how do normal order

$$\frac{1}{2}\sum_{n\neq 0} \alpha_{-n}\alpha_{+n} = \frac{1}{2}\sum_{n>0}\alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n<0}\alpha_{-n}\alpha_n = \sum_{n>0}\alpha_{-n}\alpha_n + \frac{1}{2}\underbrace{\sum_{n=1}^{\infty} n}_{\zeta(-1) = -\frac{1}{12}}$$

$[\alpha_n, \alpha_{-n}] = n$

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

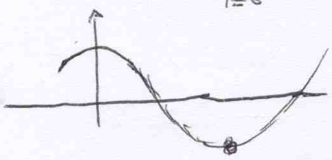
alternatively
$$\sum_{n=1}^{\infty} n = \lim_{\varepsilon\to 0} \underbrace{\sum_{n=1}^{\infty} n e^{-n\varepsilon}}$$

$$-\partial_\varepsilon \sum_{n=1}^{\infty} e^{-n\varepsilon} = -\partial_\varepsilon\left[\frac{e^{-\varepsilon}}{1-e^{-\varepsilon}}\right] = \frac{1}{\varepsilon^2} - \frac{1}{12} + \mathcal{O}(\varepsilon)$$

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right)$$

no oscillator

$$M^2 = -\frac{1}{\alpha'}\,\frac{D-2}{6} < 0 \quad \text{tachion} \quad \rightsquigarrow \text{field } T(X) \quad \left.\frac{d^2V}{dT}\right|_{T=0}$$



String field theory
B. Zwiebach

first excited state:
$$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0,p\rangle \quad \rightsquigarrow (D-2)^2 \text{ states}$$

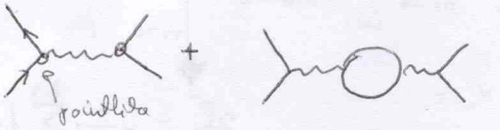
$$\underset{\substack{\uparrow\\ \text{translation}}}{SO(1,D-1)\times P} \Rightarrow \quad M\neq 0 \quad SO(D-1)$$

$$M=0 \quad SO(D-2) \Leftrightarrow \boxed{D=26}$$

$$M^2=0 \quad \left(1-\frac{D-2}{24}\right)=0$$

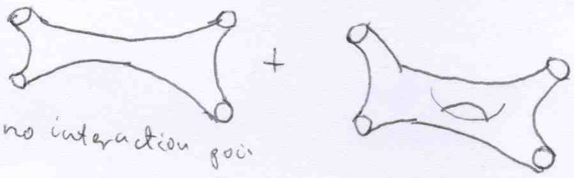
3. lecture

Particle physics:



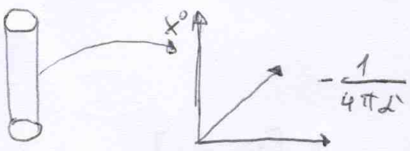
point like
↑ UV divergences

String theory



no interaction poi

free propagation



$$-\frac{1}{4\pi\alpha'}\int d^2\sigma \sqrt{-g}\, g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X^\nu\,\eta_{\mu\nu} \rightarrow \frac{-1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X\cdot\partial^\alpha X$$

$$\partial_\alpha\partial^\alpha X^\mu = 0$$

- $(\sigma,\tau)\to(\tilde\sigma,\tilde\tau)$
- Lorentz
- Weyl inv. $g_{\alpha\beta}\to\Omega^2 g_{\alpha\beta}$

$\sigma^\pm = \tau\pm\sigma$

- $\sigma^+ \to \tilde\sigma^+(\sigma^+)$
- $(\partial_+ X)^2 - (\partial_- X)^2 = 0$

light cone-gauge
$$X^\pm = \sqrt{\tfrac{1}{2}}\left(X^0 \pm X^{D-1}\right)$$

$$X^+ = X^+_L + X^+_R = \tfrac{1}{2}x^+ + \tfrac{1}{2}\alpha' p^+\sigma^+ + \tfrac{1}{2}x^+ + \tfrac{1}{2}\alpha' p^+\sigma^- = x^+ + \alpha' p^+\tau$$

