2017.09.12.
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Tananyag:

- lineáris algebra
- sajátérték-probléma és alkalmazása
- vektoranalízis

} az anyagrészek végén ZH (őszi szünet előtt/után, kb. 3 héttel utána, dec. 18.)

Január első hete → speci ZH-k
utána → vektor vizsga
13 kérdés (9 vektoralgebra, 4 vektoranalízis)
tételek (18 v.alg., 18 v.analízis)

Brounstein - Szemendjajev: Matematikai zsebkönyv (ZH-n is használható)
Elméleti fizikai példatár
Mátrixszámítás, Bolyai könyvek
Műszaki matematikai gyakorlatok
Fazekas: Felsőfokú matematikai feladatok
Differenciálgeometriai feladatgyűjtemény
„Gyöngybetűs"

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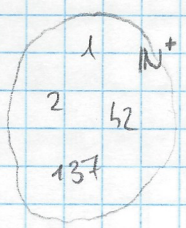
Az algebra alaptétele: minden n-edfokú egyenletnek van gyöke ($\mathbb{C}$)



$\forall a,b \in \mathbb{N}^+ \quad \exists c \in \mathbb{N}^+ : c = a+b$ zárt

$A \times B \ni (a,b)$ $\quad +: (\mathbb{N}^+ \times \mathbb{N}^+) \to \mathbb{N}^+$

$a \in A$
$b \in B$

$A \times A \to A$ (pl. vektorösszeadás)
$A \times B \to B$ (pl. vektor skalárral szorzása)

$\forall a,b,c \in \mathbb{N}^+ : \quad (a+b)+c = a+(b+c)$ / $(a\ b)\leq \neq a\ (b\leq)$ → NEM AZ ASSZOCIATIVITÁS KÉRDÉSE

$\leq \times V \to V$
1
$V \times V \to S$

asszociatív

$\forall a,b \in \mathbb{N}^+ : \quad a+b = b+a$

zártság + asszociativitás → félcsoport

2. + a. + kommutatív → kommutatív félcsoport

$(\mathbb{N}^+, +)$
$(\mathbb{N}^+, \cdot)$ } kommut. félcsoportok

$(\mathbb{N}^+, +, \cdot)$ → más struktúra

$\forall a,b \quad \exists d \quad d = a \cdot b$

$\forall a,b,c \quad (a \cdot b) \cdot c = a \cdot (b \cdot c)$

$\forall a,b \quad a \cdot b = b \cdot a$

$\exists 1 \in \mathbb{N}^+ \quad \forall a \in \mathbb{N}^+ \quad 1 \cdot a = a$ → semleges elem / neutrális elem

$5 + x = 7$ → van mo.

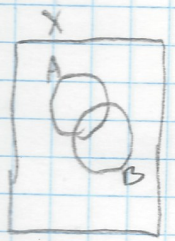
$5 + x = 5$
$5 + x = 1$
$3 + y = 3$ } nincs mo. → definiálunk: $0_5, 0_3, \ldots$ → a halmaz kibővül $\mathbb{N}$

$\exists 0 \in \mathbb{N} \quad \forall a \in \mathbb{N} \quad 0 + a = a$ neutrális elem

$\forall a \in \mathbb{N} \quad 0 \cdot a = 0$

$\forall a,b,c \in \mathbb{N} \quad (a+b) \cdot c = (a \cdot c) + (b \cdot c)$ disztributív

$(a \cdot b) + c \neq (a+c) \cdot (b+c)$ nem disztributív → $+, \cdot$ nem szimmetrikus



$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$

$(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$

X : neutrális elem a $\cap$-re

$\emptyset$: neutrális elem az $\cup$-ra

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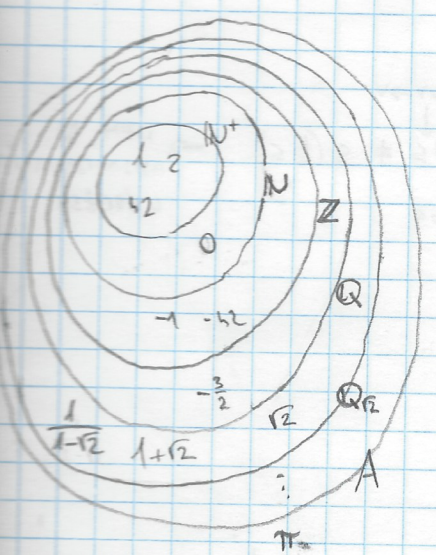

$+ : A \times A \to A$

① $\forall a,b \quad \exists c \quad c = a+b$
② $\forall a,b,c \quad (a+b)+c = a+(b+c)$ } $(\mathbb{N}^+, +)$
③ $\forall a,b \quad a+b = b+a$
④ $\exists 0 \quad \forall a \quad 0+a = a$

$\cdot : A \times A \to A$ $\Rightarrow (\mathbb{N}^+, \cdot)$ semleges elemes félcsoport

$(\mathbb{N}^+, +, \cdot) \quad \forall a,b,c \quad (a+b)\,c = ac + b \cdot c$

$3 + x = 1 \quad 3 + y = 2$
$2 + x = 0$ } → negatív számok (magában szorzásra nem zárt)

⑤ $\forall a \in \mathbb{Z} \quad \exists a' \in \mathbb{Z} \quad a + a' = 0$

Áll.: $a + x = b \quad a, b \in \mathbb{Z}$

$\rightarrow \exists x \in \mathbb{Z}$

Biz.: 4) $\exists a'$

1) $a' + (a + x) = a' + b$

2) $(a' + a) + x = a' + (a + x)$

4) $0 + x = a' + b$

3) $x = a' + b$

①, ②, ③, ④ (⑤) érvényes $\rightarrow$ (Komm.) csoport

$(\mathbb{Z}, +, \cdot)$ gyűrű $\rightarrow$ nullosztós v. nem nullosztós

($\exists a, b \neq 0$, $a \cdot b = 0$)

$3 \cdot x = 6$

$-2 \cdot x = 8$

$-2 \cdot x = 3$? $\rightarrow \mathbb{Q}$

$a \cdot x = b$

$x = (a, b)$

$(a, b) + (c, d) = (ad + bc, bd)$

$0 \cdot x = 5 \rightarrow$ az 1-5 feltételek nem teljesíthetőek, ellentmondásra vezetne

$(\mathbb{Q}, +, \cdot)$

TEST $\begin{cases} (\mathbb{Q}, +) \text{ Komm. csoport} \\ (\mathbb{Q} \setminus \{0\}, \cdot) \text{ Komm. csoport} \\ \text{disztributív} \end{cases}$

$x \cdot x = 2 \rightarrow$ testbővítés

$x \cdot x = 3$

$a + b\sqrt{2} \quad a, b \in \mathbb{Q}$

$(a_1 + b_1\sqrt{2}) + (a_2 + b_2\sqrt{2}) = (a_1 + a_2) + (b_1 + b_2)\sqrt{2}$

$(a_1 + b_1\sqrt{2}) \cdot (a_2 + b_2\sqrt{2}) = a_1 a_2 + b_1\sqrt{2}\, a_2 + a_1 b_2\sqrt{2} + b_1\sqrt{2} \cdot b_2\sqrt{2} = (a_1 a_2 + 2 b_1 b_2) + (b_1 a_2 + a_1 b_2)\cdot\sqrt{2}$

$$\frac{1}{a + b\sqrt{2}} = \frac{a - b\sqrt{2}}{(a + b\sqrt{2})(a - b\sqrt{2})} = \frac{a - b\sqrt{2}}{a^2 - (b\sqrt{2})^2} = \frac{a - b\sqrt{2}}{a^2 - 2b^2} = \left(\frac{a}{a^2 - 2b^2}\right) + \left(\frac{-b}{a^2 - 2b^2}\right)\cdot\sqrt{2}$$

$\mathbb{Q}_{\sqrt{2}}$, $\mathbb{Q}_{\sqrt{3}} \rightarrow$ nem zárt, $a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} \rightarrow \mathbb{Q}_{\sqrt{6}}$ is kell

További bővítések $\rightarrow$ $\mathbb{A}$ algebrai számok

Transzcendens számok: nem kapható meg egész együtthatós egyenlet gyökeként

Alg. + tr. $\to \mathbb{R}$

$x \cdot x = -3 \to \sqrt{-3} \to \sqrt{-1}, \sqrt{-2}, \ldots$

Ezeknél a számoknál már nem lesz olyan egyenlet, aminek nincs halmaz beli gyöke $\to$ A bővítés vége

$\Rightarrow$ Komplex számok. $\mathbb{C}$

Kommutativitást feladva új bővítés $\to \mathbb{H}$: Kvaterniók

Később: $\mathbb{O}$ Cayley *Hiperkomplex számok c. könyben

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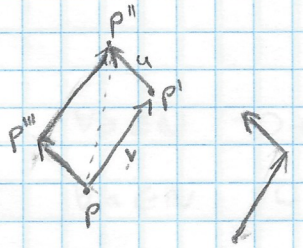
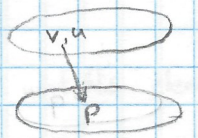




$P' = v(P)$

$P'' = u(P')$

$P''' = u(P)$

$v(P''') = P''$

$u(v(P)) = v(u(P)) \quad \forall P$

$P' = vP$

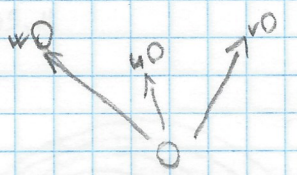
$P'' = uP' = u(vP)$

$P''' = uP$

$P'' = v(uP)$

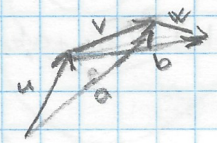
$v(uP) = u(vP)$

$\boxed{vu = uv}$



$u + v = v + u$

két eltolás eredménye = egy eltolás $\to$ zárt



$b + u = w + a$

$(w + v) + u = w + (v + u)$

$u + 0 = u \quad \forall u\text{-ra}$, 0. identitás

$u' + u = 0$

$\hookrightarrow$ (vektorok, +) csoport

paralelogramma szabály (sokszög szab. lényeg)

számok halmaza: $\mathbb{S}$ (skalárok)

vektorok halmaza: $\mathbb{V}$

$+ \quad \mathbb{V} \times \mathbb{V} \to \mathbb{V}$

zárt $\quad \forall \vec{a}, \vec{b} \in \mathbb{V} \quad \exists \vec{c} \in \mathbb{V} \quad \vec{c} = \vec{a} + \vec{b}$

asszoc. $\quad \forall \vec{a}, \vec{b}, \vec{c} \in \mathbb{V} \quad (\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$

null. $\quad \exists \vec{0} \in \mathbb{V} \quad \forall \vec{a} \in \mathbb{V} \quad \vec{0}+\vec{a} = \vec{a}$

inv. $\quad \forall \vec{a} \in \mathbb{V} \quad \exists \vec{a}' \quad \vec{a}+\vec{a}' = \vec{0}$

komm. $\quad \forall \vec{a}, \vec{b} \in \mathbb{V} \quad \vec{a}+\vec{b} = \vec{b}+\vec{a}$

- $\mathbb{S} \times \mathbb{V} \to \mathbb{V}$

  $(\lambda, \vec{a}) \mapsto \vec{b} = \lambda \vec{a}$ $\quad {}^{*}\vec{a}\lambda = \lambda\vec{a}$ konvenció

  $\forall \lambda \in \mathbb{S} \quad \forall \vec{a}, \vec{b} \in \mathbb{V} \quad \lambda(\vec{a}+\vec{b}) = \lambda\vec{a} + \lambda\vec{b}$

  $\forall \lambda, \beta \in \mathbb{S} \quad \forall \vec{a} \in \mathbb{V} \quad (\lambda+\beta)\cdot\vec{a} = \lambda\vec{a} + \beta\vec{a}$ $\quad {}^{**}+$ és $+$ más: nem disztributivitás

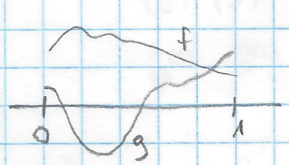
  $\forall \lambda, \beta \in \mathbb{S} \quad \forall \vec{a} \in \mathbb{V} \quad \lambda\cdot(\beta\vec{a}) = (\lambda\beta)\cdot\vec{a}$

  $\forall \vec{a} \in \mathbb{V} \quad 1\vec{a} = \vec{a}$

$\mathbb{S}(+, \cdot, 0, 1)$

Lineáris tér / vektortér: $\mathbb{V}_{\mathbb{S}}$

$\mathbb{V}_{\mathbb{R}}$ $\quad \mathbb{V}_{\mathbb{C}}$



Adott intervallumon értelmezett függvények is lineáris teret alkotnak

$$\begin{pmatrix} 1\text{ fej} \\ 1\text{ kéz} \\ 2\text{ sör} \\ 3\text{ pos} \end{pmatrix} + \begin{pmatrix} 0\text{ t} \\ 0\text{ k} \\ 3\text{ sör} \\ 5\text{ pos} \end{pmatrix} + 2\begin{pmatrix} 0 \\ 0 \\ 5 \\ 10 \end{pmatrix}$$

Számoszlopok / mátrixok is lineáris teret alkotnak

Lacks: Lineáris algebra

Matolcsi Tamás: Analízis

$\vec{u}, \vec{v}, \vec{w}$

$\alpha\vec{u} + \beta\vec{v} + \gamma\vec{w}$

$\vec{u_1}, \vec{u_2}, \vec{u_3}$ $\quad u_k$

$\alpha_1, \alpha_2, \alpha_3$ $\quad \alpha_k$

$\sum_{k=1}^{3} \alpha_k \vec{u_k}$ vektorok lineáris kombinációja

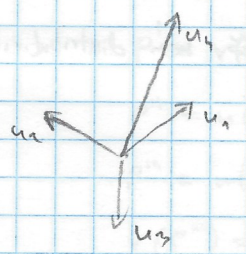
$$\vec{a} = \sum_{k=1}^{n} \alpha_k \vec{u_k}$$

$$\vec{b} = \sum_{k=1}^{n} \beta_k \vec{u_k}$$

$$\vec{a} + \vec{b} = \sum_{k=1}^{n} (\alpha_k + \beta_k) \cdot \vec{u_k}$$

$$\gamma \vec{a} = \gamma \cdot \sum_{k} \alpha_k \vec{u_k} = \sum_{k} (\gamma \alpha_k) \cdot \vec{u_k}$$

Altér: egy lin. tér részhalmaza, amelyre teljesülnek az axiómák (pl. u_k vektorok lineárkomb.-i)



$\alpha \cdot u_4 + 5 \underbrace{(u_1 + u_2 + u_3)}_{0}$

Ha 3 vektor nemtriviális (nem 0 elhatású) lineárkombinációja $\vec{0}$ lehet, akkor a három vektor lineárisan összefügg.

