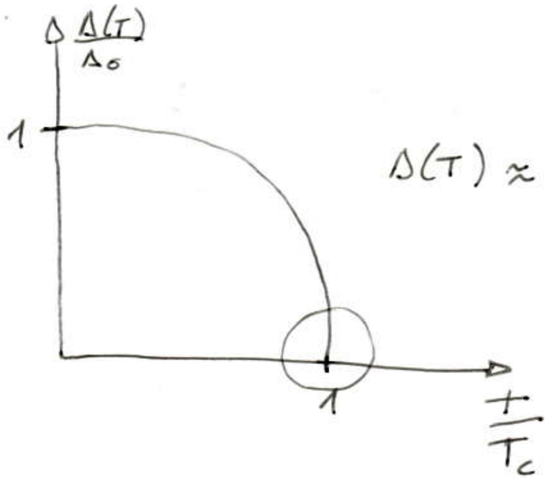
2019.04.02.

- $\frac{\Delta_0}{k_B T_c} \sim \mathcal{O}(1)$ allways! $\qquad \frac{\Delta_0}{k_B T_c} \sim \mathcal{O}(\hbar\omega_0)\cdot \underbrace{e^{-\frac{1}{g\nu_F}}}_{\text{small number}}$

there are systems, where this relation does not hold, however the other one does.

there: $\Delta_0, k_B T_c \sim \varepsilon_F$ for ultra-cold gases

- In metals the order parameter of the phase tr. is Δ.



$$\Delta(T) \approx 3.06 \cdot k_B T_c \left(1 - \frac{T}{T_c}\right)^{1/2}$$

$\boxed{\beta = \frac{1}{2}}$ in 3D

mean-field like critical exponent.

$$\varepsilon_p = \frac{p^2}{2m} - \mu \qquad \omega_n = \frac{(2n+1)\pi}{\beta\hbar}$$

$$E_p = \sqrt{\varepsilon_p^2 + \Delta^2(p)}$$

$\Delta^2(p) = \Delta^2$ ⇝ in our model Δ has no p dependance. (weak int.)

$$G(p, i\omega_n) = -\frac{\hbar(i\hbar\omega_n + \varepsilon_p)}{(\hbar\omega_n)^2 + E_p^2} = \hbar\left(\frac{u_p^2}{i\hbar\omega_n - E_p} + \frac{v_p^2}{i\hbar\omega_n + E_p}\right)$$

allways have this form.

- Numerators: $i\hbar\omega_n + \varepsilon_p \overset{!}{=} u_p^2(i\hbar\omega_n + E_p) + v_p^2(i\hbar\omega_n - E_p)$ $\quad \forall n$

$$\left.\begin{aligned} 1 &= u_p^2 + v_p^2 \\ \varepsilon_p &= E_p(u_p^2 - v_p^2) \end{aligned}\right\}$$

$$u_p = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_p}{E_p}\right)} \;; \quad v_p = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_p}{E_p}\right)}$$

these are the same, as in the BCS ansatz.

$$2u_p v_p = \sqrt{1-\frac{\epsilon_p^2}{E_p^2}} = \sqrt{\frac{E_p^2-\epsilon_p^2}{E_p^2}} = \frac{\Delta}{E_p}$$

$$F(p, i\omega_n) = F^+(p, i\omega_n) = -\hbar u_p v_p \left(\frac{1}{i\hbar\omega_n - E_p} - \frac{1}{i\hbar\omega_n + E_p}\right)$$

→ this will reproduce the usual fm of F

- Reminder: Matsubara Sum

$$\sum_n \frac{e^{i\omega_n \eta}}{i\omega_n - \frac{E}{\hbar}} = \frac{\beta\hbar}{e^{\beta E}+1}$$

- Momentum distribution with non-zero Δ:

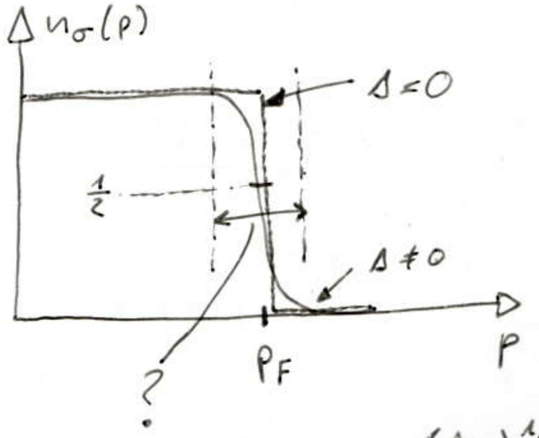
use the G from previous page...

