2019.03.26.

$$\boxed{-\left(\hbar\frac{\partial}{\partial\tau}+\varepsilon_p\right)G(p,\tau-\tau')-\frac{1}{V}\sum_q V(q)F(p-q,0)F^+(p,\tau-\tau')=\hbar\delta(\tau-\tau')}$$

- We need EoM for F too!

$$\hbar\frac{\partial}{\partial\tau}F^+(p,\tau-\tau')=-\hbar\frac{\partial}{\partial\tau}\Big[\vartheta(\tau-\tau')\langle a^+_{-p\downarrow}(\tau)a^+_{p\uparrow}(\tau')\rangle$$
$$-\vartheta(\tau'-\tau)\langle a^+_{p\uparrow}(\tau')a^+_{-p\downarrow}(\tau)\rangle\Big]=$$

$$=-\hbar\,\delta(\tau-\tau')\underbrace{\langle\{a^+_{-p\downarrow}(\tau),a^+_{p\uparrow}(\tau')\}\rangle}-$$
$$-\left\langle\hat{T}_\tau\left(\left(\hbar\frac{\partial}{\partial\tau}a^+_{-p\downarrow}(\tau)\right)a^+_{p\uparrow}(\tau')\right)\right\rangle$$

eq. time anticomm
$\tau'\to\tau$
it is $\emptyset$!

$$\Rightarrow\quad \hbar\frac{\partial}{\partial\tau}a^+_{p,\sigma}=\left[K(\tau),a^+_{p\sigma}\right]=\varepsilon_p a^+_{p\sigma}+\frac{1}{V}\sum_{p',\sigma',q}V(q)\,a^+_{p+q,\sigma}(\tau)\,a^+_{p'-q,\sigma'}(\tau)\,a_{p',\sigma'}(\tau)$$

we can insert it to the eq. above...

$$0=\left(-\hbar\frac{\partial}{\partial\tau}+\varepsilon_p\right)F^+(p,\tau-\tau')-\frac{1}{V}\sum_{p',q,\sigma'}V(q)\underbrace{\left\langle\hat{T}_\tau\,a^+_{-p+q\downarrow}(\tau)\,a^+_{p'-q\sigma'}(\tau)\,a_{p',\sigma'}(\tau)a^+_{p\uparrow}(\tau')\right\rangle}$$

- now we can apply Wick's theorem
- we only keep anomalous, eq. time stuff

$$\underbrace{\langle a^+_{-p+q\downarrow}(\tau)\,a^+_{p'-q\sigma'}(\tau)\rangle}_{\text{can be calc. from BCS grund. state}}\langle\hat{T}_\tau\,a_{p'q'}(\tau)\,a^+_{p\uparrow}(\tau')\rangle+\dots$$

$$\delta_{pp'}\,\delta_{\sigma'\uparrow}\underbrace{\langle a^+_{-(p-q)\downarrow}\,a^+_{p-q\uparrow}\rangle}\underbrace{\langle\hat{T}_\tau\,a_{p\uparrow}(\tau)\,a^+_{p\uparrow}(\tau')\rangle}$$

$$= \delta_{pp'} \delta_{\uparrow\sigma'} (-) F^{+}(p-q,0)(-) G(p,\tau-\tau')$$

- So the EoM. is:

$$\boxed{0 = \left(-\hbar \frac{\partial}{\partial \tau} + \varepsilon_p\right) F^{+}(p,\tau-\tau') - \frac{1}{V} \sum_q v(q) F^{+}(p-q,0)\, G(p,\tau-\tau')}$$