Lineárisan független vektorok maximális száma: DIMENZIÓ

$\dim V = n$

$\vec{u_1}, \vec{u_2}, \dots, \vec{u_n}$

$\vec{w}$ $\quad \exists \beta_1, \dots, \beta_n$ $\quad \vec{w} = \sum_{k} \beta_k \vec{u_k}$

$\underbrace{\vec{u_1} \dots \vec{u_n}, \vec{w}}_{n+1}$ $\quad \exists \alpha_0 : \quad \sum_{k=1}^{n} \alpha_k \vec{u_k} + \alpha_0 \vec{w} = 0$

$\alpha_0 \vec{w} = - \sum \alpha_k \vec{u_k}$ $\quad / \alpha_0 \neq 0 \Rightarrow u_k$-k lin. függetlenek

$$\vec{w} = \sum_{k} \left(- \frac{\alpha_k}{\alpha_0} \right) \cdot \vec{u_k}$$

BÁZIS → adott bázison minden vektor kifejthető

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EA.

Vektortér ≠ vektorműlet

↓ algebrai fogalom

↳ függvénytani fogalom

$+, \cdot \rightarrow$ lineáris műveletek

$V \ni \vec{a}^{(1)}, \vec{a}^{(2)}, \dots, \vec{a}^{(n)}$

$V \rightarrow \vec{v} = \sum_{k=1}^{n} \alpha_k \vec{a}^{(k)}$

$$\begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{pmatrix} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} \qquad \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha+\beta \\ 0 \\ 3\beta \\ 0 \\ 0 \end{pmatrix} \quad *$$

$\underline{a}^{(1)} \quad \underline{a}^{(2)}$

↳ $\underline{a}^{(1)}$ és $\underline{a}^{(2)}$ lineáris burka

$$\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 5 \\ 0 \\ 7 \\ 0 \\ 0 \end{pmatrix} \rightarrow \begin{pmatrix} -3\alpha+5\beta \\ 0 \\ \alpha+7\beta \\ 0 \\ 0 \end{pmatrix} \rightarrow \text{ua., mint } \begin{pmatrix} x \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix}$$

Valahány vektor lin. burka lineáris tér és a vektorok terének altere.

Triviális részterek: önmaga és az üres halmaz

Minden vektortérnek altere a nullvektort tartalmazó halmaz és önmaga. A többi: alter: valódi alterek.

$$\begin{pmatrix} 1 \\ 0 \\ 3 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ -6 \\ 0 \\ 0 \end{pmatrix} \qquad 2\vec{a}^{(1)} + \vec{a}^{(2)} = \vec{0}$$ lineárisan összefüggők

$\vec{a}^{(1)} \quad \vec{a}^{(2)}$

* $\left.\begin{matrix} \alpha+\beta=0 \\ 3\beta=0 \end{matrix}\right\} \Rightarrow \alpha=0, \beta=0 \Rightarrow$ lineárisan független vektorok

$\cos^2 x, \sin^2 x, \cos 2x$: lineárisan összefügg

$1 \cdot \cos^2 x + (-1) \cdot \sin^2 x + (-1) \cdot \cos 2x = 0$

$\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\1\\0\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\1\\0\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\1\\0\end{pmatrix}\begin{pmatrix}0\\0\\0\\0\\1\end{pmatrix}$ lin. független vektorok

$\underline{a}^{(1)}\ \underline{a}^{(2)}\ \underline{a}^{(3)}\ \underline{a}^{(4)}\ \underline{a}^{(5)}$

$$\alpha\cdot\underline{a}^{(1)}+\beta\cdot\underline{a}^{(2)}+\gamma\,\underline{a}^{(3)}+\delta\,\underline{a}^{(4)}+\varepsilon\cdot\underline{a}^{(5)}=\begin{pmatrix}\alpha\\\beta\\\gamma\\\delta\\\varepsilon\end{pmatrix}$$

$$+\times\begin{pmatrix}5\\0\\3\\-1\\0\end{pmatrix}=\begin{pmatrix}5x+\alpha\\\beta\\3x+\gamma\\-x+\delta\\\varepsilon\end{pmatrix}$$ Lehet-e $\vec{0}$?

$$\begin{array}{l}5x+\alpha=0\\ \beta=0\\ 3x+\gamma=0\\ -x+\delta=0\\ \varepsilon=0\end{array}\iff\begin{array}{l}\alpha=-5x\\ \beta=0\\ \gamma=-3x\\ \delta=x\\ \varepsilon=0\end{array}$$

pl. $x=1 \Rightarrow$ $\alpha=-5$, $\beta=0$, $\gamma=-3$, $\delta=1$, $\varepsilon=0$

Nem lehet 6 db lin. független vektor

$$x\underline{u}-5x\,\underline{a}^{(1)}+0\,\underline{a}^{(2)}-3x\,\underline{a}^{(3)}+x\,\underline{a}^{(4)}+0\,\underline{a}^{(5)}=0$$

$$\underline{u}=5\,\underline{a}^{(1)}+0\cdot\underline{a}^{(2)}+3\,\underline{a}^{(3)}+(-1)\cdot\underline{a}^{(4)}+0\,\underline{a}^{(5)}$$

Minden vektor egyértelműen állítható és a bázisvektorok lineáris kombinációjaként.

Biz. $\vec{u}=\sum_k \alpha_k\vec{a}^{(k)}$

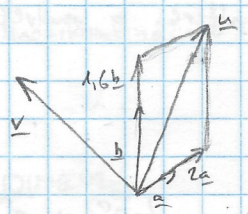
$\vec{u}=\sum_k \beta_k\cdot\vec{a}^{(k)}$

$\vec{u}-\vec{u}=\sum(\alpha_k-\beta_k)\vec{a}^{(k)}$

$\vec{0}$

$\gamma_k=0 \Rightarrow \boxed{\alpha_k=\beta_k}$

$\vec{a}^{(k)}$ bázisvektorok

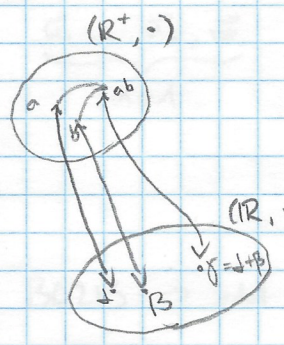


$\vec{u}=2\vec{a}+1{,}6\vec{b}$

$\underline{u}=\begin{pmatrix}2\\1{,}6\end{pmatrix}$

$\underline{v}=\begin{pmatrix}-3\\1\end{pmatrix}$

Bijekción a geometriai vektorok és a számoszlopok között $\Rightarrow$ kell. bázisvektorok



$a\cdot b=c$ $\quad\alpha=La$

$\alpha+\beta=\gamma$ $\quad\beta=Lb$

$\gamma=L(c)=L(ab)=L(a)+L(b)$

$$\boxed{\forall a,b\in\mathbb{R}^+\quad L(ab)=L(a)+L(b)}$$

Homomorfizmus: csak egyirányú a leképezés

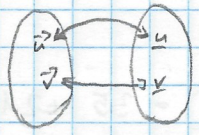
Izomorfizmus: azonos struktúra, (kölcsönösen egyértelmű, művelettartó leképezés létezik közöttük)

$\vec{u} = 2\vec{a} + 1{,}6\vec{b}$

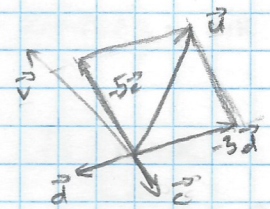
$\vec{v} = -3\vec{a} + 1\vec{b}$

$\vec{w} = \vec{u} + \vec{v} = (2-3)\vec{a} + (1{,}6+1)\vec{b} = -\vec{a} + 2{,}6\vec{b}$

$\underline{w} = \begin{pmatrix} 2 \\ 1{,}6 \end{pmatrix} + \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2{,}6 \end{pmatrix}$



Vektor → számoszlop Homomorfizmus: reprezentáció



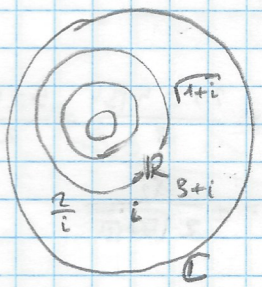
$\vec{u} = -5\vec{c} - 3\vec{d}$

$\vec{v} = -1\vec{c} + \vec{d}$

$\vec{u} \rightarrow \underline{u} = \begin{pmatrix} 2 \\ 1{,}6 \end{pmatrix}$ $\underline{u}' = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

$\vec{v} \rightarrow \underline{v} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$ $\underline{v}' = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

2017. 09. 13.
GY.



$x^2 = -3$

$x = \sqrt{-3} = \sqrt{(-1)\cdot 3} = \sqrt{-1}\cdot\sqrt{3}$

$\boxed{i^2 = -1}$

$\mathbb{C} \ni z = a + i\cdot b \qquad a, b \in \mathbb{R}$

$z_1 = a_1 + i\,b_1$

$z_2 = a_2 + i\,b_2$

$z_3 = (a_1 + ib_1) + (a_2 + ib_2) = \underbrace{(a_1 + a_2)}_{a_3 \in \mathbb{R}} + i\underbrace{(b_1 + b_2)}_{b_3 \in \mathbb{R}} = a_3 + ib_3$

$z = a\cdot 1 + bi \sim \begin{pmatrix} a \\ b \end{pmatrix}$

$\alpha z = \alpha a\cdot 1 + (\alpha b)i \sim \begin{pmatrix} \alpha a \\ \alpha b \end{pmatrix}$ komplex számok → kétdimenziós lineáris tér

$z_1 = (a_1 + ib_1)(a_2 + ib_2) = a_1a_2 + ib_1a_2 + a_1ib_2 + ib_1ib_2 = \underbrace{(a_1a_2 - b_1b_2)}_{a_3} + i\underbrace{(a_1b_2 + a_2b_1)}_{b_3}$

$a = \mathrm{Re}\, z$

$b = \mathrm{Im}\, z$

$(5+3i)(7-2i) = 35 + 21i - 10i + 6 = 41 + 11i$

Komplex konjugált $z^* = a - ib$

$\bar{z}$

$\frac{1}{z} = \frac{1}{a+ib} = \frac{1}{a+ib} \cdot \frac{a-ib}{a-ib} = \frac{a-ib}{a^2-(ib)^2} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} + i\frac{-b}{a^2+b^2}$ / ha $a=b=0$

$\Rightarrow$ nincs reciprok

$\Rightarrow (\mathbb{C}, +, \cdot)$ test

$\frac{3-2i}{5+4i} = \frac{3-2i}{5+4i} \cdot \frac{5-4i}{5-4i} = \frac{15-10i-12i-8}{25+16} = \frac{7-22i}{41} = \frac{7}{41} + \left(-\frac{22}{41}\right) \cdot i$

$\sqrt{5+3i} = x + iy$

$5 + 3i = (x+iy)^2 = x^2 + 2xiy - y^2 = (x^2 - y^2) + i(2xy)$

$\boxed{\begin{matrix} x^2 - y^2 = 5 \\ 2xy = 3 \end{matrix}} \rightarrow y = \frac{3}{2x}$ $\qquad x^2 - \left(\frac{3}{2x}\right)^2 = 5$

$4x^4 - 9 = 20x^2$

$4x^4 - 20x^2 - 9 = 0$

$x^2_{1,2} = \frac{20 \pm \sqrt{400 - 4(-36)}}{8} = \frac{20 \pm \sqrt{544}}{8} = \frac{20 \pm 4\sqrt{34}}{8} = \frac{5 \pm \sqrt{34}}{2}$

$x^2 = \frac{5+\sqrt{34}}{2} \rightarrow x = \pm\sqrt{\frac{5+\sqrt{34}}{2}}$ $\qquad y = \pm\frac{3\sqrt{2}}{2\cdot\sqrt{5+\sqrt{34}}}$

$x^2 + 4x + 5 = 0$

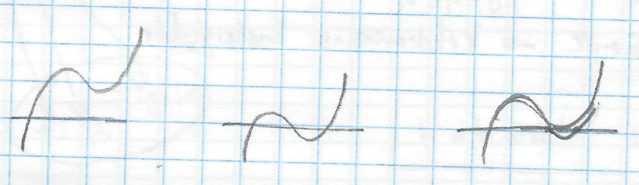
I. $x_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4 \cdot 5}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = \frac{-4 \pm 2i}{2} = -2 \pm i$

II. $(x^2 + 4x + 4) + 1 = 0$

$(x+2)^2 = -1$

$x + 2 = \pm i$

$x = -2 \pm i$

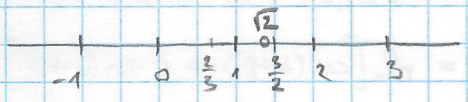


$\Rightarrow$ „A perturbáció feloldja a degenerációt"

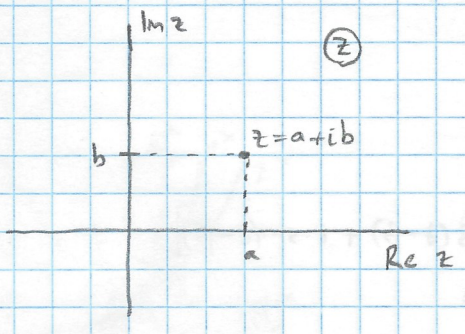
$f(x) = x^6 - 5x^5 + \dots = 0$

$= (x-1)(x^5 \dots)$

Egy n-edfokú polinomnak n gyöke van.