$\rightarrow n=0$ mass-shell

$$X^-_L = \tfrac{1}{2}x^- + \tfrac{1}{2}\alpha' p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde\alpha^-_n\, e^{-in\sigma^+}$$

$$\rightarrow \frac{1}{\sqrt{2\alpha'}}\,\frac{1}{p^+}\sum_{i=1}^{D-2}\tilde\alpha^i_{n-m}\,\tilde\alpha^i_m$$

degrees of freedom: transverse: $x^i, p_i, \alpha_n^i, \tilde{\alpha}_m^i$

$p^+, x^-, \underline{x^+, p_-}$

quantize $[x^i, p_j] = i\delta^i_j \qquad [x^-, p^+] = -i \qquad [\alpha_n^i, \alpha_m^j] = \delta^{ij}\, n\, \delta_{nm} \qquad [\tilde{\alpha}_n^i, \tilde{\alpha}_m^j] = \delta^{ij}\, n\, \delta_{nm}$

Hilbert space: Fock space: $\hat{p}^\mu |0,p\rangle = p^\mu |0,p\rangle \qquad \alpha_n^i |0,p\rangle = \tilde{\alpha}_n^i |0,p\rangle = 0 \qquad n > 0$

$\left\{ \alpha_{-n_1}^{i_1} \dots \alpha_{-n_p}^{i_n} \dots \tilde{\alpha}_{-m_1}^{j_1} \dots \tilde{\alpha}_{-m_n}^{j_n} |0,p\rangle \right\}$

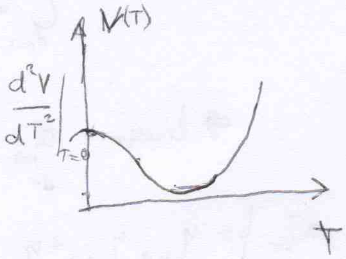
no states with negative norm!

positive def. H-space

$$M^2 = \frac{4}{\alpha'}\left(N - \frac{D-2}{24}\right) = \frac{4}{\alpha'}\left(\tilde{N} - \frac{D-2}{24}\right) \qquad N = \sum_{i=1}^{D-2} \alpha_{-n}^i \alpha_n^i \qquad \tilde{N} = \sum_{i=1}^{D-2} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i$$

String excitation:

- vacuum $\quad M^2 = -\frac{D-2}{6\alpha'} < 0 \quad \leftrightarrow \underset{\text{field}}{T(x)}$



- $N = \tilde{N} = 1$

$\alpha_{-1}^i \tilde{\alpha}_{-1}^j |0,p\rangle \qquad i,j = 1, \dots, D-2$

$SO(D-2) \qquad$ D dim. Lorentz and Poincare group

$SO(1,D-1) \Rightarrow \begin{matrix} m=0 & SO(D-2) & (p,0,0,\dots,p) \\ m\neq 0 & SO(D-1) & (p,0\dots 0) \end{matrix}$

$m=0 \qquad \boxed{D=26}$

- $N = \tilde{N} = 2 \qquad [N, \alpha_{-n}^i] = n\alpha_{-n}^i$

$\underset{i=1\dots D-2}{\alpha_{-1}^i \alpha_{-1}^j} \quad \alpha_{-2}^i \qquad \tilde{\alpha}_{-1}^i \tilde{\alpha}_{-1}^j \qquad \tilde{\alpha}_{-2}^i$

$\frac{1}{2}(D-1)(D-2) + (D-2) = \boxed{\frac{1}{2}D(D-1) - 1}$

symm. traceless reps of $SO(D-1)$

Poincare symm. of the string

$$X^\mu \to X^\nu \Lambda_\nu{}^\mu + c^\mu$$

$$\int (p^\mu x^\nu - x^\mu p^\nu) d\sigma \qquad p^\mu = \int_0^{2\pi} \frac{-1}{2\pi\alpha'} \partial_\tau X^\mu \, d\sigma$$