- we have a closed set of eq.-s for G and F^{+}

$$\varepsilon_p = \frac{\hbar^2 p^2}{2m} - \mu$$

- we introduce the quantity:

$$\Delta(p) = -\frac{1}{V} \sum_q v(q) F(p-q,0)$$

$$\Delta^{+}(p) = -\frac{1}{V} \sum_q v(q) F^{+}(p-q,0) = \Delta^{*}(p)$$

if $B = 0$ then $\Delta \in \mathbb{R}$

this will be the $\boxed{\text{gap}}$

- Using this we can get:

$$\left.\begin{aligned} -\left(\hbar \frac{\partial}{\partial \tau} + \varepsilon_p\right) G(p,\tau-\tau') + \Delta(p) F^{+}(p,\tau-\tau') &= \hbar\, \delta(\tau-\tau') \\ \Delta(p) G(p,\tau-\tau') + \left(-\hbar \frac{\partial}{\partial \tau} + \varepsilon_p\right) F^{+}(p,\tau-\tau') &= 0 \end{aligned}\right\}$$

- Set of coupled first order differential eq, with unknown Δ
- we can go to Matsubara repr. $\rightsquigarrow$ different. freq. do not mix.

$$G(p,\tau) = \frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau} G(p, i\omega_n)$$

$$F(p,\tau) = \frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau} F(p, i\omega_n)$$

$$F^{+}(p,\tau) = \frac{1}{\beta\hbar} \sum_n e^{-i\omega_n \tau} F^{+}(p, i\omega_n)$$

and ω_n are <u>fermionic</u>

$$\omega_n = \frac{(2n+1)\pi}{\beta\hbar}$$

$$\left.\begin{aligned} (i\hbar\omega_n - \varepsilon_p)\, G(p, i\omega_n) + \Delta(p) F^+(p, i\omega_n) &= \hbar \\ \Delta(p) G(p, i\omega_n) + (i\hbar\omega_n + \varepsilon_p) F^+(p, i\omega_n) &= 0 \end{aligned}\right\}$$

- the eq. for a given ω_n does not involve the others

$\rightsquigarrow$ can be solved separatly.

$$G(p, i\omega_n) = \frac{\begin{vmatrix} \hbar & \Delta(p) \\ 0 & i\hbar\omega_n + \varepsilon_p \end{vmatrix}}{\begin{vmatrix} i\hbar\omega_n - \varepsilon_p & \Delta(p) \\ \Delta(p) & i\hbar\omega_n + \varepsilon_p \end{vmatrix}} = - \frac{\hbar (i\hbar\omega_n + \varepsilon_p)}{\hbar^2\omega_n^2 + \underbrace{\varepsilon_p^2 + \Delta(p)^2}_{E(p)^2}}$$

Bogolivbov - energy

$$F^+(p, i\omega_n) = \frac{\begin{vmatrix} i\hbar\omega_n - \varepsilon_p & \hbar \\ \Delta(p) & 0 \end{vmatrix}}{\begin{vmatrix} i\hbar\omega_n - \varepsilon_p & \Delta(p) \\ \Delta(p) & i\hbar\omega_n + \varepsilon_p \end{vmatrix}} = + \frac{\hbar \Delta(p)}{(\hbar\omega_n)^2 + E(p)^2}$$

- trivial: $\boxed{\Delta = 0}$ (above T_c) $\rightsquigarrow F^+ = 0 \rightsquigarrow G(p, i\omega_n) = - \dfrac{\hbar (i\hbar\omega_n + \varepsilon_p)}{(\hbar\omega_n)^2 + \varepsilon_p^2} =$

$$= \frac{1}{i\omega_n + \frac{1}{\hbar}(e_k - \mu)}$$

this is the non-interacting GF.