$\mathbb{R}$ halmaza teljes



$a_1 + i \cdot 0$

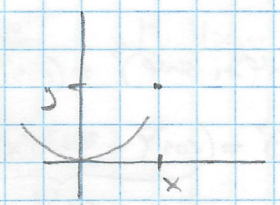
$a_2 + i \cdot 0$

$(a_1 + a_2) + i \cdot 0$

KOMPLEX SÍK

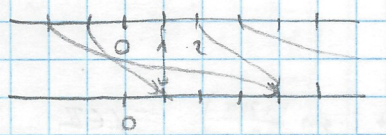
$f: \mathbb{R} \to \mathbb{R}$

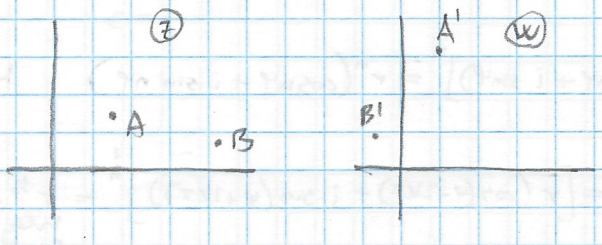
$x \mapsto y$

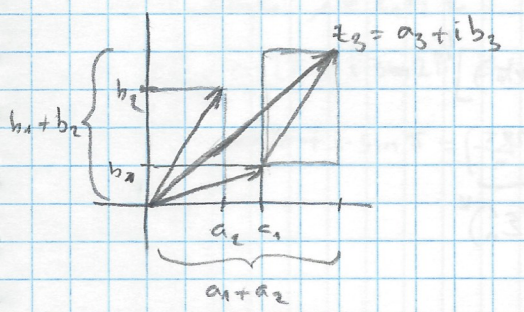


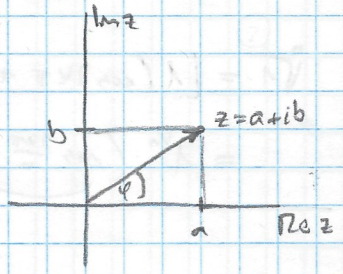
$f: \mathbb{C} \to \mathbb{C}$

$z \mapsto w = f(z)$









$r^2 = a^2 + b^2$

$a = r \cdot \cos\varphi$

$b = r \cdot \sin\varphi$

$\begin{cases} r = \sqrt{a^2+b^2} \\ \varphi = \operatorname{arctg} \frac{b}{a} \end{cases}$

$z = a + ib = r\cos\varphi + ir\sin\varphi = r(\cos\varphi + i\cdot\sin\varphi)$ $\quad r = |z|$

$\varphi = \arg z$

$z_1 = r(\cos\alpha + i\cdot\sin\alpha)$

$z_2 = R\cdot(\cos\beta + i\cdot\sin\beta)$

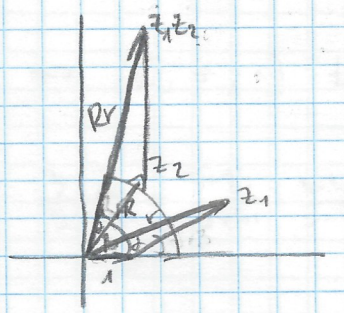
$z_1 z_2 = r(\cos\alpha + i\cdot\sin\alpha)\cdot R(\cos\beta + i\cdot\sin\beta) = rR[\cos\alpha\cos\beta + i\cdot\sin\alpha\cos\beta + \cos\alpha\cdot i\cdot\sin\beta + i\cdot\sin\alpha\cdot i\cdot\sin\beta] =$

$= rR[(\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)] = rR[\cos(\alpha+\beta) + i\cdot\sin(\alpha+\beta)] =$

$= \varrho(\cos\gamma + i\cdot\sin\gamma)$

$\varrho = rR$

$\gamma = \alpha + \beta$



$$\frac{r(\cos\alpha + i\cdot\sin\alpha)}{R(\cos\beta + i\cdot\sin\beta)} = \frac{r}{R}\cdot[\cos(\alpha-\beta) + i\cdot\sin(\alpha-\beta)]$$

$z = r(\cos\varphi + i\cdot\sin\varphi)$

$z^3 = r^3(\cos 3\varphi + i\cdot\sin 3\varphi)$

$r = 1$

$z^3 = \cos 3\varphi + i\cdot\sin 3\varphi = (\cos\varphi + i\cdot\sin\varphi)^3 = (\cos\varphi)^3 + 3(\cos\varphi)^2 i\cdot\sin\varphi +$

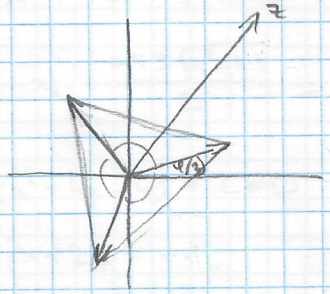
$+ 3\cos\varphi\cdot(i\cdot\sin\varphi)^2 + (i\cdot\sin\varphi)^3 = \underbrace{(\cos^3\varphi - 3\cos\varphi\sin^2\varphi)}_{\cos 3\varphi} +$

$+ i\cdot\underbrace{(3\cos^2\varphi\sin\varphi - \sin^3\varphi)}_{\sin 3\varphi}$

$\sin 3\varphi = 3\cos^2\varphi\sin\varphi - \sin^3\varphi = 3\cdot(1-\sin^2\varphi)\cdot\sin\varphi - \sin^3\varphi = 3 - 4\sin^3\varphi$

$z^n = [r(\cos\varphi + i\sin\varphi)]^n = r^n(\cos n\varphi + i\sin n\varphi)$ Moivre.

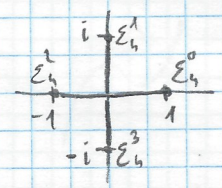
$\sqrt[n]{z} = z^{\frac{1}{n}} = [r(\cos(\varphi+2k\pi) + i\cdot\sin(\varphi+2k\pi))]^{\frac{1}{n}} = \underbrace{r^{1/n}}_{\sqrt[n]{r}}\cdot\left(\cos\frac{\varphi+2k\pi}{n} + i\cdot\sin\frac{\varphi+2k\pi}{n}\right)$ $\quad k\in\mathbb{Z}$

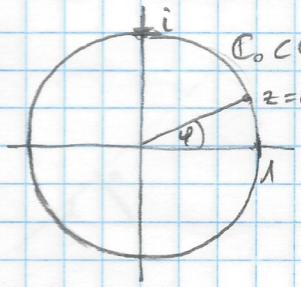


$\sqrt[n]{1} = [1(\cos 2k\pi + i\cdot\sin 2k\pi)]^{1/n} =$

$= 1^{1/n}\cdot\underbrace{\left(\cos\frac{2k\pi}{n} + i\cdot\sin\frac{2k\pi}{n}\right)}_{\varepsilon_n^k = (\varepsilon_n^1)^k}$

$\sqrt[4]{1} = [1\cdot(\cos 2k\pi + i\cdot\sin 2k\pi)]^{1/4} = \cos\frac{2k\pi}{4} + i\cdot\sin\frac{2k\pi}{4}$ $\quad k\in\mathbb{Z}$





$f: \mathbb{R} \to \mathbb{C}_0$

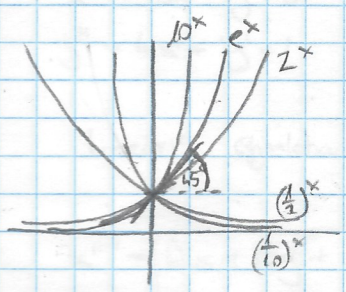
$\varphi \longmapsto z = f(\varphi) = \cos\varphi + i\cdot\sin\varphi$

$f(\alpha)\cdot f(\beta) = (\cos\alpha + i\cdot\sin\alpha)(\cos\beta + i\sin\beta) = \cos\alpha\cos\beta + i\sin\alpha\cos\beta +$

$+ \cos\alpha\, i\sin\beta + i\cdot\sin\alpha\, i\sin\beta = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i\cdot(\sin\alpha\cos\beta + \cos\alpha\sin\beta) =$

$= \cos(\alpha+\beta) + i\cdot\sin(\alpha+\beta) = f(\alpha+\beta)$

$$\boxed{f(\alpha)\cdot f(\beta) = f(\alpha+\beta)}$$

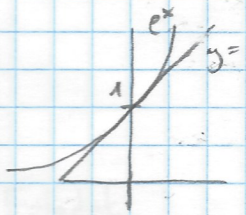


$a^x = (e^c)^x = e^{cx}$

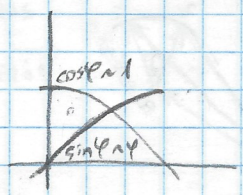
$a = e^c \qquad c = \log_e a$

$e^{c\alpha}\, e^{c\beta} = e^{c(\alpha+\beta)}$

$f(\alpha) = e^{c\alpha}$ Mi legyen a c?



$e^x \approx 1 + x$



$f(\varphi) = \cos\varphi + i\cdot\sin\varphi = e^{c\varphi}$

$\approx 1 + i\varphi = 1 + c\varphi$

$c = i$

$$\boxed{e^{i\varphi} = \cos\varphi + i\cdot\sin\varphi}$$

$z = a+ib = r(\cos\varphi + i\sin\varphi) = r\cdot e^{i\varphi}$

$(r_1 e^{i\varphi_1})(r_2 e^{i\varphi_2}) = r_1 r_2\, e^{i(\varphi_1+\varphi_2)}$

$e^{i2\pi} = \cos 2\pi + i\cdot\sin 2\pi = 1$

$e^{i\pi} = \cos\pi + i\cdot\sin\pi = -1$

2017.09.21.

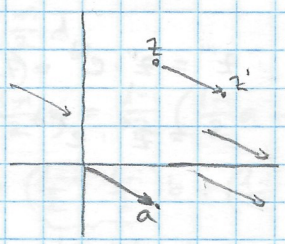
GY.

$z = x + i\cdot y \qquad z' = x' + iy'$

$\mathbb{C} \to \mathbb{C}$

$z \longmapsto z' = z + a$

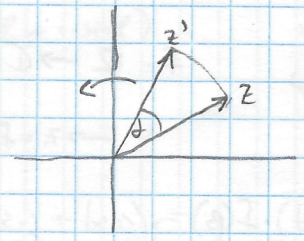
eltolás



$f(t) = \cos t + i \cdot \sin t = e^{it}$

$z \longmapsto z' = z \cdot e^{it}$

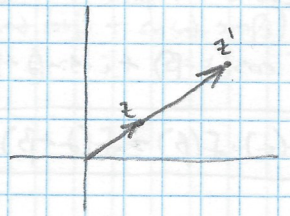
forgatás



$r \in \mathbb{R}$

$z \longmapsto z' = r \cdot z$

nagyítás

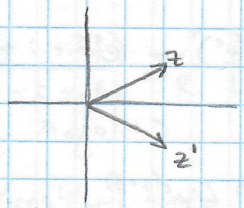


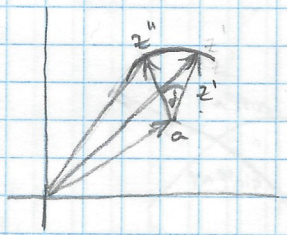
$z = x + iy$

$z^* = x - iy$

$z \longmapsto z' = z^*$

tükrözés





$z' = z - a$

$z'' = z' \cdot e^{it}$

$z''' = z'' + a$

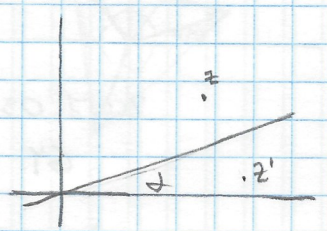
$z''' = z' + a = z' \cdot e^{it} + a = (z - a) e^{it} + a$

$a = 5 + 3i$

$z''' = [z - (5+3i)] \cdot e^{it} + a = [(x+iy) - (5+3i)](\cos t + i \sin t) + (5+3i) = [(x-5) + i(y-3)](\cos t + i \sin t) +$

$+ (5+3i) = (x-5)\cos t + i(y-3)\cos t + i(x-5)\sin t - (y-3)\sin t + 5 + 3i =$

$= \underbrace{[(x-5)\cos t - (y-3)\sin t + 5]}_{x'''} + i \cdot \underbrace{[(y-3)\cos t + (x-3)\sin t + 3]}_{y'''}$



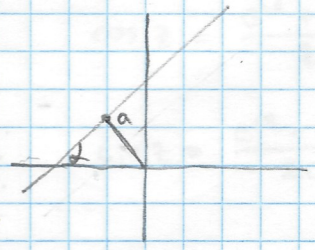
$z' = z \cdot e^{-it}$

$z'' = (z')^*$

$z''' = z'' \cdot e^{it}$

$z''' = z'' \cdot e^{it} = (z')^* \cdot e^{it} = (z \cdot e^{-it})^* \cdot e^{it} = (z^* \cdot e^{it}) \cdot e^{it} =$

$= z^* \cdot e^{2it}$



$z' = z - a$

$z'' = z' \cdot e^{i\alpha}$

$z''' = (z'')^*$

$z'''' = z''' \cdot e^{i\alpha}$

$z''''' = z'''' + a$

$x' = 2x - 3y - 5$

$y' = -x + 4y - 2$

$x'' = -3x' + 7y' - 6$

$y'' = 4x' + 2y' + 1$

lineáris egyenletrendszerek egymásba helyettesítve → lineáris egyenletrendszer

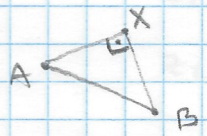
$$x''''' = \boxed{Ax + By + C = x}$$

$$y''''' = \boxed{Dx + Ey + F = y}$$

Az egyenletrendszer megoldásai: fixpontok → 0 mo.: eltolás
→ 1 mo.: forgatás
→ ∞ mo.: tükrözés

Feladat

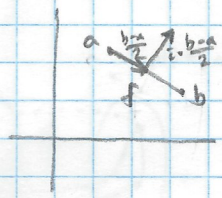
$*$



$A * B = X$

All.: $(A*B)*(C*D) = (A*C)*(B*D)$

Biz.: Komplex számokkal



$f = \frac{a+b}{2}$

$x = \frac{a+b}{2} + i \cdot \frac{b-a}{2} = \frac{1-i}{2} \cdot a + \frac{1+i}{2} \cdot b$

$U = X * Y$

$X = A * B$

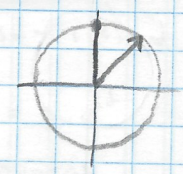
$Y = C * D$

$x = \frac{1-i}{2} a + \frac{1+i}{2} b$

$y = \frac{1-i}{2} c + \frac{1+i}{2} d$

$u = \frac{1-i}{2} x + \frac{1+i}{2} y = \frac{1-i}{2}\left(\frac{1-i}{2} a + \frac{1+i}{2} b\right) + \frac{1+i}{2}\left(\frac{1-i}{2} c + \frac{1+i}{2} d\right) =$

$= \left(\frac{1-i}{2}\right)^2 a + \underbrace{\left(\frac{1-i}{2} \cdot \frac{1+i}{2}\right)}_{\frac{1+i-i+1}{4} = \frac{1}{2}} b + \left(\frac{1+i}{2} \cdot \frac{1-i}{2}\right) c + \left(\frac{1+i}{2}\right)^2 d$

$\sqrt{i}$  $\pm \frac{1+i}{\sqrt{2}}$

$z + z^* = (x+iy) + (x-iy) = 2x = 2\,\mathrm{Re}\,z \in \mathbb{R}$

$z - z^* = (x+iy) - (x-iy) = 2iy$

$i(z - z^*) = -2y \in \mathbb{R}$

$\frac{1}{i} = \frac{1}{i} \cdot \frac{-i}{-i} = -i$

$zz^* = (x+iy)(x-iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2$

$\frac{1}{z} = \frac{1}{z} \cdot \frac{z^*}{z^*} = \frac{z^*}{|z|^2}$

$\varphi \in \mathbb{R}$ $f(\varphi) = e^{i\varphi} = \cos\varphi + i \cdot \sin\varphi$

