$\langle BCS| a^+_{k\uparrow} a^+_{-k\downarrow} |BCS\rangle \neq 0$ ⇝ pair-state

$$a^+_{k\uparrow} a^+_{-k\downarrow} = (\cancel{u_k \alpha^+_{k\uparrow}} + v_k \alpha_{-k\downarrow})(u_k \alpha^+_{-k\downarrow} - \cancel{v_k \alpha_{k\uparrow}})$$

$$u_k v_k \langle BCS|(1 - \alpha^+_{-k\downarrow}\alpha_{-k\downarrow})|BCS\rangle = u_k v_k$$

$$\langle BCS|\overbrace{a^+_{k\uparrow} a_{k\uparrow}}^{n_{k\uparrow}}|BCS\rangle = \langle BCS|(\cancel{u_k \alpha^+_{k\uparrow}} + v_k \alpha_{-k\downarrow})(\cancel{u_k \alpha_{k\uparrow}} + v_k \alpha^+_{-k\downarrow})|BCS\rangle =$$

$$= \langle BCS| v_k^2 (1 - \alpha^+_{-k\downarrow}\alpha_{-k\downarrow})|BCS\rangle = v_k^2$$

$$\langle BCS| n_{k\downarrow} |BCS\rangle = v_k^2$$

$$\langle BCS|\hat{N}|BCS\rangle = \sum_k \langle BCS| a^+_{k\uparrow} a_{k\uparrow} + a^+_{k\downarrow} a_{k\downarrow}|BCS\rangle = 2\sum_k v_k^2$$

this is the definite avg. value.

What is the fluctuation?

$$\hat{N}^2 = 4 \sum_k \hat{n}_{k\uparrow} \sum_{k'} \hat{n}_{k'\uparrow} \quad \text{if } k \neq k'$$

$$\langle BCS|\hat{n}_{k\uparrow}\hat{n}_{k'\uparrow}|BCS\rangle = v_k^2 v_{k'}^2 \quad \text{if } k \neq k'$$

$$\langle BCS|\hat{n}_{k\uparrow}\hat{n}_{k\uparrow}|BCS\rangle = (u_k^2 + v_k^2) v_k^2$$

2019.03.19.

$$\langle BCS|(\cancel{u_k \alpha^+_{k\uparrow}} + v_k \alpha_{-k\downarrow})(u_k \alpha_{k\uparrow} + v_k \alpha^+_{-k\downarrow})(u_k \alpha^+_{k\uparrow} + v_k \alpha_{-k\downarrow})(\cancel{u_k \alpha_{k\uparrow}} + v_k \alpha^+_{-k\downarrow})|BCS\rangle =$$

$$= v_k^2 \langle BCS|\alpha_{-k\downarrow}(u_k^2 \alpha_{k\uparrow}\alpha^+_{k\uparrow} + \cancel{u_k v_k \alpha^+_{-k\downarrow}\alpha^+_{k\uparrow}} + \cancel{u_k v_k \alpha_{k\uparrow}\alpha_{-k\downarrow}} + v_k^2 \alpha^+_{-k\downarrow}\alpha_{-k\downarrow})\alpha^+_{-k\downarrow}|BCS\rangle$$

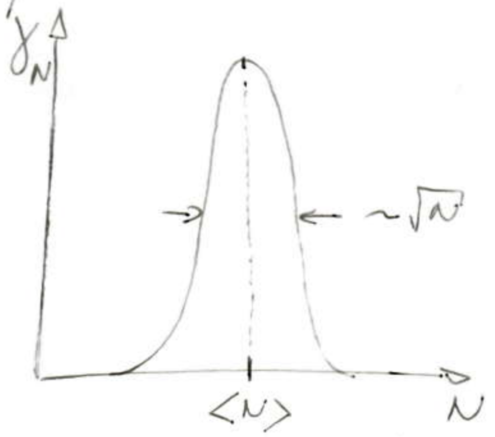
$$= v_k^2 \langle BCS| u_k^2 \underbrace{\alpha_{k\uparrow}\alpha^+_{k\uparrow}\alpha_{-k\downarrow}\alpha^+_{-k\downarrow}}_{(1-\alpha^+_{k\uparrow}\alpha_{k\uparrow})(1-\alpha^+_{-k\downarrow}\alpha_{-k\downarrow})} + v_k^2 \underbrace{\alpha_{-k\downarrow}\alpha^+_{-k\downarrow}\alpha_{-k\downarrow}\alpha^+_{-k\downarrow}}_{(1-\alpha^+_{-k\downarrow}\alpha_{-k\downarrow})(1-\alpha^+_{k\downarrow}\alpha_{k\downarrow})}|BCS\rangle$$

$$= v_k^2\,(u_k^2 + v_k^2)$$

• fluctuation of the particle number:

$$\frac{\langle N^2\rangle - \langle N\rangle^2}{\langle N\rangle^2}$$

$$|BCS\rangle = \sum_{N,\vec{k}} \gamma_{N\vec{k}}\, |N, r\rangle$$



⇝ if this is true it shrinks with $N \to \infty$

$$\langle N^2\rangle_{BCS} - \langle N\rangle^2_{BCS} =$$

$$= 4\sum_{\substack{k\neq k'\\ kk'}} v_k^2 v_{k'}^2 + 4\sum_{k}^{k=k'} v_k^2(u_k^2+v_k^2) -$$

$$-\underbrace{2\Big(\sum_k v_k^2\Big)\cdot 2\Big(\sum_{k'} v_{k'}^2\Big)}$$

$$-4\Big(\sum_{\substack{k,k'\\ k\neq k'}} v_k^2 v_{k'}^2 + \sum_k v_k^4\Big) = 4\sum_k u_k^2 v_k^2 \to 4V\int\frac{d^3k}{(2\pi)^3}\, u_k^2 v_k^2$$

$$\langle N\rangle = 2\sum_k v_k^2 \to 2V\int\frac{d^3k}{(2\pi)^3}\, v_k^2$$

$$\frac{\langle N^2\rangle_{BCS} - \langle N\rangle^2_{BCS}}{\langle N\rangle^2_{BCS}} = \frac{4V\int\frac{d^3k}{(2\pi)^3}\, u_k^2 v_k^2}{4V^2\left(\int\frac{d^3k}{(2\pi)^3}\, v_k^2\right)^2} \sim \frac{1}{V} \xrightarrow[\infty]{N} 0$$

this is the reason, why BCS can be used in the thermodynamical limit.

Normal and Anomalous Green's functions

$$K = \sum_{\vec{p},\sigma} \epsilon_p\, a^\dagger_{p,\sigma} a_{p\sigma} + \frac{1}{2V}\sum_{\substack{p,p',q\\ \sigma\sigma'}} V(q)\, a^\dagger_{p+q,\sigma}\, a^\dagger_{p'-q,\sigma'}\, a_{p'\sigma'}\, a_{p\sigma}$$

$$\epsilon_p = \frac{\hbar^2 p^2}{2m} - \mu$$

$\hat{O}(\tau) = e^{\frac{K\tau}{\hbar}} \hat{O} e^{-\frac{K\tau}{\hbar}}$, $\hat{K}(\tau) = \hat{K}(0)$

$$\hbar \frac{\partial}{\partial\tau} \hat{a}_{p,\sigma}(\tau) = \left[\hat{K}(\tau), \hat{a}_{p,\sigma}(\tau)\right] =$$
$$= -\epsilon_p a_{p\sigma} - (\dots)$$
$$(\dots) - \frac{1}{V} \sum_{p',q,\sigma'} V(q)\, a^+_{p'-q,\sigma'}(\tau)\, a_{p',\sigma'}(\tau)\, a_{p-q,\sigma}(\tau)$$

Can be prooved using:

$[O_1 O_2, O_3] = O_1 [O_2, O_3] + [O_1, O_3] O_2$

for fermionic operators:

$[O_1 O_2, O_3] = O_1 \{O_2, O_3\} - \{O_3, O_1\} O_2$

$$G(p, \tau-\tau') = -\langle T_\tau (a_{p\uparrow}(\tau) a^+_{p\uparrow}(\tau'))\rangle = \left(-\langle T_\tau (a_{p\downarrow}(\tau) a^+_{p\downarrow}(\tau'))\rangle\right)$$

$$\hbar \frac{\partial}{\partial\tau} G(p, \tau-\tau') = -\hbar \frac{\partial}{\partial\tau}\Big(\vartheta(\tau-\tau')\langle a_{p\uparrow}(\tau) a^+_{p\uparrow}(\tau')\rangle -$$
$$- \vartheta(\tau'-\tau)\langle a^+_{p\uparrow}(\tau') a_{p\uparrow}(\tau)\rangle\Big) =$$
$$= -\hbar\delta(\tau-\tau')\left(\langle a_{p\uparrow}(\tau) a^+_{p\uparrow}(\tau')\rangle + \langle a^+_{p\uparrow}(\tau') a_{p\uparrow}(\tau)\rangle\right) -$$
$$- \hbar\vartheta(\tau-\tau')\left\langle\left(\frac{\partial}{\partial\tau} a_{p\uparrow}(\tau)\right) a^+_{p\uparrow}(\tau')\right\rangle - \hbar\vartheta(\tau'-\tau)\left\langle a^+_{p\uparrow}(\tau)\left(\frac{\partial}{\partial\tau} a_{p\uparrow}(\tau)\right)\right\rangle =$$
$$= -\hbar\delta(\tau-\tau')\langle \underbrace{\{a_{p\uparrow}(\tau), a^+_{p\uparrow}(\tau)\}}_{1}\rangle - (\dots) =$$
$$= -\hbar\delta(\tau-\tau') - \langle T_\tau(\overset{\text{use eq. of motion}}{\hbar\frac{\partial}{\partial\tau} a_{p\uparrow}(\tau)})\, a^+_{p\uparrow}(\tau')\rangle =$$
$$= -\hbar\delta(\tau-\tau') - (\dots)$$