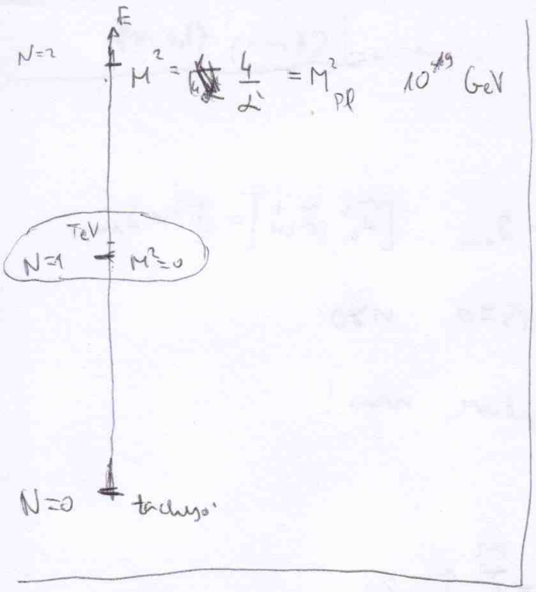
$$= M^{\mu\nu} = \underbrace{p^\mu x^\nu - p^\nu x^\mu}_{\text{orbital ang momentum}} - i\sum \frac{1}{n}\left(\alpha_{-n}^\nu \alpha_n^\mu - \alpha_{-n}^\mu \alpha_n^\nu\right) - i\sum \frac{1}{n}\left(\tilde{\alpha}_{-n}^\nu \tilde{\alpha}_n^\mu - \tilde{\alpha}_{-n}^\mu \tilde{\alpha}_n^\nu\right)$$

spin

$[M^{\rho\sigma}, M^{\tau\nu}] = \eta M - \eta M + \eta M - \eta M$

lightcone gauge: check this algebra, as they are not all independent

it is satisfied $\Leftrightarrow$ D=26



$M^2=0$ states $SO(D-2) = SO(24)$

$24 \otimes 24$ = traceless symmetric ⊕ antisymmetric ⊕ trace

$G_{\mu\nu}$ massless spin 2 graviton

$B_{\mu\nu}$ 2-form field Kalb-Ramond f.

Φ dilat.

Why is this the graviton

Feynmann – Weinberg

massless spin 2 particles ↔ gravity

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ small fluct.

$S_{EH} = \int \sqrt{-G}\,R$

⇒ linear $a_{\mu\nu}, a^+_{\mu\nu}$ creation/annihilation op

$$S_{EH} = \int \left(\partial_\mu h^\rho_\rho \partial_\nu h^{\mu\nu} - \partial^\rho h^{\mu\nu} \partial_\mu h_{\rho\nu} + \dots \right)$$

$$[a_{\mu\nu}, a^+_{\rho\sigma}] = \eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho}$$

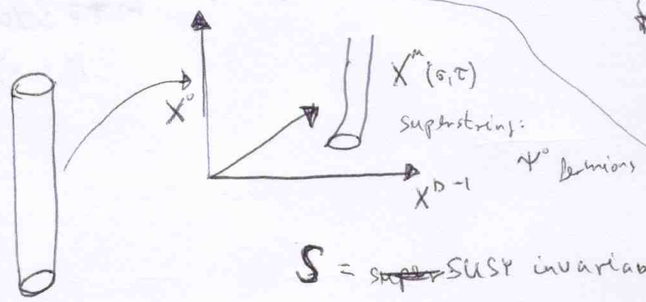
$a^+_{0i}|0\rangle$ ⇒ neg. norm.

⇒ use constraint ⇒ pos def $\mathcal{H}$-space

gauge trans.

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$$

$X^\mu \to X^\mu + \xi^\mu$ ⇒ diffeomorphism invariance



S = ~~super~~ SUSY invariant

Superstring theory: - no tachyon
- D = 10

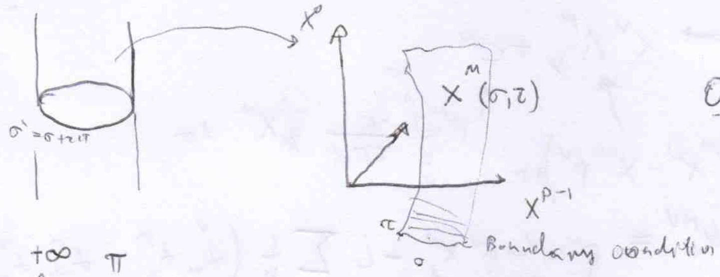
M theory (11dim)

N=2 II → A, B

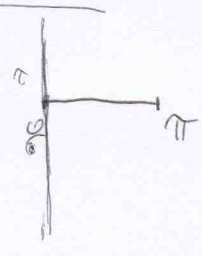
N=1 heterotic string → SO(32), $E_8 \times E_8$