$f(\varphi)^* = \cos\varphi - i \cdot \sin\varphi = \cos(-\varphi) + i \cdot \sin(-\varphi) = e^{-i\varphi}$

$f(\alpha) f(\beta) = f(\alpha+\beta)$
$e^{i\alpha} \cdot e^{i\beta} = e^{i(\alpha+\beta)}$

$\frac{e^{i\varphi} + e^{-i\varphi}}{2} = \cos\varphi$ $\qquad$ $\frac{e^{i\varphi} - e^{-i\varphi}}{2i} = \sin\varphi$

$e^{iz} = e^{i(x+iy)} = e^{ix-y} = e^{ix} \cdot e^{-y} = e^{-y}(\cos x + i \cdot \sin x) = e^{-y} \cdot \cos x + i \cdot e^{-y} \cdot \sin x$

$e^z = e^{x+iy} = e^x (\cos y + i \cdot \sin y)$

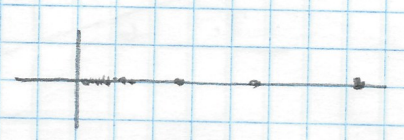
$\log_e x = \ln x$

$\ln z = \ln(r \cdot e^{i\varphi}) = \ln r + \ln e^{i\varphi} = \ln r + i\varphi$

$\ln z = \ln(r \cdot e^{i(\varphi + 2k\pi)}) = \ln r + i \cdot (\varphi + 2k\pi)$ → Komplex számnak végtelen sok logaritmusa van (0-nak nincs)

$z_1^{z_2} = \left[r_1 \cdot e^{i(\varphi_1 + 2k\pi)}\right]^{(a_2 + ib_2)} = \left[e^{\ln r_1 + i(\varphi_1 + 2k\pi)}\right]^{(a_2 + ib_2)} = e^{[\ln r_1 + i(\varphi + 2k\pi)] \cdot (a_2 + ib_2)}$

$i^i = \left(1 \cdot e^{i(\pi/2 + 2k\pi)}\right)^i = e^{-(\frac{\pi}{2} + 2k\pi)} \in \mathbb{R}$



$\log_{z_1} z_2 = \frac{\ln z_2}{\ln z_1} = \frac{\ln\left[r_2 e^{i(\varphi_2 + 2k\pi)}\right]}{\ln\left[r_1 \cdot e^{i(\varphi_1 + 2k\pi)}\right]} = \frac{\ln r_2 + i(\varphi_2 + 2k\pi)}{\ln r_1 + i(\varphi_1 + 2k\pi)}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos(3-5i) = \frac{e^{i(3-5i)} + e^{-i(3-5i)}}{2} = \frac{e^{3i+5} + e^{-3i-5}}{2} = \frac{e^{3i}e^{5} + e^{-3i}\cdot e^{-5}}{2} =$$

$$= \frac{(\cos 3 + i\cdot\sin 3)e^{5} + (\cos 3 - i\sin 3)e^{-5}}{2} = \cos 3\,\frac{e^{5}+e^{-5}}{2} + i\cdot\sin 3\cdot\frac{e^{5}-e^{-5}}{2}$$

$\sin z = 3$?

$$\frac{e^{iz} - e^{-iz}}{2i} = 3$$

$$e^{iz} - e^{-iz} = 6i$$

$$e^{iz} - \frac{1}{e^{iz}} = 6i \qquad y = e^{iz}$$

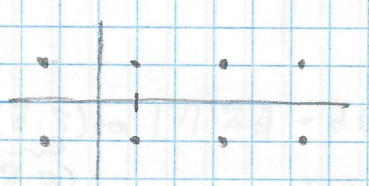
$$y - \frac{1}{y} = 6i$$

$$y^2 - 6iy - 1 = 0 \longrightarrow y_{1,2} = \frac{6i \pm \sqrt{(6i)^2 + 4}}{2} = \frac{6i \pm \sqrt{-32}}{2} = 3i \pm \sqrt{-8} = i(3 \pm \sqrt{8})$$

$$y_1 = i(3+\sqrt{8}) = e^{iz_1}$$

$$iz_1 = \ln\left[i(3+\sqrt{8})\right] = \ln i + \ln(3+\sqrt{8}) = i\left(\tfrac{\pi}{2} + 2k\pi\right) + \ln(3+\sqrt{8})$$

$$\left.\begin{aligned} z_1 &= \tfrac{\pi}{2} + 2k\pi - i\cdot\ln(3+\sqrt{8}) \\ z_2 &= \tfrac{\pi}{2} + 2k\pi - i\cdot\ln(3-\sqrt{8}) \end{aligned}\right\} z = \tfrac{\pi}{2} + 2k\pi \pm i\cdot\ln(3+\sqrt{8})$$



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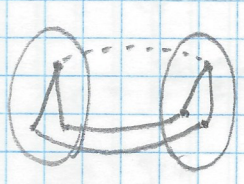


homeomorfizmus $\longrightarrow$

izomorfizmus $\longleftrightarrow$

$\vec{a}^{(n)}$ $\qquad \vec{v} = \sum_k \alpha_k \vec{a}^{(n)}$ $\qquad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}$

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

$\vec{v} \longleftrightarrow \underline{v}$ reprezentáció

Bázisvektorok: $\{\vec{e}^{(k)}\}_{k=1}^{n}$

$\vec{a} \longrightarrow \underline{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$ $\qquad \vec{a} = \sum_k a_k \vec{e}^{(k)}$

$\vec{b} \longrightarrow \underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ $\qquad \vec{b} = \sum_k b_k \vec{e}^{(k)}$

$\vec{c} = \vec{a} + \vec{b}$ $\qquad \vec{c} = \sum_k a_k \vec{e}^{(k)} + \sum_k b_k \vec{e}^{(k)} = \sum (\underbrace{a_k + b_k}_{c_k}) \vec{e}^{(k)}$

$c_k = a_k + b_k$

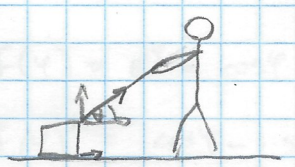
$\underline{c} = \underline{a} + \underline{b}$

$\vec{d} = \lambda \vec{a}$

$d_k = \lambda a_k$

$\underline{d} = \lambda \underline{a}$

$\vec{d} = \lambda \vec{a} = \lambda \left(\sum_k a_k \vec{e}^{(k)} \right) = \sum_k (\underbrace{\lambda a_k}_{d_k}) \vec{e}^{(k)} = \sum_k d_k \vec{e}^{(k)}$



$W = F \Delta s$

$F_s = |\vec{F}| \cdot \cos\varphi$

$W = F_s \Delta s = |\vec{F}| \cdot |\Delta \vec{s}| \cdot \cos\varphi$

$$\boxed{\vec{a}\vec{b} = |\vec{a}||\vec{b}|\cos\varphi}$$

$\vec{a}\vec{b} = \vec{b}\vec{a}$

$\vec{a}\vec{a} \geq 0$

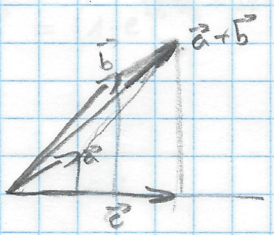
$\vec{a}\vec{a} = 0 \longrightarrow \vec{a} = \vec{0}$

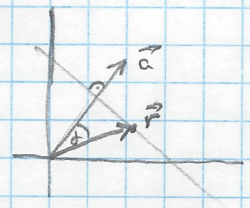
$\vec{c} = \lambda \vec{a}$

$\vec{c}\vec{b} = (\lambda \vec{a})\vec{b} = |\lambda \vec{a}||\vec{b}| \cdot \cos(\underbrace{\vec{a}, \vec{b}}_{(\vec{a}, \vec{b})}) = |\lambda||a||b| \cdot \cos\varphi \quad \Rightarrow (\lambda \vec{a})\vec{b} = \lambda(\vec{a}\vec{b})$

$|\lambda \vec{a}| = |\lambda|\,|\vec{a}|$

$(\vec{a} + \vec{b})\vec{c} = (\vec{a}\vec{c}) + (\vec{b}\vec{c})$

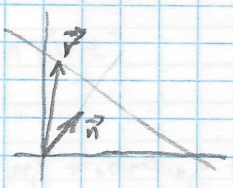




$\vec{r}\vec{a} = \text{const.}$

$|\vec{r}||\vec{a}|\cos\varphi$

$|\vec{r}| \cdot \cos\varphi$



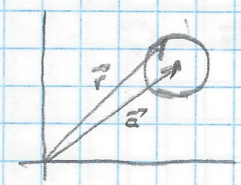
$|\vec{n}| = 1$

$\vec{n}\vec{n} = 1$

$\vec{r}\vec{n} = |\vec{r}||\vec{n}|\cos\varphi = |\vec{r}|\cdot\cos\varphi = c$

$\vec{e}\vec{e} = 1$

$\vec{e}^2 = 1$



$$(\vec{r}-\vec{a})(\vec{r}-\vec{a}) = R^2$$

V_S $\quad \cdot : V \times V \to S$

$(\vec{a}, \vec{b}) \longmapsto \lambda = \vec{a}\cdot\vec{b}$

$(\vec{a}+\vec{b})\vec{c} = (\vec{a}\vec{c}) + (\vec{b}\vec{c})$

$(\lambda\vec{a})\vec{b} = \lambda(\vec{a}\vec{b})$

$\Big\}\ (\lambda\vec{a}+\beta\vec{b})\vec{c} = \lambda(\vec{a}\vec{c}) + \beta(\vec{b}\vec{c})$

$\vec{a}\vec{a} \geq 0$

$\vec{a}\vec{a} = 0 \longrightarrow \vec{a} = \vec{0}$

$\vec{a}\vec{b} = \vec{b}\vec{a}$

Új művelet bevezetésével $\longrightarrow$ Euklideszi vektortér

$\{\vec{e}^{(k)}\}_{k=1}^{n}$

$\vec{a} = \sum_k a_k \vec{e}^{(k)}$

$\vec{b} = \sum_l b_l \vec{e}^{(l)} \equiv \sum_\ell b_\ell \vec{e}^{(\ell)}$

$\vec{a}\vec{b} = \left(\sum_k a_k \vec{e}^{(k)}\right)\left(\sum_l b_l \vec{e}^{(l)}\right) = \sum_k \sum_\ell a_k b_\ell \left(\vec{e}^{(k)}\vec{e}^{(\ell)}\right)$

$g_{k\ell} = \vec{e}^{(k)}\vec{e}^{(\ell)}$

$g_{k\ell} = g_{\ell k}$

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix}$$

$$\vec{a}\vec{b} = \sum_k \sum_\ell g_{k\ell} a_k b_\ell = \underline{a}\,\underline{b}$$

$g \to$ metrikus tenzor

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$ $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$

$\underline{a}\,\underline{b} = g_{11} a_1 b_1 + g_{12} a_1 b_2 + g_{21} a_2 b_1 + g_{22} a_2 b_2$

pl.: $a = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \underline{a}\,\underline{a} = a_1 a_1 - a_2 a_1 = a_1^2 = 1$

pl.: $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ $\quad \underline{a}\,\underline{a} = -1$

pl.: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\quad \underline{a}\,\underline{a} = a_1 a_1 - a_2 a_2 = 0$

$\vec{e}^{(1)}\vec{e}^{(1)} = 1 \quad \vec{e}^{(1)}\vec{e}^{(2)} = 0 \quad 0$

$0 \quad \vec{e}^{(2)}\vec{e}^{(2)} = 1 \quad 0$

$0 \quad 0 \quad \vec{e}^{(3)}\vec{e}^{(3)} = 1$

$$\delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$ Kronecker-delta

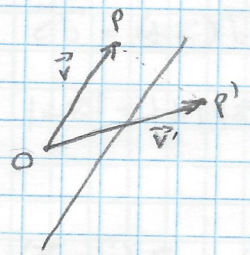
$\underline{\vec{e}^{(k)}\vec{e}^{(l)} = \delta_{kl}}$

ortogonális + normált $\longrightarrow$ ortonormált bázis (ONB)

$\vec{a}\,\vec{b} = \sum_k \sum_l a_k b_l \left(\vec{e}^{(k)}\vec{e}^{(l)}\right) = \sum_k a_k b_k$

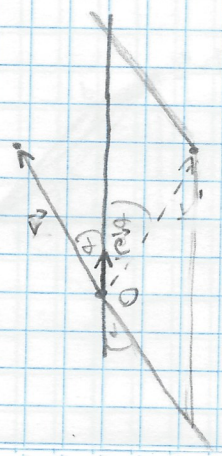
$\underline{a}\,\underline{b} = \sum_k a_k b_k$ $\longrightarrow$ A skaláris szorzat számsokasokkal való reprezentációja ONB-on

HF.



