particles are pointlike

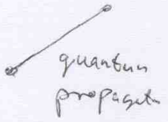
2. lecture

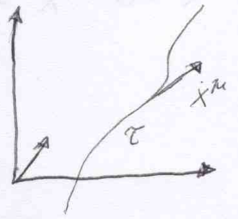
[consistent, relativistic, quantum theory]

1, movement of a free particle

2, quantise free particle

3, introduce ~~introduction~~ interaction

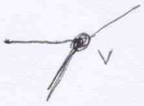




$\mathbb{R}^{1,D-1}$

$\eta = (-,+,\dots,+)$

$X^\mu(\tau)$ $\mu = 0,1,\dots D$



Calculate scattering process

1 a, $S = -m \int \sqrt{\underbrace{-\dot{X}^\mu \dot{X}^\nu \eta_{\mu\nu}}_{\dot{X}\dot{X}}}\, d\tau$

b, $S = -\frac{1}{2}\int d\tau \sqrt{-g_{\tau\tau}}\left(g^{\tau\tau}\dot{X}^2 + m^2\right)$

eq. of motion g: $\dot{X}^2 - g_{\tau\tau} m^2 = 0$

$p_\mu = \frac{\partial L}{\partial \dot{X}^\mu} = \frac{m \dot{X}^\nu \eta_{\mu\nu}}{\sqrt{\quad}}$

- Lorentz $X^\mu \to X^\nu \Lambda^\mu{}_\nu + a^\mu$
- $\tau \to \tilde{\tau}(\tau)$

$p_\mu p^\mu + m^2 = 0$ mass-shell

2, Quantization

$\Psi(X)$ $i\frac{\partial \Psi}{\partial \tau} = H\Psi$ $H = p_\mu \dot{X}^\mu - L = 0$ constraint: $\left(p_\mu p^\mu + m^2\right)\Psi = 0$

$\left(-\partial_\mu \partial^\mu + m^2\right)\Psi = 0$

KG

what sort of index can we intro

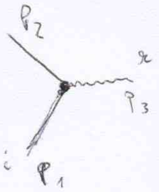
representation of the Poincare group

Wigner: 1, fix momentum $(\overset{m}{\underset{\shortparallel}{p}}, \underbrace{0,0,0,\dots 0}_{SO(D-1)})$

2, classify reps. of the little group } massive

massless $\left\{ (p, \underbrace{0,0,\dots}_{\text{little group } SO(D-2)} p) \right.$

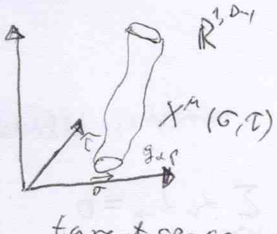
3, Interaction



$V(x^0, p_1, p_2, p_3, i, j, k)$

interacting QM $\equiv$ QFT

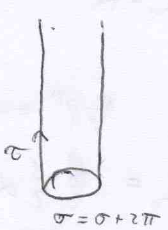
Particles are stringlike



$\mathbb{R}^{1,D-1}$

$X^\mu(\sigma,\tau)$

target space



1, free motion of a string

2, Quantize the free string

3, Interacting string

$S = -\frac{1}{\dots}\int d\dots \sqrt{(\dots)^2 \dots}$

$\dot{X} = \partial_\tau X$ $X' = \partial_\sigma X$

Symm:
- Poincaré inv.
- diff. inv $\sigma, \tau \to \tilde\sigma, \tilde\tau$
- Weyl: $g_{\alpha\beta} \to \Omega^2(\sigma)\, g_{\alpha\beta}$

e.o.m $g_{\alpha\beta} \Rightarrow T_{\alpha\beta} = 0$ Virasoro constraints $\qquad g_{\alpha\beta} = \eta_{\alpha\beta}^{\,(-,+)}$

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X \partial^\alpha X \qquad \dot X \cdot X' = 0 = \dot X^2 + X'^2$$

