- other dim∅ number:

$$\frac{\Delta_0}{k_B T_c} = \frac{\pi}{\gamma} = 1.76$$

- few numbers for different superconductors:

	T_c [K]	$\frac{\hbar\omega_D}{k_B}$ [K]	$\frac{\Delta_0}{k_B T_c}$	$\frac{C_s(T_c)-C_n(T_c)}{C_n(T_c)}$
Cd	0.56	164	1.6	1.32 - 1.40
Al	1.2	375	1.3-2.1	1.45
Sn	3.75	195	1.6	1.60
Pb	7.22	96	2.2	2.71

not so bad prediction

not so bad either.

⤳ can be improved a lot by using the true fermi surface

2019.04.30.

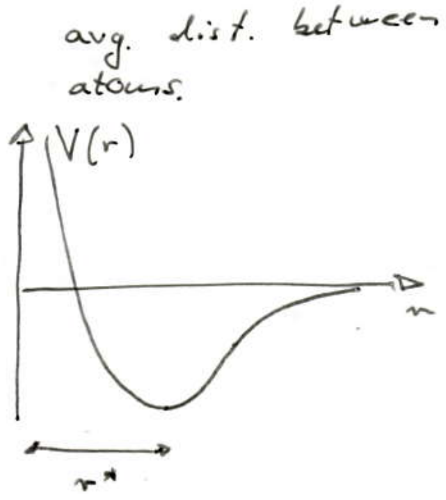
Green function formalism for Bose-condensed gases

- ^{4}He (liquid) ⟶ g.f. is needed
- ultracold trapped gases

length scales are not separable ⟶ interaction potential (r^*) avg. dist. between atoms.

↓

no approx. (Bogoliubov)



- in this case the contribution of higher order graphs is needed

- in case of a symmetry breaking (external pot.)

$G(\underline{r}_1, \underline{r}_2) \neq G(\underline{r}_1 - \underline{r}_2)$ but $G(\underline{r}_1 - \underline{r}_2, \frac{\underline{r}_1 + \underline{r}_2}{2})$ ⟶ Difficult!!

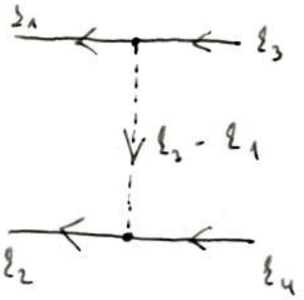
↳ Dyson-eq. turns out to be ugly...

- for ultracold gases hence Bogoliubov approx. is used
- However in case of ^4He it is useful.

⟶ We investigate homogenious system $\qquad e_k = \frac{\hbar^2 k^2}{2m}$

$$\hat{H} = \sum_k e_k \hat{a}_k^+ \hat{a}_k + \frac{1}{2V} \sum_{\substack{k_1 k_2 k_3 k_4 \\ k_1 + k_2 = k_3 + k_4}} \hat{a}_{k_1}^+ \hat{a}_{k_2}^+ V(k_1 - k_3)\, \hat{a}_{k_3} \hat{a}_{k_4}$$

- in a symmetric way



building blocks

- canonical commutation relations:

$$[a_k^+, a_{k'}] = \delta_{kk'} \quad \text{(all else commutes)}$$

$$\hat{K} = \hat{H} - \mu \hat{N}$$

$$\hat{N} = \sum_k \hat{a}_k^+ \hat{a}_k$$

- for $T < T_c$ ⤳ the $k = 0$ modes are very occupied.

