$$\varepsilon_e(0,\omega) = 1 - \frac{\omega_p^2}{\omega^2}$$ ω_p is plasma freq.

$$\varepsilon_e(k,0) = 1 + \frac{k_{TF}^2}{k^2}$$

$$k_{TF}^2 = \frac{4}{\pi}\frac{1}{k_F a_0} k_F^2$$

Thomas - Fermi

$$k_F^3 = 3\pi^2 n$$

$$v_F = \frac{\hbar k_F}{m}$$

for electrons

$\omega \ll k v_F$

we can take $\varepsilon_e(k,0)$

for ions

$k v_{ion} \ll \omega$

ion velocity

electrons

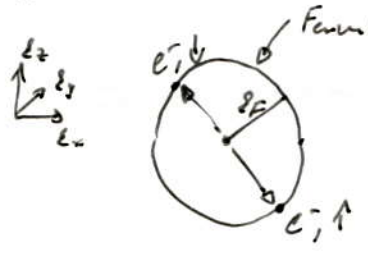
Ions move from now on...

$$\varepsilon_{ion} = 1 - \frac{\Omega_p^2}{\omega^2}$$

diffrent plasma freq. (diffrent mass, charge)

2019.03.05.

- what kind of e^--s attract eachother?
- BCS (Bardeen - Cooper - Schiffer)



- opposite spins
- $|k| = k_F$, $\vec{k} = -\vec{k}$

- Origin of the attraction:
  - Coulomb-int $\rightsquigarrow$ repelling
  - int. can change sign.

$$V(q) = \frac{4\pi e_0^2}{q^2} \rightsquigarrow \text{effective interaction: } V(k) = \frac{4\pi e_0^2}{k^2 \underbrace{\varepsilon(k)}}$$

where $e_0^2 = \dfrac{e^2}{4\pi\varepsilon_0}$

contains the possibility of attraction.

$a_0 = \frac{\hbar^2}{m e_0^2}$ first Bohr-radius

$q_{TF}^2 = \frac{4}{\pi} \frac{1}{q_F a_0} q_F^2$ Thomas-Fermi q

$q_F = (3\pi^2 n)^{1/3}$

$v_F = \frac{\hbar q_F}{m}$

$$\begin{cases} \varepsilon_e(0,\omega) = 1 - \frac{\omega_p^2}{\omega^2} \; ; \; \omega_p^2 = \frac{4\pi n e_0^2}{m} \\ \varepsilon_e(q,0) = 1 + \frac{q_{TF}^2}{q^2} \end{cases}$$

limiting cases.

- now we allow ions to move:
  - we can take $\varepsilon_e(q,0)$ for the e^--s

    $q v_i \ll \omega \ll q v_F$
  - for the ions: $\varepsilon_{ion} = 1 - \frac{\Omega_p^2}{\omega^2}$
- Ω_p^2 is the plasmon freq. of the ions.

$\Omega_p^2 = 4\pi n_i (Z e_0)^2 \cdot \frac{1}{M}$

bigger charge

ion mass

$n_i := \frac{n}{Z}$

Z tells how much of them is ionized.

$\Omega_p^2 = \left(\frac{Z m}{M}\right) \omega_p^2$

- the total ε (estimation, based on jelly model):

$$\varepsilon = 1 + \frac{q_{TF}^2}{q^2} - \frac{\Omega_p^2}{\omega^2} = \varepsilon_e(q,0)\left[1 - \frac{\omega^2(q)}{\omega^2}\right]$$

can be $\ominus$ for small ω!

and $\omega^2(q) = \frac{\Omega_p^2}{\varepsilon_e(q,0)}$

$$\omega^2(q) = \frac{Z m}{M} \omega_p^2 \frac{q^2}{q^2 + q_{TF}^2} \xrightarrow[q \to 0]{} \omega \sim |q|$$

phonon-like behaviour.

- in a certain range e^--s can attract eachother
- how long is this picture valid?

$\rightsquigarrow$ up to ω_D (Debye-freq.)

$\rightsquigarrow$ $\omega(k) = c\cdot k$

$$c^2 = \frac{Zm}{M}\frac{\omega_p^2}{k_{TF}^2} = \frac{1}{3}\frac{Zm}{M}\cdot v_F^2$$

$$\begin{cases} k_D^3 = 4\pi\cdot n \\ \omega_D = c\,k_D \\ \vartheta_D = \dfrac{\hbar\omega_D}{k_B} \end{cases}$$

$\rightsquigarrow$ char. tem. for phonons

"Room-temp" ~200 K - 300 K

$$\vartheta_D \ll T_F$$

Fermi-temp. ~6000 K

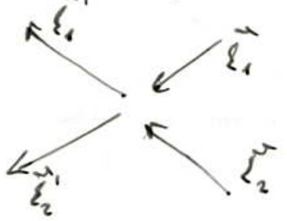
$$\omega_D \ll \frac{\hbar^2 k_F^2}{2m}$$

<u>Cooper's 1 pair problem</u>

- toy model
- 2 e^- above the non-int. Fermi-sea
- they can interact with eachother
- the other's effect is calculated by the Pauli-principle
- scattering occurs above the Fermi-sea!