Fejezzük ki $\vec{r}'$-t $\vec{r}$-vel!

a)



$\vec{e} \parallel t$, $|\vec{e}| = 1$

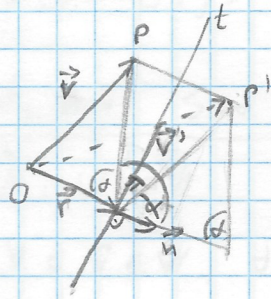
$\vec{r}' = -\vec{r} + 2 \cdot |\vec{r}| \cos\alpha \cdot \vec{e}$

$\vec{r} \cdot \vec{e} = |\vec{r}| \cdot |\vec{e}| \cdot \cos\alpha$

$\downarrow$

$\vec{r}' = -\vec{r} + \frac{2 \cdot \vec{r} \cdot \vec{e}}{|\vec{e}|} \cdot \vec{e} = -\vec{r} + 2(\vec{r} \cdot \vec{e}) \cdot \vec{e}$

b)



$\vec{r} \perp t$, $|\vec{r}| = d(O,t)$

$\vec{n} = \frac{\vec{r}}{|\vec{r}|}$

$\vec{v}' - \vec{r} = \vec{r} + (\vec{v} - \vec{r}) + |\vec{v} - \vec{r}| \cdot \cos 2\alpha \cdot \vec{n}$

$(\vec{v} - \vec{r}) \cdot \vec{n} = |\vec{v} - \vec{r}| \cdot |\vec{n}| \cdot \cos(\pi - 2\alpha) = |\vec{v} - \vec{r}| \cdot |\vec{n}| \cdot (-\cos 2\alpha)$

$\vec{v}' - \vec{r} = \vec{v} - \frac{(\vec{v} - \vec{r}) \cdot \vec{n}}{|\vec{n}|} \cdot \vec{n} = \vec{v} - (\vec{v} \cdot \vec{n} - \vec{r} \cdot \vec{n}) \cdot \vec{n} = \vec{v} - (\vec{v} \cdot \vec{n} - |\vec{r}|) \cdot \vec{n}$

$\vec{v}' = \vec{r} + \vec{v} - (\vec{v} \frac{\vec{r}}{|\vec{r}|} - |\vec{r}|) \cdot \frac{\vec{r}}{|\vec{r}|}$

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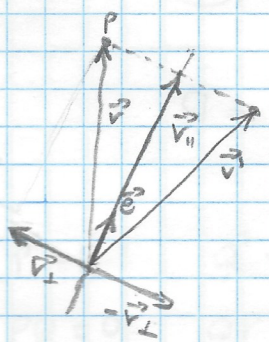


$\vec{v} = \vec{v}_\parallel + \vec{v}_\perp \quad \rightarrow \quad \vec{v}_\perp = \vec{v} - \vec{v}_\parallel$

$\vec{v}' = \vec{v}_\parallel - \vec{v}_\perp = \vec{v}_\parallel - (\vec{v} - \vec{v}_\parallel) = 2\vec{v}_\parallel - \vec{v}$

$\vec{v} + \vec{v}' = 2\vec{v}_\parallel$

$\vec{v}_\parallel = \lambda \cdot \vec{e}$

$\vec{v}_\perp = \vec{v} - \vec{v}_\parallel = \vec{v} - \lambda \vec{e}$

$\vec{v}_\perp \cdot \vec{e} \stackrel{!}{=} 0$

$(\vec{v} - \lambda \vec{e}) \vec{e} = 0$

$(\vec{v} \cdot \vec{e}) - \lambda (\vec{e} \vec{e}) = 0$

$\lambda = \vec{v}\vec{e} \quad \rightarrow \quad \boxed{\vec{v}_\parallel = (\vec{v} \cdot \vec{e}) \cdot \vec{e}}$

$\vec{v}_\parallel = P\vec{v}$ projekció

$\vec{v}' = 2\vec{v}_\parallel - \vec{v} = 2(\vec{v}\vec{e})\vec{e} - \vec{v} \quad \rightarrow \quad \vec{v}' = T\vec{v}$ tükrözés

$\vec{v}'' = 2(\vec{v}'\vec{e}) \cdot \vec{e} - \vec{v}' = 2\left([2(\vec{v}\vec{e})\vec{e} - \vec{v}]\vec{e}\right)\vec{e} - [2(\vec{v}\vec{e})\vec{e} - \vec{v}] =$

$= 2\left(\underbrace{2(\vec{v}\vec{e})(\vec{e}\vec{e})}_{\vec{v}\vec{e}} - \vec{v}\vec{e}\right)\vec{e} - 2(\vec{v}\vec{e})\vec{e} + \vec{v} = \vec{v}$

$2(\vec{v}\vec{e})\vec{e}$

$\vec{v}' = T\vec{v}$ $\qquad$ $|\vec{v} = \vec{v}$

$\vec{v}'' = T\vec{v}' = T(T\vec{v}) = T^2\vec{v} = \vec{v}$

$\forall v \in V \quad T^2\vec{v} = \vec{v} = 1\vec{v}$

$\boxed{T^2 = 1}$

$$P\vec{v}_{\parallel} = (v_{\parallel}\vec{e})\vec{e} = \left[(\vec{v}\vec{e})\vec{e}\right]\vec{e} = \left((\vec{v}\vec{e})(\vec{e}\vec{e})\right)\cdot\vec{e} = (\vec{v}\vec{e})\vec{e} = P\vec{v}$$

$\forall \vec{v}$. $P\vec{v}_{\parallel} = P(P\vec{v}) = PP\vec{v} = P^2\vec{v} = \vec{v}_{\parallel} = P\vec{v}$

$\boxed{P^2 = P}$ idempotens művelet

$\vec{v}' = 2\vec{v}_{\parallel} - \vec{v}$

$T\vec{v} = 2P\vec{v} - 1\vec{v}$

$\boxed{T = 2P - 1}$

$\boxed{P = \frac{1+T}{2}}$

$T^2 = (2P-1)^2 = (2P-1)(2P-1) = 4P^2 - 2P - 2P + 1 = 4P - 2P - 2P + 1 = 1$

$P^2 = \frac{1+T}{2}\cdot\frac{1+T}{2} = \frac{11+T1+1T+TT}{4} = \frac{1+T+T+1}{4} = \frac{1+T}{2} = P$

$Q = \frac{1-T}{2} \Rightarrow Q^2 = \frac{1-T}{2}\frac{1-T}{2} = \frac{11-1T-T1+TT}{4} = \frac{1-T}{2} = Q$

$Q\vec{v} = \frac{1-T}{2}\vec{v} = \frac{1\vec{v}-T\vec{v}}{2} = \frac{\vec{v}-\vec{v}'}{2} = \vec{v}_{\perp}$

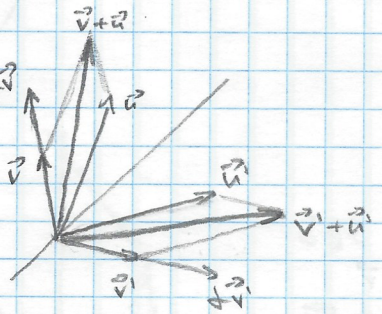
$PQ = \frac{1+T}{2}\cdot\frac{1-T}{2} = \frac{11+T1-1T-TT}{4} = \frac{1+T-T-1}{4} = 0$

$PQ\vec{v} = P(Q\vec{v})$

$0\vec{v} = \vec{0}$

$\hat{A} : \mathbb{V} \to \mathbb{V}$

$\vec{v} \mapsto \vec{u} = \hat{A}\vec{v}$



$\vec{v}' = T\vec{v}$

$\vec{u}' = T\vec{u}$

$\vec{w} = \vec{u} + \vec{v}$

$\vec{w}' = T\vec{w} = \vec{u}' + \vec{v}'$

$\boxed{T(\vec{u}+\vec{v}) = T\vec{u} + T\vec{v}}$ additiv

$\vec{u} = \lambda\vec{v}$

$T\vec{u} = \lambda T\vec{v}$

$\boxed{T(\lambda\vec{v}) = \lambda(T\vec{v})}$ homogén

LINEÁRIS OPERÁTOR

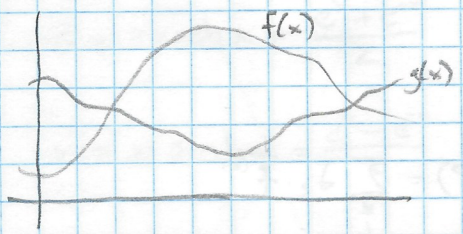
$$T(\alpha\vec{u}+\beta\vec{v}) = T(\alpha\vec{u}) + T(\beta\vec{v}) = \alpha(T\vec{u}) + \beta(T\vec{v})$$

$V_S \ni \vec{v}$ sor

$\hat{A} : V \to V$

$\forall \vec{u}, \vec{v} \in V, \ \forall \alpha, \beta \in S$

$\hat{A}(\alpha\vec{u}+\beta\vec{v}) = \alpha(\hat{A}\vec{u}) + \beta(\hat{A}\vec{v})$



$f(x) \mapsto f'(x) = Df(x)$ $\quad f' = Df$

$(f(x)+g(x))' = f'(x) + g'(x)$ $\quad D(f+g) = Df + Dg$

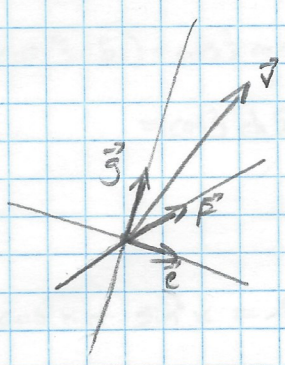
$(cf(x))' = c \cdot f'(x)$ $\quad D(cf) = c(Df)$

$D(\alpha f + \beta g) = \alpha(Df) + \beta(Dg)$

$P^2 = P$ $\quad Q^2 = Q$

$PQ = 0$ $\quad QP = 0$

$P + Q = 1$



$\vec{v}^{(1)} = (\vec{v}\vec{e})\vec{e}$ $\quad \vec{v}^{(1)} = P^{(1)}\vec{v}$

$\vec{v}^{(2)} = (\vec{v}\vec{f})\vec{f}$ $\quad \vec{v}^{(2)} = P^{(2)}\vec{v}$

$\vec{v}^{(3)} = (\vec{v}\vec{g})\vec{g}$ $\quad \vec{v}^{(3)} = P^{(3)}\vec{v}$

$\vec{e}^{(k)}$ $\quad P^{(k)}\vec{v} = (\vec{v}\vec{e}^{(k)})\vec{e}^{(k)}$

$P^{(k)}P^{(k)} = P^{(k)}$

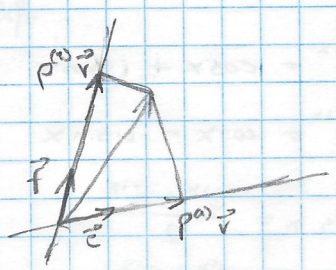
$P^{(k)}P^{(l)} = 0 \quad k \neq l$

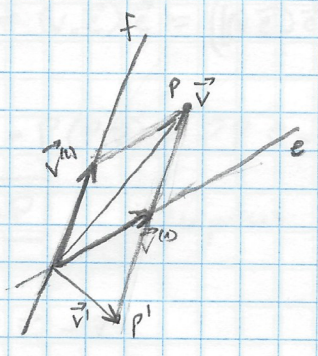
$$\delta_{kl} = \begin{cases} 1 & k = l \\ 0 & k \neq l \end{cases}$$

$$\boxed{P^{(k)}P^{(l)} = \delta_{kl}P^{(k)}}$$

$$\boxed{\sum_k P^{(k)} = I}$$

Csak akkor teljesül, ha $\boxed{\vec{e}^{(k)}\vec{e}^{(l)} = \delta_{kl}}$!





$\vec{e}^{(k)}$ lin. független vektorok

$\forall \vec{v} \quad \exists \alpha_k \qquad \vec{v} = \sum_k \alpha_k \vec{e}^{(k)}$

$\hat{A}\,\vec{e}^{(k)} = \vec{f}^{(k)}$

$$\hat{A}\vec{v} = \hat{A}\left(\sum_k \alpha_k \vec{e}^{(k)}\right) = \sum_k \hat{A}(\alpha_k \vec{e}^{(k)}) = \sum_k \alpha_k (\hat{A}\vec{e}^{(k)}) = \sum_k \alpha_k \cdot \vec{f}^{(k)}$$

$$\vec{f}^{(k)} = \sum_\ell A_{\ell k}\, \vec{e}^{(\ell)}$$

(I) $$\vec{u} = \hat{A}\vec{v} = \hat{A}\left(\sum_k \alpha_k \vec{e}^{(k)}\right) = \sum_k \alpha_k \left(\hat{A}\vec{e}^{(k)}\right) = \sum_k \alpha_k \vec{f}^{(k)} = \sum_k \alpha_k \left(\sum_\ell A_{\ell k} \cdot \vec{e}^{(\ell)}\right) =$$

$$= \sum_k \sum_\ell \alpha_k A_{\ell k}\, \vec{e}^{(\ell)} = \sum_\ell \sum_k \alpha_k A_{\ell k}\, \vec{e}^{(\ell)} = \sum_\ell \left(\sum_k A_{\ell k}\alpha_k\right) \vec{e}^{(\ell)}$$

(II) $\vec{u} = \sum_\ell \beta_\ell \vec{e}^{(\ell)}$

(I), (II) $\boxed{\beta_\ell = \sum_k A_{\ell k}\alpha_k}$

$\vec{v} = \sum_k v_k \vec{e}^{(k)}$

$\vec{u} = \sum_\ell u_\ell \vec{e}^{(\ell)}$

$\vec{u} = \hat{A}\vec{v} \longrightarrow \boxed{u_\ell = \sum_k A_{\ell k} v_k}$

2017.09.26.

GY

$e^{ix} = \cos x + i \cdot \sin x$

$e^{-ix} = \cos x - i \cdot \sin x$

$\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

$\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$x \in \mathbb{R}$

$z = i \cdot x$

$$\cos z = \cos ix = \frac{e^{i(ix)} + e^{-i(ix)}}{2} = \frac{e^{-x} + e^{x}}{2}$$

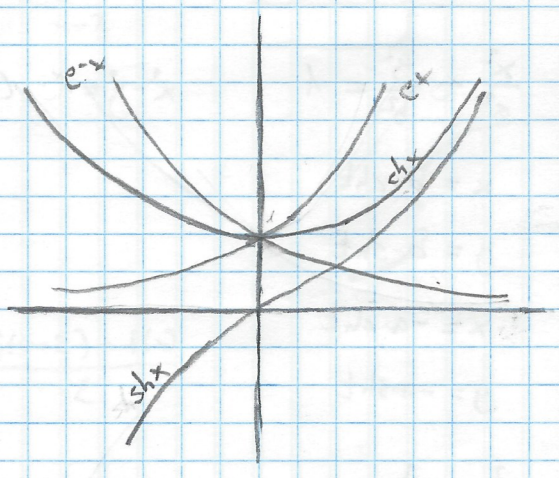
$$\sin z = \sin ix = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{e^{-x} - e^{x}}{2i}$$

$$ch\,x = \cosh x = \frac{e^{x} + e^{-x}}{2}$$

$$sh\,x = \sinh x = \frac{e^{x} - e^{-x}}{2}$$

$$th\,x = \tanh x = \frac{sh\,x}{ch\,x}$$

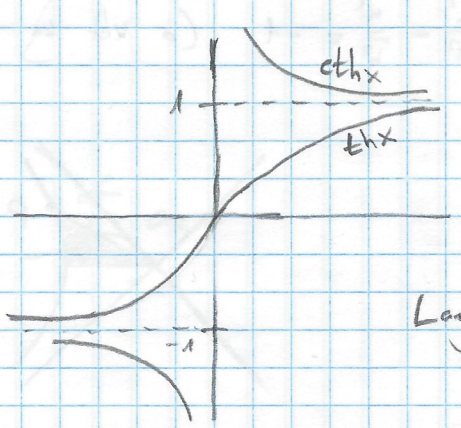
$$cth\,x = \coth x = \frac{ch\,x}{sh\,x}$$