- to get delta:

$(a-b)(a+b) = a^2 - b^2$

$$F(p, 0) = \frac{1}{\beta\hbar} \sum_n \frac{\hbar \Delta(p)}{(\hbar\omega_n)^2 + E^2(p)} = -(k_B T)\, \Delta(p) \sum_n \frac{1}{(i\hbar\omega_n)^2 - E^2(p)} =$$

Matsubara sum

$$= - \Delta(p) (k_B T) \sum_n \frac{1}{2E(p)} \left(\frac{1}{i\hbar\omega_n - E(p)} - \frac{1}{i\hbar\omega_n + E(p)} \right) e^{i\omega_n \eta} =$$

regularization factor

$$= - \frac{\Delta(p)}{2E(p)} \left(\frac{1}{e^{\beta E} + 1} - \frac{1}{e^{-\beta E} + 1} \right)$$

$$= \mathcal{F}(p, \tau = 0) = \frac{\Delta(p)}{2E(p)} \, th\left(\frac{\beta E(p)}{2}\right)$$

- we can insert this into the gap eq.:

$$\Delta(p) \overset{q \to p - p'}{=} -\frac{1}{V} \sum_{p'} v(p - p') \mathcal{F}(p', 0) = -\frac{1}{V} v(p - p') \frac{\Delta(p')}{2E(p')} \, th\left(\frac{\beta E(p')}{2}\right)$$

$\rightsquigarrow$ implicit eq. for $\Delta(p)$

$\rightsquigarrow$ in principle it can be solved numerically

- using the "Cooper pair" separable potential ($v(q)$) we can get an analytical result.

  $\rightsquigarrow$ it works well for weakly interacting e^-

$$v(p, p') = \begin{cases} -g, & \text{if } |\varepsilon_p| \leq \hbar\omega_D, \ |\varepsilon_{p'}| \leq \hbar\omega_D \\ 0, & \text{otherwise} \end{cases}$$

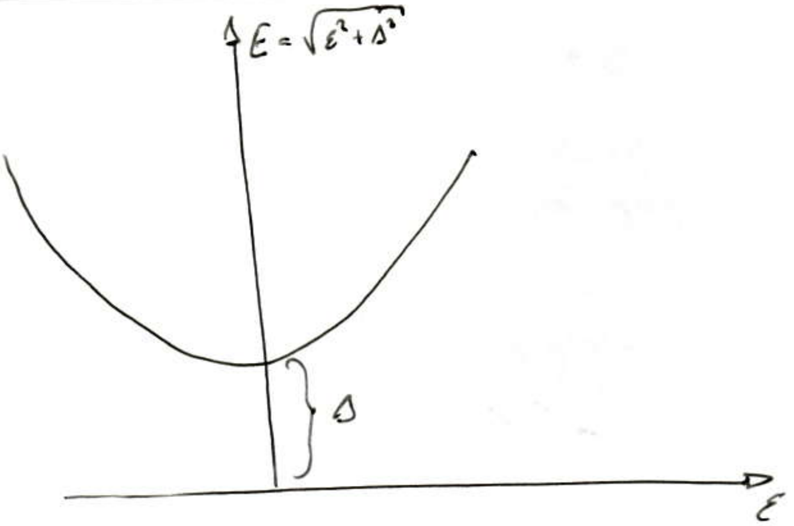
$$\to \quad \Delta(p) = \begin{cases} \Delta, & \text{if } |\varepsilon_p| \leq \hbar\omega_D \\ 0, & \text{otherwise} \end{cases}$$

- we go to integration using the DoS.

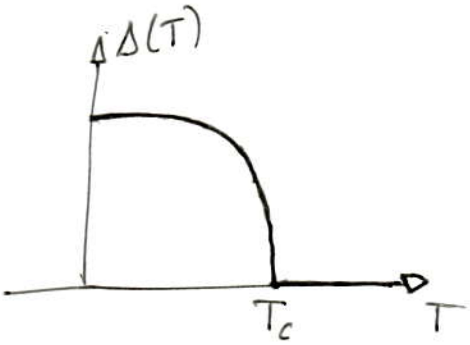
$$1 = g \int_{-\hbar\omega_D}^{\hbar\omega_D} \underbrace{\nu(\varepsilon + \mu)}_{\substack{\text{slowly changing, can} \\ \text{be replaced by } \nu_F \text{ (at Fermi-energy)}}} \frac{1}{2\sqrt{\varepsilon^2 + \Delta^2}} \, th\left(\frac{\beta\sqrt{\varepsilon^2 + \Delta^2}}{2}\right) d\varepsilon$$

$$\boxed{1 = g\nu_F \int_0^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \, th\left(\frac{\beta\sqrt{\varepsilon^2 + \Delta^2}}{2}\right)}$$

Gap - equation
- can be solved for Δ!
- scalar - eq.