Solving ~~std~~ cl. string theory:

$$\partial_\alpha \partial^\alpha X^\mu = 0 = \left(-\partial_\tau^2 + \partial_\sigma^2\right) X^\mu = 0 \qquad X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau)$$

$$\sigma^\pm = \tau \pm \sigma \qquad \partial_+ \partial_- X^\mu = 0 \qquad X^\mu = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-)$$

$$X^\mu_R \to \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\tfrac{\alpha'}{2}} \sum \tfrac{1}{n} \alpha^\mu_n e^{-in\sigma^-}$$

$$X^\mu_L \to \tfrac{1}{2}x^\mu + \tfrac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n\neq 0} \tfrac{1}{n} \tilde\alpha^\mu_n e^{-in\sigma^+}$$

constraint

$$(\partial_\pm X)^2 = 0$$

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m = 0 \qquad \partial_+ X \partial_+ X = \sum_n L_n e^{-in\sigma^+} \qquad (\alpha^\mu_n)^* = \alpha^\mu_{-n}$$

$$\tilde L_n = \frac{1}{2}\sum_m \tilde\alpha_{n-m}\cdot\tilde\alpha_m = 0 \qquad \alpha_0 = \sqrt{\tfrac{\alpha'}{2}}\, p$$

Comment:

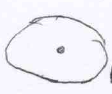
$$X^\mu \to X^\mu + a^\mu \xRightarrow{\text{Noether current}} P^\mu_\alpha = \frac{-1}{2\pi\alpha'}\partial_\alpha X^\mu \qquad \partial^\alpha P^\mu_\alpha = 0$$

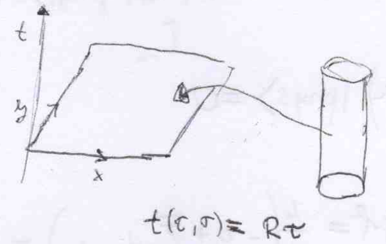
momentum of the string $= \displaystyle\int_0^{2\pi} d\sigma \frac{1}{2\pi\alpha'} \underbrace{\partial_\tau X^\mu}_{\alpha' p^\mu} = p^\mu$

constraint

$$L_0 = 0 \Rightarrow M^2 = \frac{4}{\alpha'}\sum_{n>0}\alpha_n\cdot\alpha_{-n} = \frac{4}{\alpha'}\sum_{n>0}\tilde\alpha_n\cdot\tilde\alpha_{-n} \quad \text{level matching}$$

Example: pulsating string





$R\cos\tau$

$$t(\tau,\sigma) = R\tau$$
$$y(\tau,\sigma) = R\sin\sigma\cos\tau$$
$$x(\tau,\sigma) = R\cos\sigma\cos\tau$$

$$\dot X X' = 0 = \dot X^2 + (X')^2$$
$$\partial_\tau X = \dot X = R(1, -\sin\sigma\sin\tau, -\cos\sigma\sin\tau)$$
$$\partial_\sigma X = X' = R(0, \cos\sigma\cos\tau, -\sin\sigma\cos\tau)$$

2. Quantization:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu \qquad \dot X X' = (X')^2 + (\dot X)^2 = 0 \quad \text{constraint}$$

covariant quantization: quantise → fix constraint → Qstring free

OR

fix constraint → quantise → Qstring free (light cone quantisation)

Covariant quantization: $\Pi_\mu = \frac{1}{2\pi\alpha'} \dot{X}_\mu$

Can. Commutation relation $\left[X^\mu(\sigma,\tau), \Pi_\nu(\sigma',\tau)\right] = i\,\delta(\sigma-\sigma')\,\delta^\mu{}_\nu$ all others are zero

$$X^\mu(\sigma,\tau) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-) = x^\mu + \alpha' p^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \alpha^\mu_n e^{-inp^+} + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^\mu_n e^{-in\sigma^-}$$

$$X^\mu(\sigma,\tau) = X^\mu(\sigma+2\pi,\tau) \qquad \delta(\sigma-\sigma') = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} e^{in(\sigma-\sigma')}$$

$$\underset{\uparrow\ \text{free particle}}{[x^\mu, p^\mu_\nu] = i\delta^\mu{}_\nu} \qquad \underset{\uparrow}{\left[\alpha^\mu_n, \alpha^\nu_n\right]} = \left[\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m\right] = n\,\delta_{n+m,0}\,\eta^{\mu\nu} \qquad (\alpha^\mu_n)^\dagger = \alpha^\mu_{-n}$$

