Feynman-szabályok a sűrűségfluktuációs operátorra

1. Rajzoljunk fel ∀ n kölcsönhatást tartalmazó, topológiailag különböző, két külső pontot tartalmazó gráfot. (x,τ,x',τ')

2. $2n$ db belső pontot x_i, x_i'-vel jel. $x_i = (r_i, s_i, \tau_i)$

3. $x_i \longleftarrow x_i' \;=\; -G_0(x_i, x_i')$

4. $x_i \bullet\text{- - - - -}\bullet x_i' \;=\; -\frac{1}{\hbar} v_i(x_i, x_i') = -\frac{1}{\hbar} v(r_i, r_i')\,\delta(\tau_i-\tau_i')$

5. Integrálni ∀ X_i belső pontra: $\int dX_i = \int dr_i \int d\tau_i \sum_{s_i}$

   [+ → bozonok, − → fermionok]

6. A gráf szorzandó $(2s+1)^N$, ahol N a hurkok száma. $(\pm 1)^F$, ahol F a fermion-hurkok száma.

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• Review: $D(r_1,\tau_1,r_2,\tau_2) = -\left\langle T_\tau\left(\tilde{n}(r_1,\tau_1)\,\tilde{n}(r_2,\tau_2)\right)\right\rangle$

• properties:

$\boxed{1.}$ $D(r_1,\tau_1,r_2,\tau_2) = D(r_1,\tau_1-\tau_2,r_2,0)$

proof: $\tau_1 > \tau_2$

$$D(r_1,\tau_1,r_2,\tau_2) \overset{\text{by Def.}}{=} -\mathrm{Tr}\Big[\underbrace{\hat{\rho}_K}_{\frac{e^{-\beta K}}{\langle e^{-\beta K}\rangle}} e^{\frac{K\tau_1}{\hbar}}\,\tilde{n}(r_1)\underbrace{e^{-\frac{K\tau_1}{\hbar}}\,e^{\frac{K\tau_2}{\hbar}}}_{e^{-\frac{K}{\hbar}(\tau_1-\tau_2)}}\tilde{n}(r_2)\,e^{-\frac{K\tau_2}{\hbar}} =$$

cyclic prop. of Tr

$$= -\mathrm{Tr}\Big[\hat{\rho}_K\, e^{\frac{K}{\hbar}(\tau_1-\tau_2)}\,\tilde{n}(r_1)\,e^{-\frac{K}{\hbar}(\tau_1-\tau_2)}\,e^{+\frac{K}{\hbar}0}\,\tilde{n}(r_2)\,e^{-\frac{K}{\hbar}0} =$$

$$= D(r_1,\tau_1-\tau_2,r_2,0) \quad \square$$

$\tau_2 < \tau_1$

↳ it is technically the same, but with reverse τ ordering.

$\boxed{2.}$ $D(r_1,\tau,r_2,0) = D(r_1,\tau+\beta\hbar,r_2,0) \xrightarrow{\text{Matsubara}}$ same prop. as bosonic Matsubara Greens func.!

proof:

$-\beta\hbar \leq \tau \leq 0$

$$\omega_n = \frac{2n\pi}{\beta\hbar}$$

$$D(r_1,\tau,r_2,0) = -\mathrm{Tr}\left[\rho_G\, \hat{n}(r_2)\, e^{\frac{K\tau}{\hbar}}\, \hat{n}(r_1)\, e^{-\frac{K\tau}{\hbar}}\right] =$$

cyclic prop.

$$= -\mathrm{Tr}\Big[\underbrace{\underbrace{\rho_G}_{\frac{e^{-\beta K}}{Z}}\, e^{-\frac{K\tau}{\hbar}}}_{\frac{e^{-\frac{K}{\hbar}(\tau+\beta\hbar)}}{Z}}\, \hat{n}(r_2)\, e^{\frac{K\tau}{\hbar}}\, \hat{n}(r_1)\Big] =$$

$$= -\mathrm{Tr}\left[\hat{n}(r_1)\, \frac{e^{-\frac{K}{\hbar}(\tau+\beta\hbar)}}{Z}\, \hat{n}(r_2)\, e^{\frac{K\tau}{\hbar}}\right] = -\mathrm{Tr}\left[\frac{e^{-\beta K} e^{\beta K}}{Z}\, e^{\frac{K\tau}{\hbar}}\, \hat{n}(r_1)\, e^{-\frac{K}{\hbar}(\tau+\beta\hbar)} \cdot \hat{n}(r_2)\right] = -\mathrm{Tr}\left[\frac{e^{-\beta K}}{Z}\, e^{\frac{K}{\hbar}(\tau+\beta\hbar)}\, \hat{n}(r_1)\, e^{-\frac{K}{\hbar}(\tau+\beta\hbar)}\, \hat{n}(r_2)\right] \overset{\text{insert time-ordering}}{=}$$