SUSY

Closed



Open string



$$S = -\frac{1}{4\pi\alpha'} \int_{-\infty}^{+\infty} d\tau \int_0^\pi d\sigma \left(-\partial_\tau X \partial_\tau X + \partial_\sigma X \partial_\sigma X \right)$$

e.o.f.

$$\delta S = 0 = -\frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_0^\pi d\sigma \left[-\partial_\tau X \partial_\tau \delta X + \partial_\sigma X \partial_\sigma \delta X \right]$$

$\left((\partial_\tau^2 - \partial_\sigma^2) X\right) \delta X = 0$ ⇒ e.o.m.

$$\partial_\sigma(\partial_\sigma X \delta X) - \partial_\sigma^2 X \cdot \delta X$$

$$-\frac{1}{2\pi\alpha'} \int d\tau \left[\partial_\sigma X \delta X \right]_0^\pi$$

Boundary cond: $X' \cdot \delta X = 0$

$$\partial_\sigma X^\mu \delta X^\nu \eta_{\mu\nu} = 0$$

Neumann: $\partial_\sigma X^\mu = 0$ $\quad \delta X^\mu$ anything

(static gauge $X^0 = t = R\tau$

Virasoro constraints $X' \cdot \dot{X} = 0 \quad \dot{X}^2 + X'^2 = 0 \Rightarrow \dot{\vec{X}}^2 + \vec{X}'^2 = R^2$

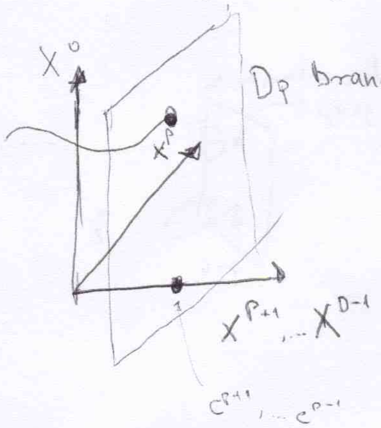
$$\Downarrow$$

$\dot{\vec{X}} \cdot \vec{X}' = 0$ $\quad \left|\frac{d\vec{X}}{dt}\right|^2 = 1$ speed of light

Dirichlet: $\delta X^\mu = 0 \quad X^\mu = c^\mu$

$\mu = \underbrace{0,\dots,p}_{N_a}, \underbrace{\dots D-1}_{D_I}$ $\quad \partial_\mu X^a = 0 \quad a = 0,\dots,p$

$X^I = c^I \quad I = p+1,\dots D-1$



D-brane ≡ surface where open string can end

D_0 brane particle

D_1 brane string

D_2 brane membrane

⋮

$SO(1,D-1) \to SO(1,p-1) \otimes SO(D-1-p)$

Classical solution (mode expansion) $\quad \sigma^\pm = \tau \pm \sigma$

recall (closed) $\quad$ no $\frac{1}{2}$

$$X_L^\mu(\sigma^+) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}_n^\mu e^{-in\sigma^+}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}x^\mu + \alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\sigma^-}$$

$\tilde{\alpha}_n^a = \alpha_n^a$

Poincare $\Lambda^\mu{}_\nu x^\nu + c^\mu$ $\to$ conserved momentum $\frac{1}{2\pi\alpha'}\int_0^\pi d\sigma\,(\partial_\tau X^\mu) = p^\mu$

Neumann

$\tilde{\alpha}_n^a = \alpha_n^a$

Dirichlet

$X^I = c^I$

$p^I = 0$

$\alpha_n^I = -\tilde{\alpha}_n^I$

$D = 26$

lightcone gauge: $X^\pm = \sqrt{\frac{1}{2}}(X^0 \pm X^p)$

$$M^2 = \frac{1}{\alpha'}\Big[\underbrace{\sum_{i=1}^{p-1}\sum_{n>0}\alpha_{-n}^i\alpha_n^i + \sum_{i=p+1}^{D-1}\sum_{n\neq 0}\alpha_{-n}^i\alpha_n^i}_{N} - \frac{D-2}{24}\Big]$$

preserve this Lorentz symm.