$n>0$ $\quad \frac{\alpha_n}{\sqrt{n}} = a_n \qquad \frac{\alpha_{-n}}{\sqrt{n}} = a^\dagger_n \qquad [\underset{\text{annihilation op.}}{a_n}, \underset{\text{creation op.}}{a^\dagger_m}] = \delta_{n,m}$

<u>Hilbert space</u>: $\underset{\alpha_n}{a_n}|0\rangle = \underset{\tilde{\alpha}_n}{\tilde{a}_n}|0\rangle = 0 \qquad n>0$

~~G~~ $\quad \alpha^{\mu_1}_{-n_1} \dots |0\rangle$

$\quad |0,p\rangle \rightarrow p^\mu$

$$\mathcal{H} = \left\{ \alpha^{\mu_1}_{-n_1} \dots \alpha^{\mu_N}_{-n_N}\, \tilde{\alpha}^{\nu_1}_{-m_1} \dots \tilde{\alpha}^{\nu_M}_{-m_M}\, |0,p\rangle \right\}$$

∞ number of different particles

positive definitness? $\quad \left[\alpha^\mu_1, \alpha^\nu_{-1}\right] = \eta^{\mu\nu} \qquad \langle p',0|\alpha^0_1 \alpha^0_{-1}|0,p\rangle = -\delta^D(p-p')$ ghosts

$\uparrow$ negative norm

Constraints:

$$L_m = \frac{1}{2}\sum_{m\in\mathbb{Z}} \alpha_{n-m}\alpha_m \qquad \langle \text{phys}|\underset{L_{-n}}{L_n}|\text{phys}\rangle = 0 \qquad \left.L_n^\dagger = L_{-n}\right\} \quad L_n|\text{phys}\rangle = 0 \quad n>0$$

$$L_0 = \sum_{m=1}^{\infty} \alpha_{-m}\alpha_m + \frac{1}{2}\alpha_0^2 \qquad (L_0 - a)|\text{phys}\rangle = 0$$

$\alpha^\mu_0 = \sqrt{\frac{\alpha'}{2}}\, p^\mu$

$$p^\mu p_\mu = -M^2 \longrightarrow M^2 = \frac{4}{\alpha'}\left(-a + \sum_{m\neq 0} \alpha_{-m}\alpha_m\right) = \frac{4}{\alpha'}\left(-a + \sum_{m\neq 0} \tilde{\alpha}_{-m}\tilde{\alpha}_m\right)$$

<u>Lightcone quantisation</u>:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\, \partial_\alpha X^\mu \partial^\alpha X_\mu$$

$\dot{X}\cdot X' = \dot{X}^2 + (X')^2 = 0 \quad \Rightarrow L_n = \tilde{L}_n = 0$

~~$\sigma \to$~~ $\tilde{\sigma}(\sigma)$ $\qquad 0 = (\partial_+ X)^2 = (\partial_- X)^2$

$$\boxed{\begin{array}{l} \sigma^+ \to \tilde{\sigma}^+(\sigma^+) \\ \sigma^- \to \tilde{\sigma}^-(\sigma^-) \end{array}} \text{ gauge symm.} \qquad \sigma^\pm = \tau \pm \sigma$$

$$\underset{\substack{\text{conformal}\\\text{gauge}}}{g_{\alpha\beta} \longrightarrow \Omega^2(\sigma)\eta_{\alpha\beta}} \qquad \begin{array}{l} \sigma \to \tilde{\sigma}(\sigma,\tau) \\ \tau \quad \tilde{\tau}(\sigma,\tau) \end{array}$$

$\downarrow$ Weyl

$\eta_{\alpha\beta}$

$\{$

$$\boxed{ds^2 = -d\sigma^+ d\sigma^-}$$

point part $X^0 \sim R\tau$ $\qquad X^{\pm} = \sqrt{\frac{1}{2}}\left(X^0 \pm X^{D-1}\right)$