(other are occupied acording to BE distribution)

$\langle a_0 \rangle = \sqrt{N_0}$, N_0 is # of bosons in the condensate
(0: $k=0$)

↓

Symmetry breaking

Symmetry breaking

$$\left.\begin{array}{l} \hat{a}_\ell' = e^{i\vartheta}\,\hat{a}_\ell \\ \hat{a}_\ell^{+\prime} = e^{-i\vartheta}\,\hat{a}_\ell^+ \end{array}\right\}$$ the $\hat{H}$ is invariant of this gauge-transformation (introducing global phases)

$\boxed{U(1) \text{ Symmetry}}$

• But $\langle a_0' \rangle = e^{i\vartheta} \underbrace{\langle a_0 \rangle}_{\sqrt{N_0}} = e^{i\vartheta}\sqrt{N_0} \neq \langle a_0 \rangle = \sqrt{N_0}$

$\rightarrow$ the ground state is less symmetric than the $\hat{H}$

$\rightsquigarrow$ has no $U(1)$ symmetry

• Canonical transformation:

$$\hat{b}_\ell = \hat{a}_\ell - \sqrt{N_0}\,\hat{\delta}_{\ell 0} \longrightarrow \langle b_\ell \rangle = 0 \;\; \forall \ell$$

$$\hat{b}_\ell^+ = \hat{a}_\ell^+ - \sqrt{N_0}\,\hat{\delta}_{\ell 0}$$

$$\left[\hat{b}_\ell^+ , \hat{b}_{\ell'}\right] = \hat{\delta}_{\ell\ell'}$$

$\rightsquigarrow$ the system is canonical

$$\left.\begin{array}{l} a_\ell = b_\ell + \sqrt{N_0}\,\delta_{\ell 0} \\ a_\ell^+ = b_\ell^+ + \sqrt{N_0}\,\delta_{\ell 0} \end{array}\right\}$$ we can put back to $\hat{H}$

$$\hat{K}_0 = \sum_\ell (e_\ell - \mu)\, a_\ell^+ a_\ell = \sum_\ell (e_\ell - \mu)(b_\ell^+ + \sqrt{N_0}\,\delta_{\ell 0})(b_\ell + \sqrt{N_0}\,\delta_{\ell 0}) =$$

$$= \sum_\ell (e_\ell - \mu)\, b_\ell^+ b_\ell + \sqrt{N_0}\,(\underset{\downarrow \atop 0}{e_0} - \mu)\left[b_0^+ + b_0\right] + N_0\,(\underset{\downarrow \atop 0}{e_0} - \mu) =$$

$$= \sum_\ell (e_\ell - \mu)\, b_\ell^+ b_\ell - \mu\sqrt{N}\,(b_0^+ + b_0) - \mu N_0$$

- Linear dependance of b_0, b_0^+ $\rightsquigarrow$ gives non-trivial thermodynamical avarages

- from now on $K_0 = \sum_{\ell}(e_\ell - \mu) b_\ell^+ b_\ell$

$$K_1' = -\mu\sqrt{N_0}\left(b_0^+ + b_0\right)$$

$$K_0' = -\mu N_0 \Big\} \text{ just a const. shift.}$$

- Interaction part:

$$\frac{1}{2V}\sum_{\substack{\ell_1 \ell_2 \ell_3 \ell_4 \\ \ell_1+\ell_2=\ell_3+\ell_4}} a_{\ell_1}^+ a_{\ell_2}^+ v(\ell_1-\ell_3)\, a_{\ell_3} a_{\ell_4} = K_{I4} + K_{I3} + K_{I2} + K_{I1} + K_{I0}$$

$$K_{I4} = \frac{1}{2V}\sum_{\substack{\ell_1 \ell_2 \ell_3 \ell_4 \\ \ell_1+\ell_2=\ell_3+\ell_4}} b_{\ell_1}^+ b_{\ell_2}^+ v(\ell_1-\ell_3)\, b_{\ell_3} b_{\ell_4}$$