$\cos ix = ch\,x \qquad ch\,ix = \cos x$

$\sin ix = i \cdot sh\,x \qquad sh\,ix = i \cdot \sin x$

$$\cos(3 - 5i) = \cos 3 \cdot \cos(-5i) - \sin 3 \cdot \sin(-5i) =$$
$$= \cos 3\, ch\, 5 + i \cdot \sin 3 \cdot sh\, 5$$



Laguerin

$$L(x) = cth\,x - \frac{1}{x}$$

$$\cos^2 x + \sin^2 x = 1 \qquad ch^2 x - sh^2 x = 1$$

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta \qquad ch(\alpha + \beta) = ch\alpha\, ch\beta + sh\alpha\, sh\beta$$

$$\sin(\alpha + \beta) = \sin\alpha \sin\beta + \cos\alpha \cos\beta \qquad sh(\alpha + \beta) = sh\alpha\, ch\beta + ch\alpha \cdot sh\beta$$

$$tg(\alpha + \beta) = \frac{tg\alpha + tg\beta}{1 - tg\alpha \cdot tg\beta} \qquad th(\alpha + \beta) = \frac{th\alpha + th\beta}{1 + th\alpha \cdot th\beta}$$

$$(\cos x)' = -\sin x \qquad (ch\,x)' = sh\,x$$

$$(\sin x)' = \cos x \qquad (sh\,x)' = \cos x$$

$$(tg\,x)' = \frac{1}{\cos^2 x} \qquad (th\,x)' = \frac{1}{ch^2 x}$$

$$(ctg\,x)' = -\frac{1}{\sin^2 x} \qquad (cth\,x)' = -\frac{1}{sh^2 x}$$

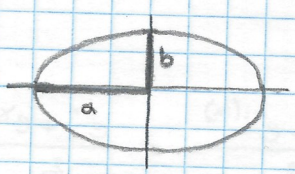
$x(t)$
$y(t)$

$x^2 + y^2 = r^2$

$x(t) = r \cdot \cos t$

$y(t) = r \cdot \sin t$

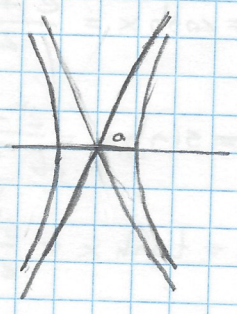
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$x(t) = a \cdot \cos t$

$y(t) = b \cdot \sin t$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\quad x^2 - y^2 = 1 \quad (a,b=1)$



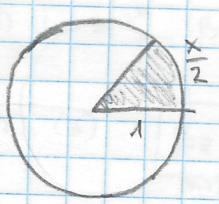
$x = a \cdot \operatorname{ch} t$

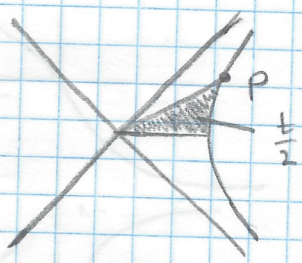
$y = a \cdot \operatorname{sh} t$

$x = -a \cdot \operatorname{ch} t$

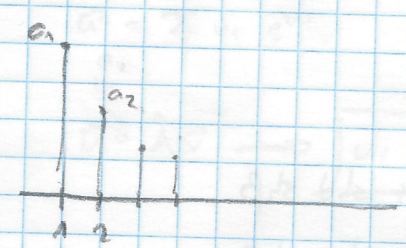
$y = -a \cdot \operatorname{sh} t$

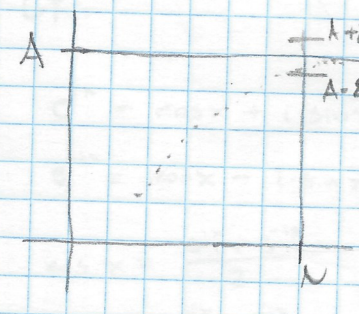
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ (a két ág az előzőek másik oldalán)





$P(\operatorname{ch} t, \operatorname{sh} t)$

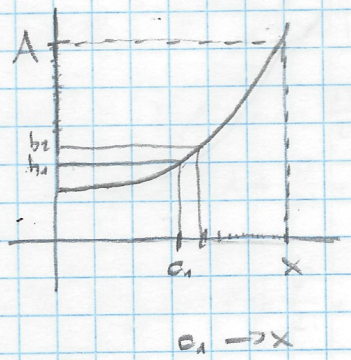




$\forall \varepsilon \quad \exists N \quad \forall n > N(\varepsilon) \quad a_n \in [A-\varepsilon, A+\varepsilon]$

$\exists A \quad \forall \varepsilon \quad \exists N \quad \forall n > N$

$a_n \xrightarrow[\infty]{} A \qquad \lim_{n \to \infty} a_n = A$



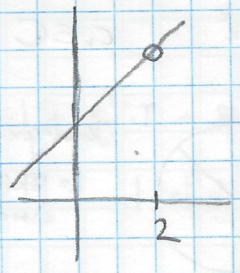
$a_1 \to x$

$b_k = f(a_k)$

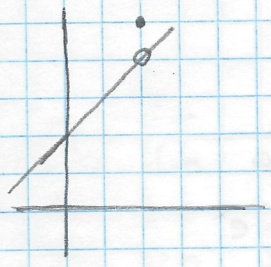
$a_n \to x$

$f(a_n) \to A$

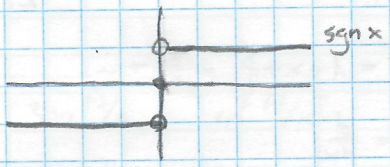
$\lim f(x) = A$

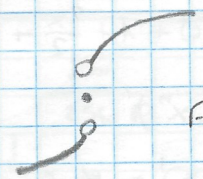


$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

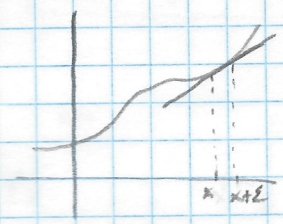


$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \neq 2 \\ 5 & x = 2 \end{cases}$$





Fourier



$$f'(x) = \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon) - f(x)}{\varepsilon}$$

Sorok → $1+2+3+\dots$

$1 + \frac{1}{2} + \frac{1}{3} + \dots$

$1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\sum_{k=1}^{\infty} a_k =$$

Függvénysorok. $(f+g)(x) = f(x) + g(x)$

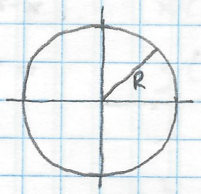
$$\sum_{k=1}^{n} f_k(x) \qquad \sum_{k=0}^{\infty} a_k x^k$$

$$\sum_k a_k \cdot \cos(kx) \qquad \text{Fourier-sorok}$$

$$1 + x + x^2 + \dots + x^n = \frac{1-x^{n+1}}{1-x} \quad \begin{cases} \to \infty & x > 1 \\ \to \frac{1}{1-x} & x < 1 \end{cases}$$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \dots = e^x$$

$\sum_{k=0}^{\infty} c_k z^k \qquad c_k \in \mathbb{C}$



Konvergenciakör

↳ $|z|<R$ - re a sor konvergens

$1 + 2z + 3z^2 + 4z^3 \dots$

$1 + z + z^2 + \dots z^n + \dots$

$R = 1$

$R = \infty$

$\exp(z) = e^z = 1 + z + \frac{z^2}{2!} + \dots + \frac{z^n}{n!} + \dots$

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$D = \frac{d}{dx}$

$De^x = 0 + 1 + \frac{2x}{2} + \frac{3x^2}{3 \cdot 2!} + \dots + \frac{n \cdot x^{n-1}}{n \cdot (n-1)!} + \dots = 1 + x + \frac{x^2}{2} + \dots = e^x$

$(\sum c_n x^n)'$

$(Ke^x)' = Ke^x$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \dots$

$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \dots$

$\operatorname{ch} x = \frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$

$\operatorname{sh} x = \frac{e^x - e^{-x}}{2} = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$

$e^{ix} = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \dots$

$e^{-ix} = 1 - ix - \frac{x^2}{2!} + i\frac{x^3}{3!} + \frac{x^4}{4!} - i\frac{x^5}{5!} - \dots$

$\cos x = \frac{e^{ix} + e^{-ix}}{2} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

$\sin x = \frac{e^{ix} - e^{-ix}}{2i} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

Algebra (struktúra): lineáris műveletek + számmal való szorzás + egymással való szorzás

pl. lineáris operátorok

$\alpha \in \mathbb{R} \quad T \quad T^2 = I$

$$e^{\alpha T} = I + \alpha T + \frac{(\alpha T)^2}{2!} + \frac{(\alpha T)^3}{3!} + \frac{(\alpha T)^4}{4!} + \dots = I + \alpha T + \frac{\alpha^2}{2!} \cdot I + \frac{\alpha^3}{3!} \cdot T + \frac{\alpha^4}{4!} \cdot I + \dots =$$

$$= I\left(1 + \frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} + \dots\right) + T\left(\alpha + \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} + \dots\right) = I \cdot \mathrm{ch}\,\alpha + T \cdot \mathrm{sh}\,\alpha$$

$$e^{P} = I + P + \frac{P^2}{2!} + \frac{P^3}{3!} + \dots = I + P\left(1 + \frac{1}{2!} + \frac{1}{3!} + \dots\right) = I + (e-1)P$$

$(P^2 = P)$

2017. 09. 28.

GY

$\xrightarrow{\text{nap}}$ $\xrightarrow{\text{árn}}$

2 kg	1 kg	0 kg
3 l	2 l	0 l
2 cs	0 cs	0 cs
3 s	5 s	10 s
0 M	0 M	1 M
0 s	-3 s	-5 s

$\downarrow$ árn

360 Ft/kg	200 Ft/l	360 Ft/cs	300 Ft/s	10^7 Ft/M
1 €/kg	1 €/l	0 €/cs	1 €/s	

$\downarrow$ ország

$\xrightarrow{\text{nap}}$

2540 Ft	2175 Ft	10002875 Ft
8 €	8 €	110 €

$\downarrow$ ország

$\xrightarrow{\text{árn}}$ $\xrightarrow{\text{ország}}$ $\xrightarrow{\text{ország}}$

$\downarrow$ nap $\quad \downarrow$ árn $\quad \downarrow$ nap

$A_{kl} \qquad B_{lm} \qquad C_{km}$

$k \in \{1,2,3\}, \quad k \in \{H, K, Sz\}$

$l \in \{1, \dots\}, \quad l \in \{\text{tej, kenyér, ...}\}$

$m \in \{1, \dots\}, \quad m \in \{HU, A, CN, \dots\}$

$$\sum_k A_{kl} B_{lm} = C_{km}$$

MÁTRIXOK $\quad \underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{C}}$

A_{kl}: mátrixelemek → lehetnek mátrixok is (hipermátrix)

Mátrixreprezentáció (izomorfizmus)

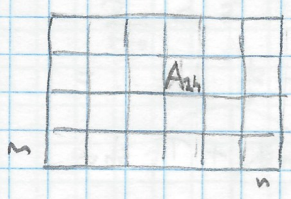
Sorvektor, oszlopvektor

$\square \cdot \square = \square$ zárt

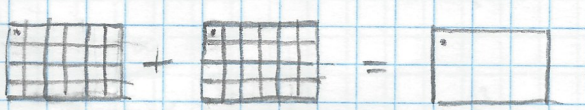
Kompatibilis mátrixok → össze lehet szorozni

$$\begin{array}{|cccc|} 1 & 2 & 3 & 0 \\ 4 & \cdot & \cdot & \cdot \\ 0 & & & \\ 0 & & & \end{array} \cdot \begin{array}{|cccc|} 7 & 7 & 7 & 0 \\ 5 & \cdot & \cdot & \\ 0 & \cdot & & \\ 0 & & & \end{array}$$

végtelen mátrixok: mind kompatibilis

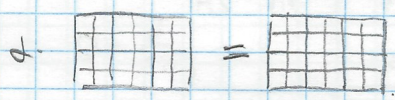


$\underline{\underline{A}}$ $(m \times n)$



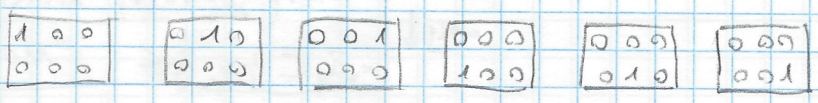
$\underline{\underline{A}} + \underline{\underline{B}} = \underline{\underline{C}}$

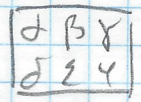
$C_{ke} = A_{ke} + B_{ke}$ $\quad k \in \{1, \dots, m\}$ $\quad e \in \{1, \dots, n\}$

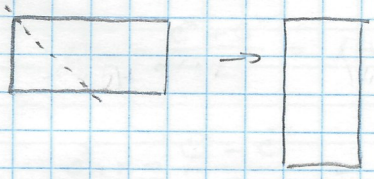


$\lambda \underline{\underline{A}} = \underline{\underline{D}}$

$D_{ke} = \lambda A_{ke}$



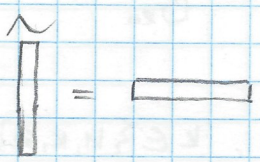




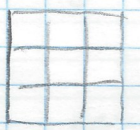
transzponálás

$\underline{\underline{\tilde{A}}}$ $\quad \underline{\underline{\tilde{\tilde{A}}}} = \underline{\underline{A}}$

$(\underline{\underline{\tilde{A}}})_{ke} = A_{ek}$



$\underline{\underline{A}}^T$, $\underline{\underline{A}}^*$

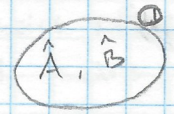


Négyzetes mátrixok: önmagával való szorzásra zárt
lineáris műveletekre is zárt
} → Algebra

2017.09.28.
EA

$V \ni \vec{v}$

$\hat{A} : V \to V$



$\hat{A}(\vec{u} + \vec{v}) = (\hat{A}\vec{u}) + (\hat{A}\vec{v})$

$\hat{A}(\lambda \vec{v}) = \lambda (\hat{A}\vec{v})$

$\hat{A}(\lambda \vec{v} + \beta \vec{u}) = \lambda (\hat{A}\vec{v}) + \beta (\hat{A}\vec{u})$

$+ : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$

$\hat{A} + \hat{B} = \hat{C}$

$\hat{C}\vec{v} = \hat{A}\vec{v} + \hat{B}\vec{v}$

$(\hat{A}+\hat{B})\vec{v} = (\hat{A}\vec{v}) + (\hat{B}\vec{v})$

$S \times \mathcal{O} \to \mathcal{O}$

$\lambda \hat{A} = \hat{D}$

$\hat{D}(\vec{v}) = \lambda(\hat{A}\vec{v})$

$(\lambda\hat{A})\vec{v} = \lambda \cdot (\hat{A}\vec{v})$

$\cdot : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$

$\hat{A}\hat{B} = \hat{F}$

$\hat{F}\vec{v} = \hat{A}(\hat{B}\vec{v})$

$\hat{C}(\hat{B}\hat{A}) = (\hat{C}\hat{B})\hat{A} \qquad \forall \hat{A}, \hat{B}, \hat{C} \in \mathcal{O}$