$$n_\sigma(p) = G(p, \tau = 0-\eta) = \frac{1}{\beta\hbar}\sum_n G(p, i\omega_n)\, e^{i\omega_n \eta} =$$

$\sigma = \uparrow, \downarrow$

$$= \frac{u_p^2}{e^{\beta E_p}+1} + \underbrace{\frac{v_p^2}{e^{-\beta E_p}+1}}_{v_p^2\left(1-\frac{1}{e^{\beta E_p}+1}\right)} = \frac{u_p^2 - v_p^2}{e^{\beta E_p}+1} + v_p^2$$

- Special case $T=0$: $n_\sigma(p) = v_p^2 = \frac{1}{2}\left(1-\frac{\epsilon_p}{E_p}\right)$



⇝ no jump
⇝ no sharp Fermi-surface
(sometimes no surface at all...)

- the unknown width: $\sim\left(\frac{\Delta_0}{e_F}\right)^{1/2}$, and $e_F = \mu = \frac{\hbar^2 k_F^2}{2m}$

$$\frac{\epsilon_p}{E_p} \xrightarrow{\Delta \to 0} \begin{cases} +1 & \frac{p^2}{2m} > \mu \\ -1 & \frac{p^2}{2m} < \mu \end{cases}$$

- the boundaries are defined somehow by:

$$\Delta = \pm\left(\frac{p^2}{2m} - \mu\right)$$

$$\downarrow$$

$$\frac{1}{2}\,\frac{\sqrt{2}\,\Delta \pm \Delta}{\sqrt{2}\,\Delta} = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

$p = \hbar k$

$$\Delta^2 = \left(\frac{\hbar^2 k^2}{2m} - \frac{\hbar^2 k_F^2}{2m}\right)^2 \approx 4\left(\frac{\hbar^2}{2m}\right)^2 \underbrace{(k-k_F)^2}_{\delta k^2} \underset{k \approx k_F}{\underset{\uparrow}{(k+k_F)^2}}$$

$$\delta k = \frac{\Delta m}{\hbar^2 k_F} = \frac{\Delta}{\hbar v_F}$$

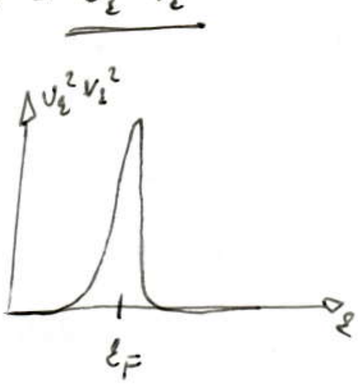
- How far away are the partners in a Cooper-pair?

$$\frac{1}{\delta k} = \frac{\hbar v_F}{\Delta} = \frac{\hbar^2 k_F^2}{2m}\cdot\frac{1}{\Delta}\cdot\frac{2}{k_F} = \underbrace{\frac{E_F}{\Delta}}_{\text{huge number!}}\cdot\underbrace{\frac{2}{k_F}}_{\text{inter-particle distance}}$$

huge number! → the partners are very far away!

- What are the numbers of the pairs?
  - Only a few % of e^- are paired
  - At higher temp. it is even less.

$$\underset{\overset{\|}{(u_k^2+v_k^2)v_k^2}}{\langle a^+_{k\downarrow} a^+_{-k\uparrow} a_{-k\downarrow} a_{k\uparrow}\rangle} - \underset{\overset{\|}{v_k^2}}{\langle a^+_{k\uparrow} a_{k\uparrow}\rangle}\,\underset{\overset{\|}{v_k^2}}{\langle a^+_{-k\downarrow} a_{-k\downarrow}\rangle} = \underline{u_k^2 v_k^2}$$



$$\frac{N_{CP}}{V} = \frac{1}{V}\sum_k u_k^2 v_k^2 = \int \frac{d^3k}{(2\pi)^3}\,\frac{\Delta^2_{(k)}}{4E_k^2}$$

$$\Delta(\varepsilon) = \begin{cases} \Delta_0 & \text{if } -\hbar\omega_D \leq \frac{\hbar^2 \varepsilon^2}{2m} - \mu \leq \hbar\omega_D \\ 0 & \text{otherwise} \end{cases}$$ ← from the gap-eq.