$$K_{I3} = \frac{\sqrt{N_0}}{2V}\sum_{\ell_1 \ell_2 \ell_3 \ell_4}{}' v(\ell_1-\ell_3)\Big[\delta_{\ell_1 0} b_{\ell_2}^+ b_{\ell_3} b_{\ell_4} + b_{\ell_1}^+ \delta_{\ell_2 0} b_{\ell_3} b_{\ell_4} + b_{\ell_1}^+ b_{\ell_2}^+ \delta_{\ell_3 0} b_{\ell_4} + b_{\ell_1}^+ b_{\ell_2}^+ b_{\ell_3} \delta_{\ell_4 0}\Big]$$

$$K_{I2} = \frac{N_0}{2V}\sum_{\ell_1 \ell_2 \ell_3 \ell_4}{}' v(\ell_1-\ell_3)\Big[\delta_{\ell_1 0}\delta_{\ell_2 0} b_{\ell_3} b_{\ell_4} + \delta_{\ell_1 0} b_{\ell_2}^+ \delta_{\ell_3 0} b_{\ell_4} + \delta_{\ell_1 0} b_{\ell_2}^+ b_{\ell_3} \delta_{\ell_4 0} + b_{\ell_1}^+ \delta_{\ell_2 0}\delta_{\ell_3 0} b_{\ell_4} + b_{\ell_1}^+ \delta_{\ell_2 0} b_{\ell_3} \delta_{\ell_4 0} + b_{\ell_1}^+ b_{\ell_2}^+ \delta_{\ell_3 0}\delta_{\ell_4 0}\Big]$$

$$K_{I1} = \frac{N_0^{3/2}}{2V} v(0)\left[2b_0^+ + 2b_0\right]$$

$$K_{I0} = \frac{N_0^2}{2V} v(0)$$

$\rightsquigarrow$ much more terms in the Hamiltonian

$$K = K_0 + K_0' + K_1' + K_{I4} + K_{I3} + K_{I2} + K_{I1} + K_{I0}$$

- defining the Green's functions:

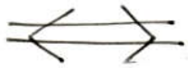
$$G_{11}(q,\tau) = -\langle T_\tau\, b_q(\tau)\, b_q^+(0)\rangle$$ the full Green's func.

$$\hat{O}(\tau) = e^{\frac{K\tau}{\hbar}}\, \hat{O}\, e^{-\frac{K\tau}{\hbar}}$$

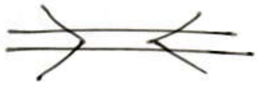
$$\langle \hat{O} \rangle = \mathrm{Tr}\left(\hat{\rho}_G\, \hat{O}\right) \qquad \hat{\rho}_G = \frac{e^{-\beta K}}{Z_G}$$

$$G_{12}(q,\tau) = -\langle T_\tau\, b_q(\tau)\, b_{-q}(0)\rangle$$ non-trivial Gf.

$\rightsquigarrow$ symmetry breaking $\rightarrow$



$$G_{21}(q,\tau) = -\langle T_\tau\, b_q^+(\tau)\, b_{-q}^+(0)\rangle \rightarrow$$



$$G_{22}(q,\tau) = -\langle T_\tau\, b_{-q}^+(\tau)\, b_{-q}(0)\rangle \rightarrow$$

- Green's func. is now a 2×2 matrix

$$b_{q,\alpha} = \begin{cases} b_q & \alpha = 1 \\ b_{-q}^+ & \alpha = 2 \end{cases}$$

$$G_{\alpha\beta}(q,\tau) = -\langle T_\tau\, b_{q\alpha}(\tau)\, b_{q\beta}^+(0)\rangle$$

we can summarize the 4 possibilities

• properties:

$$G_{11}(-k,-\tau) = G_{22}(k,\tau)$$

/there is some cc. relation between G_{12}, G_{21} .../

• Matsubara-frequencies:

$$G_{\alpha\beta}(k, i\omega_n) = \int_0^{\beta\hbar} G_{\alpha\beta}(k,\tau)\, e^{i\omega_n \tau}\, d\tau \qquad \omega_n = \frac{2n\pi}{\beta\hbar}$$

$$G_{\alpha\beta}(k,\tau) = \frac{1}{\beta\hbar}\sum_n G_{\alpha\beta}(k, i\omega_n)\, e^{-i\omega_n\tau}$$