Lorentz $SO(1,D-1) \to SO(D-2)$ $\qquad \left(A_+, A_-, A_i\right)$ $\quad \left(A^+, A^-, A^i\right)$ $SO(D-2)$

$A_+ = -A^-$ $\qquad A_- = -A^+$ $\qquad A_i = A^i$

$A\cdot B = -A^+B^- - A^-B^+ + A^iB^i$

Solving for X^+ $\qquad X^+ = X^+_L(\sigma^+) + X^+_R(\sigma^-) \xrightarrow{g.\ trafo} \left.\begin{array}{l} X^+_L(\sigma^+) = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+\sigma^+ \\ X^+_R(\sigma^-) = \frac{1}{2}x^+ + \frac{1}{2}\alpha' p^+\sigma^- \end{array}\right\} X^+ = x^+ + \alpha' p^+\tau$

$\tau \to \tau + a$ shift x^+

Solve X^- $\qquad \partial_+\partial_- X^- = 0$ $\qquad X^- = X^-_L(\sigma^+) + X^-_R(\sigma^-)$

$(\partial_+X)^2 = 0 = (\partial_+X)^2 \Rightarrow 2\,\partial_+X^-\underbrace{\partial_+X^+}_{\frac{1}{2}\alpha' p^+} = \sum_{i=1}^{D-2}\partial_+X^i\partial_+X^i$

$$\partial_+X^-_L(\sigma^+) = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}\partial_+X^i\partial_+X^i$$

$(\partial_-X)^2 = 0$ $\qquad \partial_-X^-_R(\sigma^-) = \frac{1}{\alpha' p^+}\sum_{i=1}^{D-2}\partial_-X^i\partial_-X^i$

$X^- = X^-_L(\sigma^+) + X^-_R(\sigma^-)$

$\to \frac{1}{2}x^- + \frac{1}{2}\alpha' p^-\sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{\alpha_n^-}{n}e^{-in\sigma^+}$

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha'}}\frac{1}{p^+}\sum_{m=-\infty}^{\infty}\sum_{i=1}^{D-2}\alpha^i_{n-m}\alpha^i_m$$

$n=0$ (?)

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+}\sum_{i=1}^{D-2}\left(\frac{1}{2}\alpha' p^ip^i + \sum_{n\neq 0}\alpha^i_n\alpha^i_{-n}\right)$$

u.e.z. $\alpha \to \tilde{\alpha}$

$p^\mu p_\mu = -M^2 \Rightarrow M^2 = 2p^+p^- - \sum_{i=1}^{D-2}p^ip^i = \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n =$

$= \frac{4}{\alpha'}\sum_{i=1}^{D-2}\sum_{n>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n$

Physical degrees of freedom:

$$\boxed{x^i,\ p_i,\ \alpha^i_n,\ \tilde{\alpha}^i_n,\ p^+,\ x^-}$$

Quantization $\qquad [x^i, p_j] = i\delta^i_j \qquad [x^-, p^+] = -i \qquad [\alpha^i_n, \alpha^j_m] = [\tilde{\alpha}^i_n, \tilde{\alpha}^j_m] = n\,\delta_{n+m}\,\delta^{ij}$

build up the Fock space $|0,p\rangle$ $\qquad \alpha^i_n|0,p\rangle = \tilde{\alpha}^j_n|0,p\rangle = 0 \quad n>0$

$\mathcal{H} = \left\{\alpha^{i_1}_{-n_1}\cdots\alpha^{i_N}_{-n_N}\tilde{\alpha}^{j_1}_{-m_1}\tilde{\alpha}^{j_M}_{-n_M}|0,p\rangle\right\}$ positive definit

$$M^2 = \frac{4}{\alpha'}\left(\underbrace{\sum_{i=1}^{D-2}\sum_{n>0}\alpha^i_{-n}\alpha^i_n}_{N} - a\right) = \frac{4}{\alpha'}\left(\underbrace{\sum_{i=1}^{D-2}\sum_{m>0}\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n}_{\tilde{N}} - a\right)$$