$\hat{A}\hat{B} \neq \hat{B}\hat{A}$ (nem általában)

$\lambda(\hat{A}+\hat{B}) \stackrel{?}{=} \lambda\hat{A} + \lambda\hat{B}$

$[\lambda(\hat{A}+\hat{B})]\vec{v} = \lambda(\hat{C}\vec{v}) = \lambda[(\hat{A}+\hat{B})\vec{v}] = \lambda[\hat{A}\vec{v} + \hat{B}\vec{v}] = \lambda(\hat{A}\vec{v}) + \lambda(\hat{B}\vec{v}) = (\lambda\hat{A})\vec{v} + (\lambda\hat{B})\vec{v} =$

$= [(\lambda\hat{A}) + (\lambda\hat{B})]\vec{v}$

$[\underbrace{(\lambda\hat{A})}_{\hat{D}}\hat{B}]\vec{v} = (\hat{D}\hat{B})\vec{v} = \hat{D}(\hat{B}\vec{v}) = \hat{D}\vec{u} = (\lambda\hat{A})\vec{u} = \lambda(\hat{A}\vec{u}) = \lambda(\hat{A}(\hat{B}\vec{v})) = \lambda(\hat{A}\hat{B})\vec{v}$

$[\underbrace{(\hat{A}+\hat{B})}_{\hat{C}}\hat{G}]\vec{v} = (\hat{C}\hat{G})\cdot\vec{v} = \hat{C}(\hat{G}\vec{v}) = \hat{C}\vec{u} = (\hat{A}+\hat{B})\vec{u} = \hat{A}\vec{u} + \hat{B}\vec{u} = \hat{A}(\hat{G}\vec{v}) + \hat{B}(\hat{G}\vec{v}) =$

$= (\hat{A}\hat{G})\vec{v} + (\hat{B}\hat{G})\vec{v} = (\hat{A}\hat{G} + \hat{B}\hat{G})\vec{v}$

$\begin{cases} \lambda(\hat{A}+\hat{B}) = \lambda\hat{A} + \lambda\hat{B} \\ (\lambda\hat{A})\hat{B} = \lambda(\hat{A}\hat{B}) \\ (\hat{A}+\hat{B})\hat{G} = \hat{A}\hat{G} + \hat{B}\hat{G} \end{cases}$

$\hat{A}(\lambda\hat{B}) = \lambda(\hat{A}\hat{B})$

$\hat{H}(\hat{A}+\hat{B}) = \hat{H}\hat{A} + \hat{H}\hat{B}$

$W \ni \vec{a}$

$\vec{a} + \vec{b}$

$\lambda \vec{a}$

* $W \times W \to W$

$a * b$

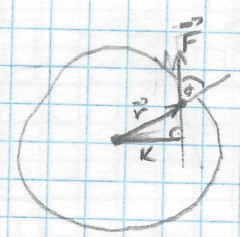
$(\lambda \vec{a}) * \vec{b} = \lambda (\vec{a} * \vec{b})$

$(\vec{a} + \vec{b}) * \vec{c} = \vec{a} * \vec{c} + \vec{b} * \vec{c}$

$(\lambda \vec{a} + \beta \vec{b}) * \vec{c} = \lambda (\vec{a} * \vec{c}) + \beta (\vec{b} * \vec{c})$

$\vec{c} * (\lambda \vec{a} + \beta \vec{b})$

Operátorok → (asszociatív) algebrát alkotnak



$M = K \cdot F$

$K = r \cdot \sin \alpha$

$M = r \cdot F \cdot \sin \alpha$

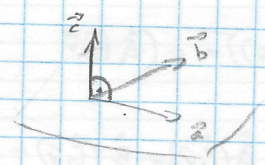
$\vec{M} = \vec{r} \times \vec{F}$

$\vec{L} = m \vec{r} \times \vec{v}$

$\vec{F} = q \vec{v} \times \vec{B}$

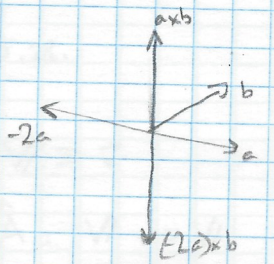
$\vec{a} \times \vec{b} = \vec{c}$ $\quad V \times V \to V$

$|\vec{z}| = |\vec{a}| |\vec{b}| \cdot \sin \alpha$



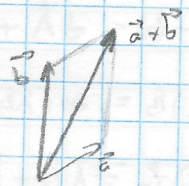
$\vec{b} \times \vec{a} = - \vec{a} \times \vec{b}$ antikommutatív

A 3 (vagy 7) dimenziós tér művelete



$(\beta \vec{a}) \times \vec{b} = \beta (\vec{a} \times \vec{b})$

$(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$



LIE - ALGEBRA : $(V, +, \times)$ ilyet alkot

$$\{\vec{e}^{(k)}\}_{k=1}^{n}$$

$$\vec{a} = \sum_k \alpha_k \vec{e}^{(k)}$$

$$\vec{b} = \sum_l \beta_l \vec{e}^{(l)}$$

$$\vec{c} = \vec{a} * \vec{b} = \left(\sum_k \alpha_k \vec{e}^{(k)}\right)\left(\sum_l \beta_l \vec{e}^{(l)}\right) = \sum_k \sum_l \alpha_k \beta_l \left(e^{(k)} * e^{(l)}\right)$$

$$\vec{e}^{(k)} * \vec{e}^{(l)} = \sum_m C_m^{(kl)} \vec{e}^{(m)} \qquad n^3 \text{ db } C$$

↳ nemkezdi állandó ⇒ önkényes

ha $a*b = -b*a$

$$C_m^{kl} = -C_m^{lk}$$

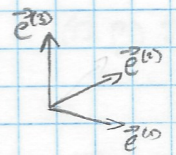
$$C_3^m = -C_3^m$$

$$C_3^{11} = -C_3^{11} = 0$$

$$\vec{c} = \sum_k \sum_l \alpha_k \beta_l \left(\sum_m C_m^{(kl)} \vec{e}^{(m)}\right) = \sum_k \sum_l \sum_m \alpha_k \beta_l C_m^{(kl)} \cdot \vec{e}^{(m)} =$$

$$= \sum_m \left(\sum_k \sum_l C_m^{(kl)} \alpha_k \beta_l\right) \vec{e}^{(m)} = \sum_m \gamma_m \vec{e}^{(m)}$$

$$\boxed{\gamma_m = \sum_k \sum_l C_m^{(kl)} \alpha_k \beta_l}$$



$$\vec{e}^{(k)} \vec{e}^{(l)} = \delta_{kl}$$

$$\vec{e}^{(1)} \times \vec{e}^{(2)} = \vec{0} = 0\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)}$$

$$\vec{e}^{(1)} \times \vec{e}^{(3)} = \vec{e}^{(2)} = 0\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 1\cdot\vec{e}^{(3)}$$

$$\vec{e}^{(2)} \times \vec{e}^{(3)} = -\vec{e}^{(1)} = 0\,\vec{e}^{(1)} - 1\,\vec{e}^{(2)} + 0\cdot\vec{e}^{(3)}$$

$$\vec{e}^{(2)} \times \vec{e}^{(1)} = -\vec{e}^{(3)} = 0\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} - 1\cdot\vec{e}^{(3)}$$

$$\vec{e}^{(3)} \times \vec{e}^{(2)} = \vec{0} = 0\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)}$$

$$\vec{e}^{(3)} \times \vec{e}^{(1)} = \vec{e}^{(2)} = 1\cdot\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 0\cdot\vec{e}^{(3)}$$

$$\vec{e}^{(3)} \times \vec{e}^{(3)} = \vec{e}^{(1)} = 0\,\vec{e}^{(1)} + 1\cdot\vec{e}^{(2)} + 0\cdot\vec{e}^{(3)}$$

$$\vec{e}^{(1)} \times \vec{e}^{(2)} = -\vec{e}^{(3)} = -1\cdot\vec{e}^{(1)} + 0\cdot\vec{e}^{(2)} + 0\cdot\vec{e}^{(3)}$$

$$\vec{e}^{(3)} \times \vec{e}^{(3)} = \vec{0} = 0\,\vec{e}^{(1)} + 0\,\vec{e}^{(2)} + 0\,\vec{e}^{(3)}$$

Levi-Civita-szimbólum

$$\vec{e}^{(k)} \times \vec{e}^{(l)} = \sum_m \varepsilon_{klm} \vec{e}^{(m)}$$

$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$$

$$\text{többi} = 0$$

$$\gamma_m = \sum_k \sum_l \varepsilon_{klm}\, \alpha_k \beta_l$$

$$\vec{a} \rightarrow \underline{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$\vec{b} \rightarrow \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$$

$$\vec{c} \rightarrow \underline{c} = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

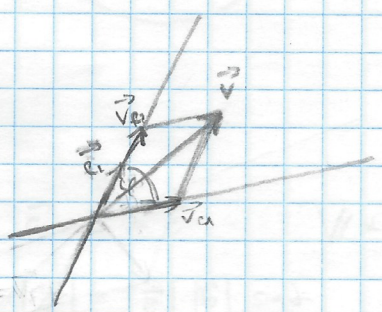
$$c_m = \sum_k \sum_l \varepsilon_{klm}\, a_k b_l$$

$$c_1 = \varepsilon_{11} a_1 b_1 + \varepsilon_{12} a_1 b_2 + \varepsilon_{13} a_1 b_3 + \varepsilon_{21} a_2 b_1 + \varepsilon_{22} a_2 b_2 + \varepsilon_{23} a_2 b_3 + \varepsilon_{31} a_3 b_1 + \varepsilon_{32} a_3 b_2 + \varepsilon_{33} a_3 b_3$$

$$c_1 = \varepsilon_{231} a_2 b_3 + \varepsilon_{321} a_3 b_2 = a_2 b_3 - a_3 b_2$$

$$\boxed{\begin{aligned} c_1 &= a_2 b_3 - a_3 b_2 \\ c_2 &= a_3 b_1 - a_1 b_3 \\ c_3 &= a_1 b_2 - a_2 b_1 \end{aligned}}$$

HF. Ferde vetítés



$$|\vec{e}_1| = |\vec{e}_2| = 1$$

$$\vec{v} = \vec{v}_{e_1} + \vec{v}_{e_2}$$

$$\vec{v}' = \vec{v}_{e_1} - \vec{v}_{e_2} = \vec{v}_{e_1} - (\vec{v} - \vec{v}_{e_1}) = 2\vec{v}_{e_1} - \vec{v}$$

$$\vec{v}_{e_1} = \alpha \vec{e}_1 \qquad \vec{v}_{e_2} = \beta \cdot \vec{e}_2$$

$$\alpha \vec{e}_1 + \beta \vec{e}_2 = \vec{v} \quad / \cdot \vec{e}_1 \,,\, \cdot \vec{e}_2$$

(1) $\alpha + \beta(\vec{e}_1 \vec{e}_2) = \vec{v}\vec{e}_1$

(2) $\alpha(\vec{e}_1 \vec{e}_2) + \beta = \vec{v}\vec{e}_2$

$$\alpha + \cos\varphi \cdot \beta = \vec{v}\vec{e}_1$$

$$\cos\varphi \cdot \alpha + \beta = \vec{v}\vec{e}_2$$

$$\cos^2\varphi \cdot \alpha + \beta \cdot \cos\varphi = (\vec{v}\vec{e}_2) \cdot \cos\varphi$$

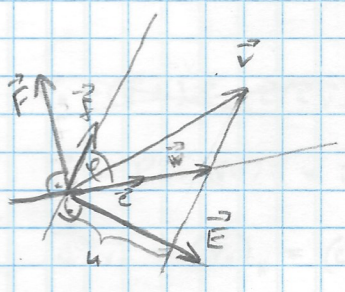
$$\alpha(1 - \cos^2\varphi) = \vec{v}\vec{e}_1 - \vec{v}\vec{e}_2 \cdot \cos\varphi$$

$$\alpha = \frac{\vec{v}\vec{e}_1 - \vec{v}\vec{e}_2 \cdot \cos\varphi}{\sin^2\varphi} \qquad \beta = \frac{\vec{v}\vec{e}_2 - \vec{v}\vec{e}_1 \cdot \cos\varphi}{\sin^2\varphi}$$

$$\cos\varphi = \vec{e}_1 \vec{e}_2$$

$$\sin^2\varphi = 1 - (\vec{e}_1 \vec{e}_2)^2$$

2017.10.05.
GY.



$$\vec{v}\vec{E} = |v||E|\cdot\cos\varphi = |E|\cdot u$$

$$u = \frac{\vec{v}\vec{E}}{|\vec{E}|}$$

$$|\vec{w}| = \frac{u}{\sin\varphi} = \frac{\vec{v}\vec{E}}{|\vec{e}|\cdot\sin\varphi}$$

$$\vec{w} = \frac{\vec{e}}{|\vec{e}|}\cdot|w| = \frac{\vec{e}(\vec{v}\vec{E})}{|\vec{e}|\cdot|\vec{E}|\cdot\sin\varphi} = \frac{\vec{e}(\vec{v}\vec{E})}{(\vec{e}\cdot\vec{E})} = \hat{P}\vec{v}$$

Legyen $|\vec{E}| = \frac{1}{|\vec{e}|\cdot\sin\varphi}$

$$\vec{E}\vec{e} = |\vec{E}||\vec{e}|\cdot\cos\left(\tfrac{\pi}{2}-\varphi\right) = |\vec{E}|\cdot|\vec{e}|\cdot\sin\varphi = 1$$

$\vec{E}\vec{f} = 0$	$\vec{F}\vec{e} = 0$
$\vec{E}\vec{e} = 1$	$\vec{F}\vec{f} = 1$