$$= D(r_1,\tau+\beta\hbar,r_2,0)$$

⤳ we can do $\mathcal{F}$ on this and go to such representation:

$$\left.\begin{aligned} D(r_1,r_2,i\omega_n) &= \int_0^{\beta\hbar} e^{i\omega_n\tau}\, D(r_1,\tau,r_2,0)\, d\tau \\ D(r_1,\tau_1,r_2,\tau_2) &= \frac{1}{\beta\hbar}\sum_n D(r_1,r_2,i\omega_n)\, e^{-i\omega_n(\tau_1-\tau_2)} \end{aligned}\right\}$$

Homogenious system

$$\varphi_{\xi, m_s}(r, s) = \frac{1}{\sqrt{V}} e^{i\xi r} \chi_{m_s}(s)$$

$$\downarrow$$

$$\hat{\Psi}(r, s) = \frac{1}{\sqrt{V}} \sum_{\xi, s} e^{i\xi r} \hat{a}_{\vec{\xi}, s}$$

$$\hat{n}(r) = \sum_s \Psi^+(r, s)\Psi(r, s) = \frac{1}{V} \sum_{\substack{\xi, \xi' \\ s}} e^{i(\xi - \xi')r} \hat{a}^+_{\xi', s} \hat{a}_{\xi, s} \overset{q = \xi - \xi'}{=}$$

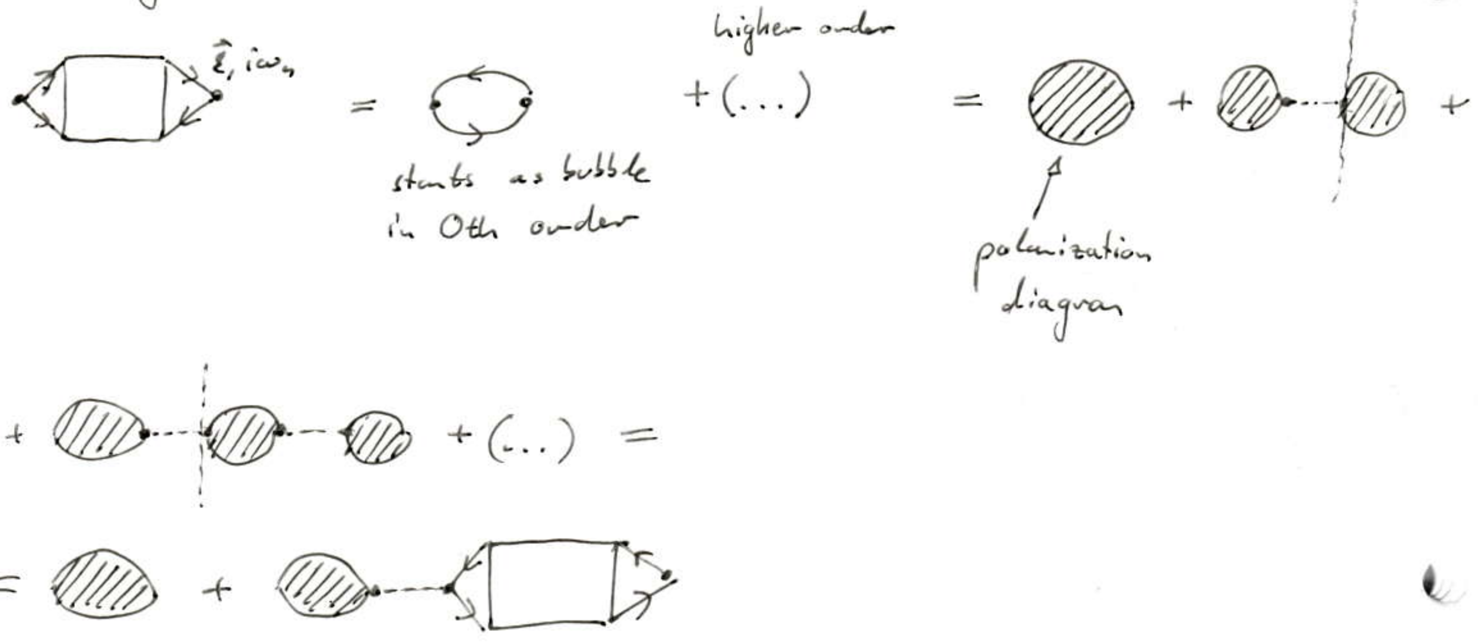
$$= \frac{1}{V} \sum_{\xi, q, s} e^{i\vec{q}\vec{r}} a^+_{\xi+q, s} a_{\xi, s} = \frac{1}{V} \sum_q e^{i\vec{q}\vec{r}} \hat{n}_{q, s}$$

