}$:



$$\Sigma_{11}^B(\vec{q}, i\omega_n) = \frac{1}{\hbar}\left[\underbrace{n_0\cancel{v(0)} - \cancel{\mu^B}}_{\text{eq. for } \mu^B} + n_0\underbrace{v(-\vec{q})}_{v(\vec{q})}\right] = \frac{n_0}{\hbar}v(\vec{q})$$

$$\leftarrow \boxed{-\Sigma_{12}} \rightarrow \;\approx$$



$$= \frac{n_0}{\hbar} v(q)$$

$$\Sigma_{12}^{B} = \frac{n_0}{\hbar} v(q)$$

$$\Sigma_{22}^{B} = \frac{n_0}{\hbar} v(q)$$

$$\Sigma_{21}^{B} = \frac{n_0}{\hbar} v(q)$$

$\leadsto$ self energy does not bring bose-freq. ($i\omega_n$) dependency in Bogoliubov approximation.

$$D^B(q, i\omega_n) = \left[i\omega_n - \frac{1}{\hbar}\left(e_q + n_0 v(q)\right)\right]\left[i\omega_n + \frac{1}{\hbar}\left(e_q + n_0 v(q)\right)\right] - + \left[\frac{n_0}{\hbar} v(q)\right]^2$$

$$G_{11}^{B}(q, i\omega_n) = \frac{i\omega_n + \frac{1}{\hbar}\left(e_q + n_0 v(q)\right)}{D^B(q, i\omega_n)}$$

$$G_{12}^{B}(q, i\omega_n) = - \frac{\frac{n_0}{\hbar} v(q)}{D^B(q, i\omega_n)} \qquad \omega_n = \frac{2\pi n}{\beta\hbar}$$

- <u>One-particle excitations</u> can be found by locating singularities in the retarded Gf.

$\hookrightarrow$ analytic continuation $i\omega_n \to \omega + i\varepsilon$

$\downarrow$

singularites $\Rightarrow \boxed{D = 0}$

$$D^B(q, \omega)\Big|_{\omega = \frac{E_q}{\hbar}} = \left(\frac{E_q}{\hbar}\right)^2 - \frac{1}{\hbar}\left(e_q + n_0 v(q)\right)^2 + \left(\frac{n_0}{\hbar} v(q)\right)^2 \stackrel{!}{=} 0$$

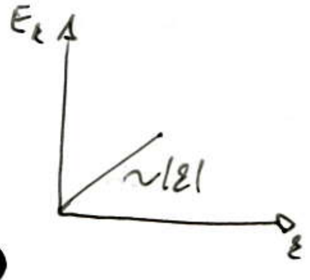
E_q : Bogoliubov-excitation energy

$$E_k = \sqrt{(e_k + n_0 V(k))^2 - (n_0 V(k))^2} = \sqrt{e_k^2 + 2 e_k n_0 V(k)}$$

- how does this behaves if $k \to 0$? (long wavelength limit)

$$E_k \underset{k\to 0}{\sim} \sqrt{2 \frac{\hbar^2 k^2}{2m} n_0 \underbrace{V(0)}_{\frac{4\pi\hbar^2 a}{m}}} \sim k$$

- same behaviour as Quantum gases.
- phonon - like behaviour



$$c_B = \sqrt{\frac{n_0 V(0)}{m}}$$ $\rightsquigarrow$ speed of sound in Bogoliubov-approx.

- other way is to diagonalize the Hamiltonian with Bogoliubov-transformation and neglecting the 3rd, 4th order in a, a^+.

$\rightsquigarrow$ similar to canonical transformation in fermions on BCS ansatz. $\quad \left| \hat{a}_k = u_k \hat{b}_k + v_k \hat{b}_{-k}^+ \right|$

$$G_{11} = \frac{u_k^2}{i\omega_n - \frac{E_k}{\hbar}} + \frac{(-v_k^2)}{i\omega_n + \frac{E_k}{\hbar}} = \frac{i\omega_n (u_k^2 - v_k^2) + \frac{E_k}{\hbar}(u_k^2 + v_k^2)}{\underbrace{(i\omega_n - \frac{E_k}{\hbar})(i\omega_n + \frac{E_k}{\hbar})}_{-\omega_n^2 - \frac{E_k^2}{\hbar^2}}}$$

partial fraction decomposition

must agree with what we derived on the last page

$$\left.\begin{aligned} u_k^2 - v_k^2 &= 1 \\ u_k^2 + v_k^2 &= \frac{e_k + n_0 V(k)}{E_k} \end{aligned}\right\} \rightsquigarrow \delta_{ij} = \int d^3r \, (u_i^* u_j - v_i^* v_j)$$

normalisation in the two component stuff.

$$u_k = \sqrt{\frac{1}{2}\left(1 + \frac{e_k + n_0 V(k)}{E_k}\right)}$$

$$v_k = \sqrt{\frac{1}{2}\left(-1 + \frac{e_k + n_0 V(k)}{E_k}\right)}$$

$$2 u_k v_k = \frac{n_0 v(k)}{E_k}$$

$$G_{12} = - u_k v_k \left(\frac{1}{i\omega_n - \frac{E_k}{\hbar}} - \frac{1}{i\omega_n + \frac{E_k}{\hbar}} \right)$$

- usually the collective excitations are interesting, too
- due to linearity in $|k|$

$\Downarrow$

the poles of density-density corr. and 1 part G.f. are the same!

$\rightsquigarrow$ In Bose-condensed systems 1 particle and collective excitations are basically the same.

- Going beyond the B. approx to explain interaction between normal and condensed atoms involves including graphs like  .