$\hbar\omega_D >> \Delta$, exp small factor!

$$= \frac{\Delta_0^2 \nu_F}{4} \int_{-\hbar\omega_D}^{\hbar\omega_D} d\varepsilon \, \frac{1}{\varepsilon^2 + \Delta^2} = \frac{\nu_F \Delta_0^2}{4} \left[\frac{1}{\Delta_0} \operatorname{arctg}\left(\frac{\varepsilon}{\Delta_0}\right) \right]_{-\hbar\omega_D}^{\hbar\omega_D} = \frac{\nu_F \Delta_0^{\cancel{2}}}{4} \frac{\pi}{\cancel{\Delta_0}} = \boxed{\frac{\pi}{4} \nu_F \Delta_0}$$

density of Cooper-pairs

⇓

$$N_{cp} = V \cdot \frac{\pi}{4} \nu_F \Delta_0$$

$\nu_F = \frac{n}{3 e_F}$ can be written like this

$n = \frac{N}{V}$ where N is the total num. of e^-

$$\longrightarrow \boxed{\frac{N_{cp}}{N} = \frac{\pi}{12} \cdot \frac{\Delta_0}{e_F}}$$

this is an extremely small number

- Δ_0 is small
- e_F is big

$\frac{N_{cp}}{N} \sim 10^{-3} - 10^{-4}$ in a typical metal.

- Delta as a func. of temperature:

$\Delta(T)$ around $\underline{T \lesssim T_c}$

- Gap eq. (before Matsubara sum):

• restriction: $-\hbar\omega_D \leq \frac{\hbar^2 \varepsilon^2}{2m} - \mu \leq \hbar\omega_D$

$$\Delta = \frac{g}{\beta\hbar} \int' \frac{d^3q}{(2\pi)^3} \sum_n \frac{\hbar\Delta}{(\hbar\omega_n)^2 + \varepsilon_q^2 + \Delta^2}$$

- close to T_c we can expand for small Δ:

$$1 = \frac{g}{\beta} \int' \frac{d^3q}{(2\pi)^3} \sum_n \frac{1}{(\hbar\omega_n)^2 + \varepsilon_q^2} \left[1 - \frac{\Delta^2}{(\hbar\omega_n)^2 + \varepsilon_q^2} + \mathcal{O}\left(\frac{\Delta^4}{\varepsilon_q T_c^4}\right) \cdots \right]$$

$$\frac{1}{\hbar\beta} g \int^{1} \frac{d^3q}{(2\pi)^3} \sum_n \frac{\hbar}{(\hbar\omega_n)^2 + \varepsilon_q^2}\Bigg|_{T=T_c} = g\,\nu_F \ln\left[\frac{\hbar\omega_D}{k_B T_c}\cdot\frac{2\gamma}{\pi}\right]$$

• now T_c is replaced by T:

$$\frac{1}{\hbar\beta} g\int^1 \frac{d^3q}{(2\pi)^3}\sum_n \frac{\hbar}{(\hbar\omega_n)^2+\varepsilon_q^2} = g\,\nu_F \underbrace{\ln\left[\frac{\hbar\omega_D}{k_BT_c}\cdot\frac{2\gamma}{\pi}\cdot\frac{T_c}{T}\right]}_{\ln\left(\frac{\hbar\omega_D}{k_BT_c}\frac{2\gamma}{\pi}\right)+\ln\left(\frac{T_c}{T}\right)} = 1 + g\nu_F\ln\left(\frac{T_c}{T}\right)$$

in the leading term.

• the second term:

$\beta \Rightarrow \beta_c$, since it's a conv. close to T_c

$$-\frac{1}{\beta} g\int^1\frac{d^3q}{(2\pi)^3}\sum_n \frac{\Delta^2}{((\hbar\omega_n)^2+\varepsilon_q^2)^2} = -g\left(\frac{\Delta}{k_BT_c}\right)^2 \int^1 \frac{d^3q}{(2\pi)^3}\sum_n \frac{(k_BT_c)^3}{((\hbar\omega_n^c)^2+\varepsilon_q^2)^2} =$$

$$= -g\left(\frac{\Delta}{k_BT_c}\right)^2 \nu_F \sum_n 2 \underbrace{\int_0^{\hbar\omega_D} d\varepsilon\, \frac{1}{((\hbar\omega_n^c)^2+\varepsilon^2)^2}}$$

$(k_BT_c)^3$

$$\left[\frac{\frac{\hbar\omega_n^c\, \varepsilon}{(\hbar\omega_n^c)^2+\varepsilon^2} + \mathrm{arctg}\left(\frac{\varepsilon}{\hbar\omega_n^c}\right)}{2(\hbar\omega_n^c)}\right]_0^{\hbar\omega_D} = \frac{\pi}{2}\,\frac{\mathrm{sgn}(\hbar\omega_n^c)}{(\hbar\omega_n^c)^3} = \frac{\pi}{2}\,\frac{1}{|\hbar\omega_n^c|^3}$$

it can be $\ominus$!

◦ trick:

⇝ for some (several!) n-s are $\ll \hbar\omega_D$,
and we count only those!

⇝ first part is $\emptyset$ at both bounds.