- for a 2nd order phase transition $T = T_c \to \Delta = 0$



- At T_c lets say Δ is infinitesimally small

$$1 = g\nu_F \int_0^{\hbar\omega_D} \frac{d\varepsilon}{\varepsilon} \, th\left(\frac{\beta_c \varepsilon}{2}\right) \rightsquigarrow \beta(g, \nu, \omega_D)!$$

(dependance)

$$x = \frac{\beta_c \varepsilon}{2} \rightsquigarrow Q = \frac{\beta_c \hbar\omega_D}{2} = \frac{\hbar\omega_D}{2 k_B T_c} \quad \text{(upper limit)}$$

$\hbar\omega_D \sim$ room temperature
$E_F \sim$ few thousand kelvin
$k_B T_c \sim$ few kelvin.

$\Bigg\}$ we have a splitting of energy scales.

$\rightsquigarrow Q \gg 1$

$$1 = g\nu_F \int_0^Q \frac{dx}{x}\, th(x) = g\nu_F \left\{ \underbrace{\left[\ln(x)\, th(x) \right]_0^Q}_{\ln Q - \phi} - \underbrace{\int_0^{} dx \ln x \frac{1}{\cosh^2(x)}}_{} \right\}$$

expanding...

$Q \to \infty$

since $\frac{1}{\cosh^2}$ goes to ϕ quickly

$$\int_0^\infty \frac{\ln x \, dx}{\cosh^2(x)} = -\ln\left(\frac{4\gamma}{\pi}\right)$$

γ : Euler const.

$\ln \gamma = C = 0.577...$

$$C = \lim_{\ell \to \infty} \left[\sum_{n=1}^{\ell} \frac{1}{n} - \ln(\ell) \right]$$

$$1 = g\nu_F \ln\left(\frac{4\gamma}{\pi} Q\right) =$$

$$1 = g\nu_F \ln\left(\frac{4\gamma}{\pi} \frac{\hbar\omega_D}{2k_B T_c}\right)$$

$$\boxed{k_B T_c = \frac{\gamma}{\pi} 2\hbar\omega_0 \cdot e^{-\frac{1}{g\nu_F}}}$$

- the exponential factor makes $k_B T_c \ll \hbar\omega_D$ if g is a small number. (interaction is weak)

- What is Δ at $T = 0$? "$th(\frac{1}{0}) = 1$"

$$1 = g\nu_F \int_0^{\hbar\omega_D} \frac{d\varepsilon}{\sqrt{\varepsilon^2 + \Delta^2}} \cdot 1 = g\nu_F \left[\ln\left(\varepsilon + \sqrt{\varepsilon^2 + \Delta^2}\right)\right]_0^{\hbar\omega_D}$$

$\Delta(T=0) \ll \hbar\omega_0 \to$ upper limit

$$1 = g\nu_F \left(\ln(2\hbar\omega_0) - \ln(\Delta)\right) = g\nu_F \ln\left(\frac{2\hbar\omega_D}{\Delta(T=0)}\right)$$

$$\Delta(T=0) = \boxed{\Delta_0 = 2\hbar\omega_D e^{-\frac{1}{g\nu_F}}}$$

small!!

- Famous relation:

$$\frac{\Delta_0}{k_B T_c} = \frac{\pi}{\gamma} \approx 1.76 \to \text{independent of } g$$

$\cdot$ — " — $\hbar\omega_D$

$\to k_B T_c$ is in the order of the gap! (in case of weakly int. e^-)