• What is the perturbation? (for int. picture)

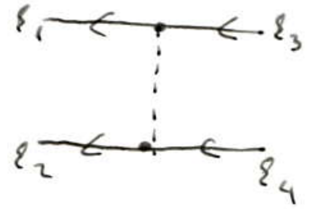
$$K_0 = \sum_k (e_k - \mu)\, b_k^+ b_k$$

$\rightsquigarrow$ all the others are perturbation

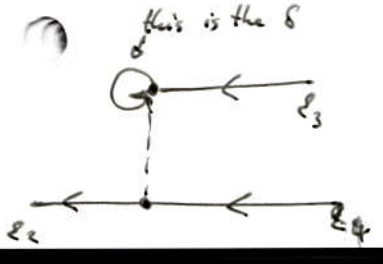
$$\left.\begin{aligned} G_{11}^{(0)}(k, i\omega_n) &= \frac{1}{i\omega_n - \frac{1}{\hbar}(e_k - \mu)} \\ G_{22}^{(0)}(k, i\omega_n) &= \frac{1}{-i\omega_n - \frac{1}{\hbar}(e_{-k} - \mu)} \\ G_{12}^{(0)}(k, i\omega_n) &= G_{21}^{(0)}(k, i\omega_n) = 0 \end{aligned}\right\}$$

non-interacting G.f.
$\rightsquigarrow$ diagonal

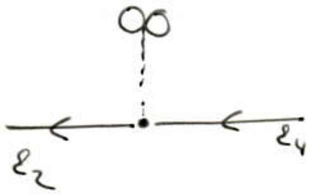
• the interaction terms:



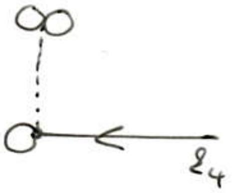
$$\frac{1}{2V}\, b_{k_1}^+ b_{k_2}^+\, v(k_1 - k_3)\, b_{k_3} b_{k_4} \qquad k_1 + k_2 = k_3 + k_4$$



$$\frac{\sqrt{N_0}}{2V}\, \delta_{k_1 0}\, b_{k_2}^+ \underbrace{v(k_1 - k_3)}_{v(-k_3)}\, b_{k_3} b_{k_4}$$

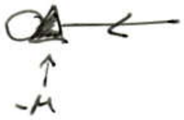


$$\frac{N_0}{2V}\, b^+_{q_2}\, \underbrace{V(q_1-q_3)}_{V(0)}\, b_{q_4}\, \delta_{q_1,0}\, \delta_{q_3,0}$$



$$\frac{N_0^{3/2}}{2V}\, V(0)\, b_{q_4}\, \delta_{q_1,0}\, \delta_{q_2,0}\, \delta_{q_3,0}$$

$$\frac{N_0^2}{2V}$$

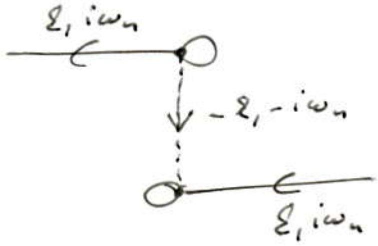


$$(-\mu)\sqrt{N_0}\, b_0$$

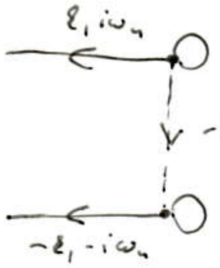
$$(-\mu)\sqrt{N_0}\, b_0^+$$

$$(-\mu) N_0$$

- let's look at this figure:



$\rightsquigarrow$ this only gives contribution to G_{11}, G_{22}



$\rightsquigarrow$ this gives G_{12} contributions