$\leadsto$ rearrange

$$\hbar\delta(\tau-\tau') = -\left(\hbar\frac{\partial}{\partial\tau} + \epsilon_p\right) G(p, \tau-\tau') + \frac{1}{V}\sum_{p',q,\sigma'} V(q) \langle T_\tau (a^+_{p'-q,\sigma'}(\tau)\, a_{p',\sigma'}(\tau) \cdot$$
$$\cdot a_{p-q,\uparrow}(\tau)\, a^+_{p\uparrow}(\tau'))\rangle$$

$$\langle T_\tau (a^+_{p'-q,\sigma'}(\tau)\, a_{p',\sigma'}(\tau)\, a_{p-q\uparrow}(\tau)\, a^+_{p\uparrow}(\tau')) \rangle \underset{\uparrow}{=} (\dots)$$

Wick - theorem

- thermodynamical avg.:

$$\langle \hat{Q} \rangle = \sum_i \langle Q_i | \frac{1}{Z} e^{-\beta \hat{K}} \hat{O} | Q_i \rangle$$

BCS and excited states
⇝ quasi - particle grnd and excited states

$$\rightarrow \quad a_{p'\sigma'}\, a_{p-q\uparrow} \neq 0 \quad \text{(might)!}$$

orthogonal states on the 2 sides for all terms!

$$\langle a_{p\uparrow} a_{q\uparrow} \rangle = \langle (u_p \alpha_{p\uparrow} + v_p \alpha^+_{-p\downarrow})(u_q \alpha_{q\uparrow} + v_p \alpha^+_{-q\downarrow}) \rangle = 0$$

$$\langle a_{p\downarrow} a_{q\uparrow} \rangle = \langle (u_p \alpha_{p\downarrow} - v_p \alpha^+_{-p\uparrow})(u_q \alpha_{q\uparrow} + v_q \alpha^+_{-q\downarrow}) \rangle = \delta_{p,-q} \underbrace{\langle a_{-p\downarrow} a_{p\uparrow} \rangle}_{\neq \emptyset !}$$

anomalous avarage

⇝ only in opposite spin and momenta case.

$$(\dots) = \langle a_{p'\sigma'}(\tau)\, a_{p-q,\uparrow}(\tau) \rangle \langle T_\tau (a^+_{p'-q,\sigma'}(\tau)\, a^+_{p\uparrow}(\tau')) \rangle +$$

+ Hartree - Fock types terms

other contractions $(\langle a^+ a \rangle \langle T a a^+ \rangle)$

$$= \delta_{p',q-p}\, \delta_{\sigma'\downarrow} \langle a_{-(p-q)\downarrow}(\tau)\, a_{p-q,\uparrow}(\tau) \rangle \langle T_\tau (a^+_{p'-q\sigma'}(\tau)\, a^+_{p\uparrow}(\tau')) \rangle =$$

anomalous - type Green's func. with op. spin momenta

$$= \delta_{p',q-p}\, \delta_{\sigma'\downarrow} \langle a_{-(p-q)\downarrow}(\tau)\, a_{p-q\uparrow}(\tau) \rangle \langle T_\tau (a^+_{-p\downarrow}(\tau)\, a^+_{p\uparrow}(\tau')) \rangle + (\dots)$$

- in the EoM of the Gf. a new type of Gf. emerges

not the fm of the normal Gf.

we define new ones

$$\left.\begin{aligned} F(p,\tau-\tau') &= -\langle T_\tau (a_{p\uparrow}(\tau)\, a_{-p\downarrow}(\tau'))\rangle \\ F^+(p,\tau-\tau') &= -\langle T_\tau (a^+_{-p\downarrow}(\tau)\, a^+_{p\uparrow}(\tau'))\rangle \end{aligned}\right\} \text{ anomalous Green's func.}$$

$$= \delta_{p',q-p}\, \delta_{\sigma',\downarrow}\, F(p-q,0)(-)F^+(p,\tau-\tau') + \text{other terms}$$

- with this the EoM of G:

$$\hbar\,\delta(\tau-\tau') = -\left(\hbar\frac{\partial}{\partial\tau} + \varepsilon_p\right) G(p,\tau-\tau') - \frac{1}{V}\sum_q V(q)\,F(p-q,0)\,F^+(p,\tau-\tau')$$

+ other terms

- it is enough to take the evaluation of around the Fermi lvl.
  (that's where $V(q)$ is important)

- if we neglect the HF-like terms we can introduce an effective m^* mass to "compensate"...

$\rightsquigarrow$ $m \mapsto m^*$
$\mu \mapsto \mu^*$

we consider the effect to be mean-field-like, so they just change other parameters.