where $\hat{n}_{q, s} = \sum_{\xi, s} \hat{a}^+_{\xi+q, s} \hat{a}_{\xi, s}$

- Momentum & Matsubara represcentation:

$$D(r_1, \tau_1, r_2, \tau_2) = \frac{1}{\beta\hbar} \sum_n \int \frac{d^3\xi}{(2\pi)^3} e^{i\vec{\xi}(\vec{r}_1 - \vec{r}_2)} e^{-i\omega_n(\tau_1 - \tau_2)} D(\vec{\xi}, i\omega_n)$$

- Something simular to Dyson-eq.:



$$\Rightarrow \quad -D(q, i\omega_n) = -\hbar \Pi(q, i\omega_n) + (-\hbar\Pi(q, i\omega_n))\left(-\frac{1}{\hbar}V(q)\right)(-D(q, i\omega_n))$$

partial summation.

$$\boxed{D(q, i\omega_n) = \frac{\hbar\,\Pi(q, i\omega_n)}{\varepsilon(q, i\omega_n)}}$$

dielectric func.

$$\boxed{\varepsilon(q, i\omega_n) = 1 - v(q)\Pi(q, i\omega_n)}$$

$\to$ we can have a lowest order approx of Π as the bubble.

Spectral func.:

$$\rho^0(r_1, t_1, r_2, t_2) = \langle [\tilde{n}(r_1, t_1), \tilde{n}(r_2, t_2)] \rangle$$

where $n(r, t) = e^{i\frac{K}{\hbar}t}\, n(r)\, e^{-i\frac{K}{\hbar}t}$

$w_n = e^{-\beta K_n}$

$\{|n\rangle\}_n$ orthogonal representation

$\underbrace{\hat{K}}|n\rangle = K_n|n\rangle$

full, with interaction! not K_0!

$$\rho^0(r_1, t_1, r_2, t_2) = \sum_{n,m} w_n \Big(\langle n| \big(e^{i\frac{K}{\hbar}t_1}\, \tilde{n}(r_1)\, e^{-i\frac{K}{\hbar}t_1} \underbrace{\,}_{|m\rangle\langle m|} e^{i\frac{K}{\hbar}t_2}\, \tilde{n}(r_2)\, e^{-i\frac{K}{\hbar}t_2} -$$

$$- e^{i\frac{K}{\hbar}t_2}\, \tilde{n}(r_2)\, e^{-i\frac{K}{\hbar}t_2} \underbrace{\,}_{|m\rangle\langle m|} e^{+i\frac{K}{\hbar}t_1}\, \tilde{n}(r_1)\, e^{-i\frac{K}{\hbar}t_1}\big)|n\rangle\Big) =$$

$$= \sum_{n,m} w_n \Big(e^{i\frac{(K_n - K_m)(t_1 - t_2)}{\hbar}} \langle n|\tilde{n}(r_1)|m\rangle\langle m|\tilde{n}(r_2)|n\rangle -$$

$$- e^{-i\frac{(K_n - K_m)(t_1 - t_2)}{\hbar}} \langle n|\tilde{n}(r_2)|m\rangle\langle m|\tilde{n}(r_1)|n\rangle \Big)$$

change $n \to m$ in 2nd line

$$w_m = w_n e^{\beta(K_n - K_m)}$$

$$\downarrow$$

$$= \sum_{n,m} w_n \left(1 - e^{\beta(K_n - K_m)}\right) e^{i\frac{(K_n - K_m)(t_1 - t_2)}{\hbar}} \overbrace{\langle n | \hat{n}(r_1) | m \rangle \langle m | \hat{n}(r_2) | n \rangle}^{A^D_{nm}(r_1, r_2)}$$

$\rightsquigarrow$ spectral decomposition of the spectral func.

