$$\frac{1}{\hbar\beta} g \int^{1} \frac{d^3q}{(2\pi)^3} \sum_n \frac{\hbar}{(\hbar\omega_n)^2 + \varepsilon_q^2}\Big|_{T=T_c} = g\,\nu_F \ln\left[\frac{\hbar\omega_D}{k_B T_c}\cdot\frac{2\gamma}{\pi}\right]$$

- now T_c is replaced by T:

$$\frac{1}{\hbar\beta} g \int^{1} \frac{d^3q}{(2\pi)^3} \sum_n \frac{\hbar}{(\hbar\omega_n)^2 + \varepsilon_q^2} = g\,\nu_F \underbrace{\ln\left[\frac{\hbar\omega_D}{k_B T_c}\cdot\frac{2\gamma}{\pi}\cdot\frac{T_c}{T}\right]}_{\ln\left(\frac{\hbar\omega_D}{k_B T_c}\frac{2\gamma}{\pi}\right) + \ln\left(\frac{T_c}{T}\right)} = 1 + g\nu_F \ln\left(\frac{T_c}{T}\right)$$

in the leading term.

- the second term:

$\beta \Rightarrow \beta_c$, since it's a cons. close to T_c

$$-\frac{1}{\beta} g \int^{1} \frac{d^3q}{(2\pi)^3} \sum_n \frac{\Delta^2}{\left((\hbar\omega_n)^2 + \varepsilon_q^2\right)^2} = -g\left(\frac{\Delta}{k_B T_c}\right)^2 \int^{1} \frac{d^3q}{(2\pi)^3} \sum_n \frac{(k_B T_c)^3}{\left((\hbar\omega_n^c)^2 + \varepsilon_q^2\right)^2} =$$

$$= -g\left(\frac{\Delta}{k_B T_c}\right)^2 \underbrace{\nu_F}_{(k_B T_c)^3} \sum_n 2 \underbrace{\int_0^{\hbar\omega_D} d\varepsilon \frac{1}{\left((\hbar\omega_n^c)^2 + \varepsilon^2\right)^2}}_{\left[\frac{\frac{\hbar\omega_n^c\,\varepsilon}{(\hbar\omega_n^c)^2+\varepsilon^2} + \mathrm{arctg}\left(\frac{\varepsilon}{\hbar\omega_n^c}\right)}{2(\hbar\omega_n^c)}\right]_0^{\hbar\omega_D}}$$

$$= \frac{\pi}{2}\,\frac{\mathrm{sgn}(\hbar\omega_n^c)}{2(\hbar\omega_n^c)^3} = \frac{\pi}{4}\,\frac{1}{|\hbar\omega_n^c|^3}$$

it can be $\ominus$!

- trick:

$\rightsquigarrow$ for some (several!) n-s are $\ll \hbar\omega_D$, and we count only those!

$\rightsquigarrow$ first part is $\emptyset$ at both bounds.

2019.04.09.

$$= -g\left(\frac{\Delta}{k_B T_c}\right)^2 (k_B T_c)^3 \sum_n \left(\nu_F \cdot \frac{\pi}{2}\,\frac{1}{|\hbar\omega_n^c|^3}\right)$$

$$\frac{\pi}{4}\frac{1}{|\hbar\omega_n^c|^3} = \frac{\pi}{4}\frac{\beta_c^3\,\cancel{\hbar^3}}{|2n+1|^3\,\pi^3\cdot\cancel{\hbar^3}}$$

• so the 2nd term altogether:

$$-\frac{1}{\pi^2}\frac{g}{\cancel{2}}\nu_F\left(\frac{\Delta}{k_B T_c}\right)^2 \underbrace{\sum_{n=-\infty}^{\infty}\frac{1}{|2n+1|^3}}_{\cancel{2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}} = -\frac{1}{\pi^2}g\nu_F\left(\frac{\Delta}{k_B T_c}\right)^2\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} =$$

we can do this...