$\hat{P}$ forbids scattering in the sea. (Projector)

$$\hat{H} = -\frac{\hbar^2}{2m}(\Delta_1 + \Delta_2) + \hat{P}\,V(\vec{r}_1 - \vec{r}_2)$$



$$k = \frac{k_1 - k_2}{2}\ ;\ k' = \frac{k_1' - k_2'}{2}\ ;\ K = k_1 + k_2\ ;\ K' = k_1' + k_2'$$

$\vec{K} = \vec{K}' = 0$

conserved

they are on the other sides of Fermi-sea. } easiest, lowest energy case of superconductivity

there is p-wave SC. too!

$V(r) = \sum_q v(q)\, e^{iqr}$

the rest.

$\Psi(\vec{r}_1, s_1, \vec{r}_2, s_2) = e^{i\vec{K}\left(\frac{\vec{r}_1+\vec{r}_2}{2}\right)} \varphi(\vec{r}_1 - \vec{r}_2)\, \chi(s_1, s_2)$

total wf.

Center of Mass

singlet: $\frac{1}{\sqrt{2}}\left(\alpha(s_1)\beta(s_2) - \beta(s_1)\alpha(s_2)\right)$
(energetically favorable)

$\varphi(\vec{r}_1 - \vec{r}_2) \overset{!}{=} \varphi(\vec{r}_2 - \vec{r}_1)$ so the total wf. is anti-symmetric.

$\varphi(\vec{r}_1 - \vec{r}_2) = \sum_{\vec{\ell}} c(\vec{\ell})\, e^{+i\vec{\ell}(\vec{r}_1 - \vec{r}_2)}$

$\boxed{c(\vec{\ell}) = c(-\vec{\ell})}$ $\boxed{c(\ell) = 0 \text{ if } \ell < \ell_F}$

$-\frac{\hbar^2}{m}\Delta\varphi(r) + P\, v(r)\, \varphi(r) = E\varphi(r) = (-\Delta + 2E_F)\,\varphi(r)$

reduced mass of 2 e^--s...

no binding case.
(e^--s are on the Fermi-sea...)

$$\hat{P}\, v(r)\varphi(r) = \sum_{\substack{k',q \\ |\vec{q}+\vec{k}'|>k_F}} c(k')\, v(q)\, e^{i(\vec{q}+\vec{k}')\vec{r}} = \sum_{\substack{k',k \\ k>k_F}} c(k')\, v(\vec{k}-\vec{k}')\, e^{i\vec{k}\vec{r}}$$

$$\hat{H}\varphi(r) = \sum_{k'} \frac{\hbar^2 k'^2}{m} c(k') e^{i\vec{k}'\vec{r}} + \sum_{k',k''} c(k'')\, v(\vec{k}'-\vec{k}'')\, e^{i\vec{k}'\vec{r}} = (-\Delta+2E_F)\sum_{k'} c(k') e^{i\vec{k}'\vec{r}}$$

all k'-s must agree.

$$\frac{\hbar^2 k^2}{m} c(k) + \sum_{k'} v(\vec{k}-\vec{k}')\, c(\vec{k}') = (-\Delta+2E_F)\, c(k)$$

• let's suppose we have a <u>separable potential</u>

$$v(k,k') = \begin{cases} -\dfrac{v}{V}, & \text{if } \dfrac{\hbar^2 k^2}{2m} < E_F + \hbar\omega_D \ \& \ \dfrac{\hbar^2 k'^2}{2m} < E_F + \hbar\omega_D \\[2ex] 0, & \text{otherwise} \end{cases}$$

coupling const.

NO translational invariance!

$$A := -\frac{v}{V}\sum_{k'}{}' c(k')$$

$$\rightsquigarrow \sum_{k'}{}' c(k') = \sum_{k'} c(k')$$

& all the restrictions

$$E_F < \frac{\hbar^2 k'^2}{2m} < E_F + \hbar\omega_D$$

$$\frac{\hbar^2 k^2}{m} c(k) + A = (-\Delta + 2E_F)\, c(k)$$

$$c(k) = -\frac{A}{\frac{\hbar^2 k^2}{m} + \Delta - 2E_F}$$

$$A = -\frac{v}{V}\sum_{k'}{}' -\frac{A}{\frac{\hbar^2 k'^2}{m} + \Delta - 2E_F}$$

$$\boxed{1 = \frac{v}{V}\sum_{\ell'}{}' \frac{1}{\frac{\hbar^2 \ell'^2}{m} + \Delta - 2E_F}}$$ Gap-eq. for Δ

- continuous approximation:

$$E = \frac{\hbar^2 \ell^2}{2m} \qquad \ell = \frac{1}{\hbar}\sqrt{2mE}$$

$$\frac{1}{V}\sum_{\ell} \longrightarrow \int \frac{4\pi \ell^2 d\ell}{(2\pi)^3} = \int \nu(E)\, dE$$ d.o.s.