$$\rho^D(r_1, t_1, r_2, t_2) = \sum_{n,m} w_n \left(1 - e^{\beta(K_n - K_m)}\right) e^{i\frac{(K_n - K_m)(t_1 - t_2)}{\hbar}} A^D_{nm}(r_1, r_2)$$

$\downarrow$ only depends on $t_1 - t_2$! $\to \mathcal{F}$

$\to$ by $\mathcal{F}$ we get δ-s!

$$\left.\begin{aligned}
\rho^D(r_1, r_2, \omega) &= \int_{-\infty}^{\infty} dt\, \rho^D(r_1, t, r_2, 0)\, e^{i\omega t} \\
\rho^D(r_1, t_1, r_2, t_2) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \rho^D(r_1, r_2; \omega)\, e^{-i\omega(t_1 - t_2)}
\end{aligned}\right\}$$

$$\rho^D(r_1, r_2, \omega) = \sum_{n,m} w_n \left(1 - e^{\beta(K_n - K_m)}\right) A^D_{nm}(r_1, r_2)\, \delta\left(\omega - \frac{K_n - K_m}{\hbar}\right)$$

$\to$ the system has δ peaks at the excitation energies.
(when 2 lvl-s are connected by A_{nm} matrix)

Special case: homogenious sys

- Side remark: non-hom sys is hard to calculate.

$$\left.\begin{aligned}
\rho^D(r_1, r_2, t_1, t_2) &= \int \frac{d^3 q}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \rho^D(q, \omega)\, e^{-i\omega(t_1 - t_2)}\, e^{iq(r_1 - r_2)} \\
\rho^D(q, \omega) &= \int d^3 r \int_{-\infty}^{\infty} dt\, \rho^D(\vec{r}, t, \vec{0}, 0)\, e^{-i\vec{q}\vec{r}}\, e^{i\omega t}
\end{aligned}\right\}$$

$$A^0_{nm}(r_1, r_2) = \langle n|\hat{n}(r_1)|m\rangle\langle m|\hat{n}(r_2)|n\rangle$$

non homo sys. we can choose such a basis:

$$\hat{P}|n\rangle = p_n|n\rangle$$

$\hookrightarrow$ total momentum of sys.

$$\hat{n}(r) = \frac{1}{V}\sum_q e^{-i\vec{q}\vec{r}}\underbrace{\hat{n}_{\vec{q}}}$$

transforms as a vector of momentum $\vec{q}$

$$A^0_{nm}(r_1, r_2) = \frac{1}{V^2}\sum_{\ell,\ell'}\underbrace{\langle n|\hat{n}(\ell)|m\rangle}\underbrace{\langle m|\hat{n}(\ell')|n\rangle}e^{-i\vec{\ell}\vec{r}_1}e^{-i\vec{\ell}'\vec{r}_2} =$$

$\ell + p_m \stackrel{!}{=} p_n$ $\quad$ $\ell' + p_n \stackrel{!}{=} p_m$

on the scalar product will be zero $\longrightarrow$ $\boxed{\ell = -\ell'}$

$$= \frac{1}{V}\sum_\ell A^0_{nm}(\ell)e^{-i\ell(r_1 - r_2)}$$

where $A^0_{nm}(\ell) = \frac{1}{V}\langle n|\hat{n}(\ell)|m\rangle\underbrace{\langle m|\hat{n}(-\ell)|n\rangle}_{\langle n|\hat{n}(\ell)|m\rangle^*} =$

$$A^0_{nm}(\ell) = \frac{|\langle n|\hat{n}(\ell)|m\rangle|^2}{V}$$

$$A^0_{nm} \in \mathbb{R}^+$$

$\langle m|\hat{n}(-\ell)|n\rangle^* =$

$= \langle n|\hat{n}(-\ell)^+|m\rangle =$

$= \langle n|\left(\sum_q a^+_{q-\ell}\, a_q\right)^+|m\rangle =$

$= \langle n|\left(\sum_q a^+_q\, a_{q-\ell}\right)|m\rangle =$

now shift $q \to q+\ell$

$= \langle n|\hat{n}(\ell)|m\rangle$

$$S^0(\ell,\omega) = \int d^3r\, e^{-i\ell r}\, S^0(\vec{r},\vec{0},\omega) =$$

$$= 2\pi\sum_{n,m} w_n\left(1 - e^{\beta(K_n - K_m)}\right)\delta\left(\omega - \frac{K_n - K_m}{\hbar}\right)A^0_{nm}(\ell)$$