→ contribution of ⊖ even numbers is the same as the ⊕ since $|.|$

$$= -\frac{1}{\pi^2}g\nu_F\left(\frac{\Delta}{k_B T_c}\right)^2\underbrace{\left[\sum_{n=1}^{\infty}\frac{1}{n^3} - \sum\frac{1}{(2n)^3}\right]}_{\left(1-\frac{1}{8}\right)\sum_{n=1}^{\infty}\frac{1}{n^3} = \frac{7}{8}\zeta(3)} =$$

$$= -\frac{1}{\pi^3}g\nu_F\left(\frac{\Delta}{k_B T_c}\right)^2\cdot\frac{7}{8}\zeta(3)$$

• Altogether:

$$0 = -g\nu_F\underbrace{\ln\left(\frac{T_c+\Delta T}{T_c}\right)} - \frac{7}{8\pi^3}g\nu_F\left(\frac{\Delta(T)}{k_B T_c}\right)^2\zeta(3)$$

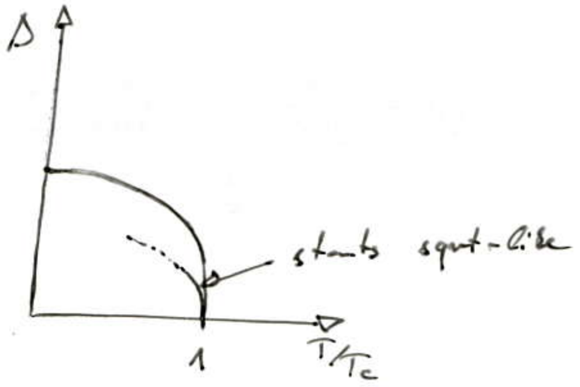
/ where $\Delta T = T - T_c$ /

$$\ln\left(1+\frac{\Delta T}{T_c}\right)\xrightarrow{\frac{\Delta T}{T_c}\ll 1}\frac{\Delta T}{T_c} = \frac{T-T_c}{T_c}$$

$\Delta(T)$ can be expressed:

$T \lesssim T_c$

$$\boxed{\Delta(T) = k_B T_c\underbrace{\sqrt{\frac{8\pi^2}{7\zeta(3)}}}_{\sim 3.0632}\sqrt{1-\frac{T}{T_c}}}$$



- this approx is okay for weakly int. superconductors.
- otherwise there are problems with the cutting of ω-s in the integral...

Thermodynamical stability of the SC state

$T < T_c$

- SC state is stable
- $\underline{\Delta = 0}$ is always a solution of the (original) gap-eq.

$$\downarrow$$

$$\Delta = \frac{g}{\beta\hbar} \int \frac{d^3 q}{(2\pi)^3} \sum_n \frac{\hbar\Delta}{(\hbar\omega_n)^2 + \xi_q^2 + \Delta^2}$$

- $\underline{(\Omega_{SC} - \Omega_N) < 0}$
  where Ω is the grand canonical potential (stability = more $\ominus$)

$$\Delta = -\frac{g}{V} \sum_p{}' \langle \hat{a}_{-p\downarrow} \hat{a}_{p\uparrow} \rangle \leftarrow \text{same gap eq.}$$

- Ω can be obtained only using the free Hamiltonian...

$$\Omega_{SC} = \underset{\Omega_N}{\underset{\parallel}{\overset{\text{non-int. part}}{\overset{\downarrow}{\Omega_0}}}} + \int_0^1 d\lambda \, \frac{1}{\lambda} \langle \lambda H_1 \rangle_\lambda$$