$$E_F = \frac{p_F^2}{2m}$$
$$p_F = \left(2m\frac{E_F}{\hbar^2}\right)^{1/2}$$

$$\nu(E) = \frac{4\pi \ell^2}{(2\pi)^3}\left(\frac{d\ell}{dE}\right) = m^{3/2} E^{1/2} \frac{1}{\sqrt{2}\pi^2 \hbar^3}$$

if the region is small $\rightsquigarrow$ $\nu(E) \sim$ const. can be pulled out.

$$\nu(E_F) = \nu_F = \frac{m^{3/2} E_F^{1/2}}{\sqrt{2}\pi^2 \hbar^3} = \frac{m p_F}{2\pi^2 \hbar^3}$$ integral can be perfomed

$$1 = v\int_{E_F}^{E_F+\hbar\omega_D} dE\, \nu(E) \frac{1}{\Delta + 2E - 2E_F} \underset{\mathcal{E} = E - E_F}{=} v\nu_F \underbrace{\int_0^{\hbar\omega_D} \frac{1}{\Delta + 2\mathcal{E}} d\mathcal{E}}_{\frac{1}{2}\left[\ln(\Delta + 2\mathcal{E})\right]_0^{\hbar\omega_D}} =$$

$$= \frac{v\nu_F}{2} \ln\left(\frac{\Delta + 2\hbar\omega_D}{\Delta}\right) = 1$$

$$e^{\frac{2}{v\nu_F}} = 1 + \frac{2\hbar\omega_D}{\Delta}$$

- small attraction: exp is huge! $(\sim \frac{1}{v})$

$$\boxed{\Delta \cong 2\hbar\omega_D e^{-\frac{2}{v\nu_F}}}$$

- even for infinitesimally small attraction there is a bound state

- binding is lower by exponencially small factor

$\Delta << \hbar\omega_D$

- non-analytic dependance of the coupling const.
  $\rightarrow$ non-perturbative ...

- medium of the other e^--s
  - what happens if we only have 2?

$\rightsquigarrow E_F = 0$ $\quad$ cutoff $\quad \Delta = 0$

$$V \int_0^{E_c} d\varepsilon \frac{\nu(\varepsilon)^{\sim \varepsilon^{1/2}}}{\varepsilon} = 2 V \nu(E_c)$$

Lower limit $\quad V_c = \frac{1}{2\nu(E_c)}$

$\Delta \neq 0$

$$1 = V \int_0^{E_c} d\varepsilon \frac{\nu(\varepsilon)}{\Delta + \varepsilon}$$

it cannot be solved for Δ, if $V < V_c$

- in the previous one there was no lower bound for the coupling const.

$\downarrow$

here there is a critical value, below which there is no bound state.

- if there is an attraction, the sharp boundary of the Fermi sea disappears,

More pairs

- we build up the grnd. state from pairs.

$i = (\vec{r}_i, s_i)$, N : even.

$$\Psi(1,2,\dots N) = \underset{\substack{\uparrow \\ \text{anti-sym} \\ \text{operator}}}{\hat{A}} \left\{ \varphi(1,2)\varphi(3,4)\dots \varphi(N-1,N) \right\}$$

$\varphi(i,j) = \varphi(\vec{r}_i - \vec{r}_j)\,\alpha(s_i)\,\beta(s_j)$ since it is under the $\hat{A}$ operator

$$\hat{A}\,\varphi(i,j) = \varphi(\vec{r}_i - \vec{r}_j)\underbrace{\left[\alpha(s_i)\beta(s_j) - \alpha(s_j)\beta(s_i)\right]}_{\sqrt{2}\,\chi(s_i, s_j)}$$

$$\varphi(\vec{r}_i - \vec{r}_j) = \sum_{\ell_i} c(\vec{\ell}_i)\, e^{i\vec{\ell}_i(\vec{r}_i - \vec{r}_j)} \qquad c(\ell) = c(-\ell)$$

$$\Psi(1,2,\dots,N) = \sum_{\ell_1}\sum_{\ell_2}\dots\sum_{\ell_{N-1}} c(\ell_1)\,c(\ell_2)\dots c(\ell_{N-1}) \cdot SD$$

$$\underset{\substack{\uparrow \\ \text{Slater - determinant}}}{SD} = \hat{A}\left\{ e^{i\vec{\ell}_1\vec{r}_1}\alpha(s_1)\, e^{-i\vec{\ell}_1\vec{r}_2}\beta(s_2)\dots e^{i\vec{\ell}_{N-1}\vec{r}_{N-1}}\alpha(s_{N-1})\, e^{-i\vec{\ell}_{N-1}\vec{r}_N}\beta(s_N)\right\}$$

- we can convert this to 2nd quant. form.

$$|\hat{\Psi}_N\rangle = \sum_{\ell_1}\sum_{\ell_2}\dots\sum_{\ell_{N-1}} c_{\ell_1} c_{\ell_2}\dots c_{\ell_{N-1}}\, \hat{a}^+_{\ell_1\uparrow}\hat{a}^+_{-\ell_1\downarrow}\dots \hat{a}^+_{\ell_{N-1}\uparrow}\hat{a}^+_{-\ell_{N-1}\downarrow}|\varnothing\rangle$$