H_1 is the interaction:

$$\hat{H}_1 = \frac{1}{2V} \sum_{\substack{q \\ k,k' \\ \sigma,\sigma'}} V(q)\, \hat{a}^+_{k+q,\sigma} \hat{a}^+_{k'-q,\sigma'} \hat{a}_{k'\sigma'} \hat{a}_{k\sigma}$$

$$\langle H_1 \rangle = -\frac{g}{2V} \sum_{\substack{q \\ k,k' \\ \sigma,\sigma'}} \langle \hat{a}^+_{k+q\sigma} \hat{a}^+_{k'-q\sigma'} \rangle \underbrace{\langle \hat{a}_{k',\sigma'} \hat{a}_{k,\sigma} \rangle}_{\delta_{k,k'}\,\delta_{\sigma',-\sigma}}$$

$$= -\frac{g}{V}\sum_{\ell}\langle \hat{a}^{+}_{-\ell\uparrow}\hat{a}^{+}_{\ell\downarrow}\rangle \sum_{\ell'}\langle \hat{a}_{\ell'\uparrow}\hat{a}_{-\ell\downarrow}\rangle = -\frac{V}{g}\Delta^2$$

$$\boxed{\Delta = -\frac{g}{V}\sum_{p}{}'\langle \hat{a}_{p\downarrow}\hat{a}_{-p\uparrow}\rangle}$$

using the Wick-theorem and keeping the anomalous average.
(We leave out the Hartree-Fock terms, that can be used to renormalize the m, μ for the electrons...)

$$\Omega_s - \Omega_N = \int_0^1 \frac{d\lambda}{\lambda}\langle \lambda H_1\rangle_\lambda$$

$\lambda g = g'$ ⇝ running coupling constant

$$\frac{1}{\lambda} = \frac{g}{g'}$$

$$d\lambda = \frac{dg'}{g}$$

$$= \int_0^g \frac{dg'}{g'}(-)\frac{V}{g'}\Delta^2(g') =$$

$$= -V\int_0^g \frac{dg'}{g'^2}\Delta^2(g')$$

- in place of g' we introduce Δ', that is a monotonous func. of g'

$$dg' = \left(\frac{dg'}{d\Delta'}\right)d\Delta'$$

$$= -V\int_0^\Delta d\Delta'\cdot\Delta'^2\cdot\frac{1}{g'^2}\cdot\left(\frac{dg'}{d\Delta'}\right) = +V\int_0^\Delta d\Delta'\cdot\Delta'^2\cdot\underbrace{\left(\frac{d\frac{1}{g'}}{d\Delta'}\right)}_{}$$

this can be obtained from the gap-eq.

$$\hat{\Delta}(g) = \frac{g}{V}\sum_q{}'\frac{1}{\beta}\sum_n\frac{\hat{\Delta}(g)}{(\hbar\omega_n)^2+\xi_q^2+\Delta^2(g)}$$

$$\frac{1}{g} = \frac{1}{V}\sum_q{}'\frac{1}{\beta}\sum_n\frac{1}{(\hbar\omega_n)^2+\xi_q^2+\Delta^2(g)} \quad \Big/ \frac{d}{d\Delta}$$

$$\frac{d\frac{1}{g}}{d\Delta} = \frac{-1}{\beta}\int^{1}\frac{d^3q}{(2\pi)^3}\sum_n \frac{2\Delta(g)}{\left((\hbar\omega_n)^2 + E_q^{\,2}\right)^2}$$

$\longrightarrow$

$$\Omega_s - \Omega_N = \underbrace{-\frac{2V}{\beta}\int_0^{\Delta} d\Delta' \int^{1}\frac{d^3q}{(2\pi)^3}\sum_n \frac{\Delta'^3}{\left((\hbar\omega_n)^2 + E_q^{\,2}\right)^2}}_{>0} < 0$$

everything is positive

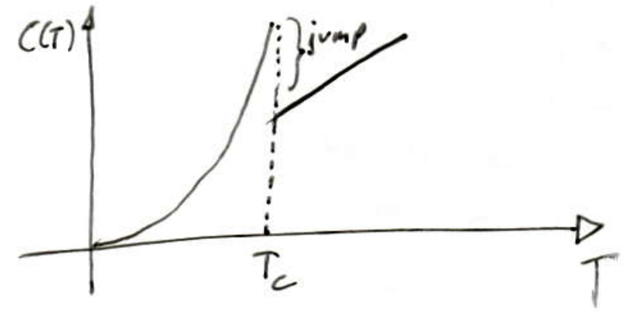
• estimating this integral close to T_c:

$$\frac{\Omega_s - \Omega_N}{V} \approx -\frac{2}{\beta}\underbrace{\int_0^{\Delta} d\Delta'\,\Delta'^3}_{\frac{\Delta^4}{4}}\underbrace{\int^{1}\frac{d^3q}{(2\pi)^3}\sum_n \frac{1}{\left[(\hbar\omega_n)^2 + \epsilon_q^{\,2}\right]^2}}_{\beta\nu_F\frac{\pi}{4}\sum_n\frac{1}{|\hbar\omega_n^c|^3}}$$

and $\Delta(T) \approx \text{const.}\left(1 - \frac{T}{T_c}\right)^{1/2}$

$$\frac{\Omega_s - \Omega_N}{V} \approx -\frac{4}{7}\frac{\pi^2}{\zeta(3)}(k_B T_c)^2\,\nu_F\left(1 - \frac{T}{T_c}\right)^2$$

Heat capacity of the superconductors



$$\frac{C_s(T_c) - C_N(T_c)}{C_N(T_c)} = ?$$

$$d\Omega = -S\,dT - p\,dV - N\,d\mu + \underbrace{\sum_i \Lambda_i\, d\lambda_i}$$

other intrinsic extensive pairs.

$$dF = -S\,dT - p\,dV + \mu\, dN + \sum_i \Lambda_i\, d\lambda_i$$

$N_s = N_N$ $\qquad\qquad$ $F = E - TS$

$\mu_s = \mu_N$ $\qquad\qquad$ $\Omega = E - TS - \mu N$

$$\Omega_s - \Omega_N = F_s - F_N \underbrace{- (\mu N)_s + (\mu N)_N}_{-\mu \underbrace{(N_s - N_N)}_{\varnothing}}$$

$$\Omega_s - \Omega_N = F_s - F_N \leadsto C = -T\left(\frac{\partial^2 F}{\partial T^2}\right)_{V,N}$$

$$C_s(T_c) - C_N(T_c) = -T_c \frac{\partial^2}{\partial T^2}\underbrace{\left(F_s - F_N\right)_{V,N}}_{\Omega_s - \Omega_N}\Bigg|_{T_c} =$$

$$= -T_c \frac{\partial^2}{\partial T^2}\left[-V\frac{4}{7}\frac{\pi^2}{\zeta(3)}(k_B T_c)^2 \nu_F \left(1 - 2\frac{T}{T_c} + \frac{T^2}{T_c^2}\right)\right]_{T=T_c} =$$

$$= \boxed{V\frac{8}{7}\frac{\pi^2}{\zeta(3)} k_B^2 T_c\, \nu_F = C_s(T_c) - C_N(T_c)} > 0$$

$$C_N(T_c) = V\frac{2\pi^3}{3}\nu_F k_B T_c \quad \text{(for a normal gas)}$$

$$\frac{C_s(T_c) - C_N(T_c)}{C_N(T_c)} = \frac{12}{7}\frac{1}{\zeta(3)} \approx \underline{\underline{1.42613}}$$

$\leadsto$ universal number for weakly int. SC-s...

• other dim∅ number:

$$\frac{\Delta_0}{k_B T_c} = \frac{\pi}{\gamma} = 1.76.$$

• few numbers for different superconductors:

	T_c [K]	$\frac{\hbar\omega_D}{k_B}$ [K]	$\frac{\Delta_0}{k_B T_c}$	$\frac{C_s(T_c)-C_n(T_c)}{C_n(T_c)}$
Cd	0.56	164	1.6	1.32 - 1.40
Al	1.2	375	1.3 - 2.1	1.45
Sn	3.75	195	1.6	1.60
Pb	7.22	96	2.2	2.71

not so bad prediction

not so bad either.

⇝ can be improved a lot by using the true fermi surface