$\Rightarrow$ $\vec{E}, \vec{F}$ és $\vec{e}, \vec{f}$ reciprokbázisa

$$\vec{w} = \vec{e}(\vec{v}\vec{E}) = P\vec{v}$$

$$\vec{z} = \vec{f}(\vec{v}\vec{F}) = Q\vec{v}$$

$$\vec{w} + \vec{z} = \vec{v}$$

$$P^2\vec{v} = P(P\vec{v}) = P\vec{w} = \vec{e}(\vec{w}\vec{E}) = \vec{e}\left[\vec{e}(\vec{v}\vec{e})\cdot\vec{E}\right] = \vec{e}\cdot(\vec{e}\vec{E})\cdot(\vec{v}\vec{E}) = \vec{e}(\vec{v}\vec{E}) = \vec{w} = P\vec{v}$$

$$P^2 = P, \quad Q^2 = Q$$

$$(QP)\vec{v} = Q(P\vec{v}) = Q\vec{w} = \vec{f}(\vec{w}\vec{F}) = \vec{f}\left[\vec{e}(\vec{v}\vec{E})\vec{F}\right] = \vec{f}\cdot(\vec{v}\vec{E})\cdot(\underbrace{\vec{e}\vec{F}}_{0}) = \vec{0}$$

$\vec{E}^{(1)}\vec{e}^{(1)} = 0$	$\vec{E}^{(1)}\vec{e}^{(2)} = 0$
$\vec{E}^{(2)}\vec{e}^{(1)} = 1$	$\vec{E}^{(2)}\vec{e}^{(2)} = 1$

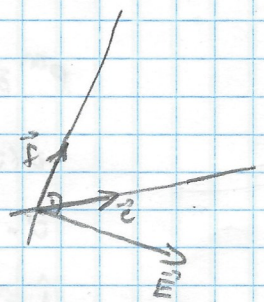
$\Rightarrow$ $\boxed{\vec{E}^{(k)}\vec{e}^{(l)} = \delta_{kl}}$

$\vec{e}^{(k)}$ $k = 1, \dots, n$

$\vec{E}^{(k)}$ $\vec{E}^{(k)}\vec{e}^{(l)} = \delta_{kl}$

$$\vec{v} = \sum_k \lambda_k \vec{e}^{(k)}$$

$$\lambda_k = \vec{v}\vec{E}^{(k)}$$



$$\vec{E} = \gamma\vec{e} + \delta\vec{f}$$

$$\vec{E}\vec{f} = \gamma\vec{e}\cdot\vec{f} + \delta\vec{f}\vec{f} = 0$$

$$\vec{E}\vec{e} = \gamma\vec{e}\vec{e} + \delta\vec{F}\cdot\vec{e} = 1$$

$\downarrow$

γ, δ

$\vec{a}\vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\varphi$

ONB

$\vec{e}^{(k)}\vec{e}^{(l)} = \delta_{kl}$

$\vec{a} = \sum_k a_k \vec{e}^{(k)}$

$\vec{a}\vec{e}^{(l)} = \left(\sum_k a_k \vec{e}^{(k)}\right)\cdot\vec{e}^{(l)} = \sum_k (a_k \vec{e}^{(k)})\cdot\vec{e}^{(l)} = \sum_k a_k (\vec{e}^{(k)}\vec{e}^{(l)}) = \sum_k a_k \delta_{kl}$

$b_l = \sum_k a_k \delta_{kl}$

$b_1 = \sum_k a_k \delta_{k1} = a_1 \underbrace{\delta_{11}}_{1} + a_2 \underbrace{\delta_{21}}_{0} + a_3 \underbrace{\delta_{31}}_{0} = a_1$

$b_2 = \sum_k a_k \delta_{k2} = a_1 \delta_{12} + a_2 \delta_{22} + a_3 \delta_{32} = a_2$

$\sum_k a_k \cdot \delta_{kl} = a_l$

$$\boxed{\begin{aligned} \vec{a} &= \sum_k a_k \vec{e}^{(k)} \\ a_k &= \vec{a}\cdot\vec{e}^{(k)} \end{aligned}}$$

NONB

$\vec{e}^{(k)}\vec{E}^{(l)} = \delta_{kl}$

$\vec{a} = \sum_k a_k \vec{e}^{(k)}$

$\vec{a}\vec{E}^{(l)} = \left(\sum_k a_k \vec{e}^{(k)}\right)E^{(l)} = \sum_k a_k \delta_{kl} = a_l$

$a_k = \vec{a}\cdot\vec{E}^{(k)}$

$$\boxed{\begin{aligned} \vec{a} &= \sum_k a_k \vec{e}^{(k)} \\ a_k &= \vec{a}\vec{E}^{(k)} \end{aligned}}$$

$\vec{a} = \sum_k a_k \vec{e}^{(k)} = \sum_k (\vec{a}\cdot\vec{E}^{(k)})\cdot\vec{e}^{(k)} = \sum_k \vec{e}^{(k)}\cdot(\vec{a}\vec{E}^{(k)}) = \sum_k \vec{e}^{(k)}(\vec{E}^{(k)}\vec{a})$

$P^{(k)}\vec{a} = \vec{e}^{(k)}(\vec{E}^{(k)}\vec{a})$

$\vec{c}(\vec{b}\vec{z}) = (\vec{c}\circ\vec{b})\vec{z}$ diadikus szorzás

$\vec{a}\circ\vec{b} = \hat{A}$

$\hat{A}\vec{z} = \vec{a}(\vec{b}\vec{z})$

$\hat{P}^{(k)} = \vec{e}^{(k)}\circ\vec{E}^{(k)}$

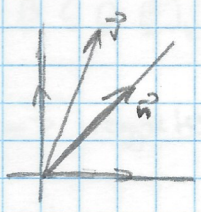
$\sum \hat{P}^{(k)} = \hat{1}$

ONB $\vec{e}^{(k)} \cdot \vec{e}^{(l)} = \delta_{kl}$

~~$\vec{v}_{\parallel} = (\vec{v}\vec{e})\vec{e} = \vec{e}(\vec{e}\vec{v}) = (\vec{e}\circ\vec{e})\vec{v}$~~

$\vec{v}_{\parallel} = P = (\vec{v}\vec{n})\vec{n} = \vec{n}(\vec{n}\vec{v}) = (\vec{n}\circ\vec{n})\vec{v}$

$\vec{e}^{(k)}$/ $\vec{e}^{(k)}\vec{v}_{\parallel} = (v_{\parallel})_k = (\vec{e}^{(k)}\vec{n})(\vec{n}\vec{v})$



$$\vec{a}\vec{b} = \left(\sum_k a_k \vec{e}^{(k)}\right)\left(\sum_l b_l \vec{e}^{(l)}\right) = \sum_k \sum_l a_k b_l \left(\vec{e}^{(k)}\vec{e}^{(l)}\right) =$$

$$= \sum_k a_k \left(\sum_l b_l \delta_{kl}\right) = \sum_k a_k b_k = \underline{a}\,\underline{b}$$

$$\vec{e}^{(k)}\vec{v}_{\parallel} = n_k(\vec{n}\vec{v}) = n_k(\underline{n}\,\underline{v}) = n_k \sum_l n_l v_l = \sum_l \underbrace{n_k n_l}_{P_{kl}} v_l = \sum_l P_{kl} v_l = (v_{\parallel})_k$$

$(\underline{n}\circ\underline{n})_{kl} = n_k n_l$

$\hat{A} = \vec{a}\circ\vec{b}$

$A = \underline{a}\circ\underline{b}$ $\qquad A_{kl} = a_k b_l$ $\qquad \sum_l A_{kl} B_{lm} = C_{km}$

$\sum_l A_{kl} B_l = C_k$

$\vec{v}' = 2(\vec{v}\vec{e})\vec{e} - \vec{v}$

$\underline{v}' = 2(\underline{v}\,\underline{e})\,\underline{e} - \underline{v}$

$v_k' = 2(\underline{v}\,\underline{e})e_k - v_k = 2\left(\sum_l v_l e_l\right)e_k - v_k = \sum_l 2e_k e_l v_l - v_k = \sum_l 2e_k e_l v_l - \sum_l \delta_{kl} v_l =$

$= \sum_l \underbrace{(2e_k e_l - \delta_{kl})}_{T_{kl}} v_l$

$\vec{v}' = \hat{T}\vec{v}$

$\underline{v}' = \underline{\underline{T}}\,\underline{v}$

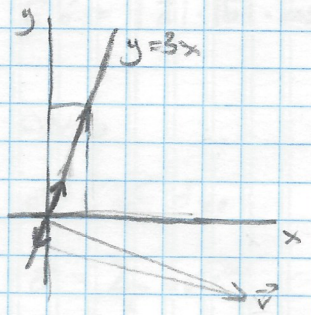
$v_k' = \sum_l T_{kl} v_l$

| □ |

$\hat{T} = 2\hat{P} - \hat{1}$

$\underline{\underline{T}} = 2\underline{\underline{P}} - \underline{\underline{1}}$

1



$\underline{n} = \begin{pmatrix}1\\3\end{pmatrix} \cdot \frac{1}{\sqrt{10}}$

$|\underline{e}|^2 = \underline{e}\,\underline{e} = \sum_k e_k^2$

$P_{kl} = n_k n_l$

$\underline{\underline{P}} = \begin{bmatrix} \square \end{bmatrix} = \frac{1}{10}\begin{pmatrix}1\\3\end{pmatrix}\begin{pmatrix}1 & 3\end{pmatrix} = \frac{1}{10}\cdot\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}$

$\underline{v}_{\parallel} = \underline{\underline{P}}\,\underline{v} = \frac{1}{10}\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-1\\-3\end{pmatrix}$

$\underline{\underline{T}} = 2\underline{\underline{P}} - \underline{\underline{1}}$

$\underline{\underline{T}} = \frac{2}{10}\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix} - \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{5}\left[\begin{pmatrix}1 & 3\\ 3 & 9\end{pmatrix} - \begin{pmatrix}5 & 0\\ 0 & 5\end{pmatrix}\right] = \frac{1}{5}\begin{pmatrix}-4 & 3\\ 3 & 4\end{pmatrix}$

$\underline{v}' = \underline{\underline{T}}\,\underline{v} = \frac{1}{5}\begin{pmatrix}-4 & 3\\ 3 & 4\end{pmatrix}\begin{pmatrix}5\\-2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-26\\7\end{pmatrix}$

HF. Gyakorolni → vetítés + fordé vektés + tükrözés

2017. 10. 05.

EA

$V \quad \hat{A} \quad V \to V$

$\vec{u}, \vec{v} \in V \quad \begin{cases} \hat{A}(\vec{v}+\vec{c}) = \hat{A}\vec{v} + \hat{A}\vec{c} \\ \hat{A}(\lambda\vec{v}) = \lambda(\hat{A}\vec{v}) \end{cases}$

$\hat{A}(\lambda\vec{c} + \beta\vec{v}) = \lambda(\hat{A}\vec{u}) + \beta(\hat{A}\vec{v})$

$\mathbb{O}_V \ni \hat{A} \qquad \mathbb{O}\times\mathbb{O}\to\mathbb{O}$

$\hat{A} + \hat{B} = \hat{C}$

$\hat{C}\vec{v} = (\hat{A}+\hat{B})\vec{v}$

$S\times\mathbb{O}\to\mathbb{O}$

$\lambda\hat{A} = \hat{D}$

$\hat{D}\vec{v} = (\lambda\hat{A})\vec{v} = \lambda(\hat{A}\vec{v})$

$\mathbb{O}\times\mathbb{O}\to\mathbb{O}$

$\hat{A}\hat{B} = \hat{F}$

$\hat{F}\vec{v} = (\hat{A}\hat{B})\vec{v} = \hat{A}(\hat{B}\vec{v})$

$\lambda(\hat{A}+\hat{B}) = (\lambda\hat{A}) + (\lambda\hat{B})$

$(\hat{A}+\hat{B})\hat{C} = (\hat{A}\hat{C}) + (\hat{B}\hat{C})$

$(\lambda\hat{A})\hat{B} = \lambda(\hat{A}\hat{B})$

$(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C})$

⇒ asszociatív algebrát alkotnak

$\underline{\underline{D}} = \underline{\underline{A}}\,\underline{\underline{B}}$

$D_{kl} = \sum_m A_{km} B_{ml}$

$\underline{\underline{E}} = \underline{\underline{B}}\cdot\underline{\underline{C}} \qquad \underline{\underline{F}} = \underline{\underline{D}}\,\underline{\underline{C}}$

$\underline{\underline{G}} = \underline{\underline{A}}\cdot\underline{\underline{E}}$

} $\underline{A/II.}$ $\underline{\underline{F}} = \underline{\underline{G}}$

asszociativitás

Biz.:

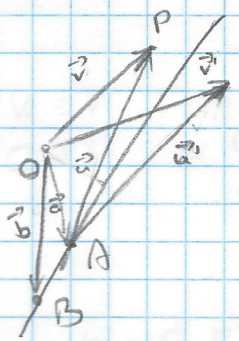
$$F_{kl} = (\underline{\underline{D}}\,\underline{\underline{C}})_{kl} = \sum_m D_{km} C_{ml} = \sum_m \left(\sum_p A_{kp} \cdot B_{pm}\right) C_{ml} = \sum_m \sum_p A_{kp} \cdot B_{pm} C_{ml} =$$

$$= \sum_p A_{kp} \sum_m B_{pm} C_{ml} = \sum_p A_{kp} (\underline{\underline{B}}\,\underline{\underline{C}})_{pl} = \sum_p A_{kp} \cdot E_{pl} = (\underline{\underline{A}}\,\underline{\underline{E}})_{kl} = G_{kl}$$

$$\Rightarrow (\underline{\underline{A}}\,\underline{\underline{B}})\,\underline{\underline{C}} = \underline{\underline{A}}\,(\underline{\underline{B}}\,\underline{\underline{C}}) .$$

$$(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}}\,\underline{\underline{E}})_{kl} = \sum_m \sum_p \sum_s \sum_t A_{km} B_{mp} C_{ps} D_{st} E_{tl}$$

$\Rightarrow$ négyzetes Mátrixok is asszociatív algebrát alkotnak



$$\vec{v}' = \vec{a} + \vec{a}' = \vec{a} + T\vec{a} = \vec{a} + 2(\vec{u}\vec{c})\vec{e} - \vec{u} = \vec{a} + (2(\vec{v}-\vec{c})\vec{e})\vec{e} - (\vec{v}-\vec{a}) =$$

$$= \vec{a} + 2(\vec{v}\vec{e} - \vec{a}\vec{e})\vec{e} - \vec{v} + \vec{a} = \underbrace{[2\vec{a} - 2(\vec{a}\vec{e})\vec{e}]}_{\vec{c}} + \underbrace{[2(\vec{v}\vec{e})\vec{e} - \vec{v}]}_{T\vec{v}}$$

$$\boxed{\vec{v}' = T\vec{v} + \vec{c}}$$

$$\vec{c}(\vec{a}) = 2\vec{a} - 2(\vec{a}\vec{c})\vec{c}$$

$$\vec{c}(\vec{b}) = 2\vec{b} - 2(\vec{b}\vec{c})\vec{c} = 2(\vec{a} + \lambda\vec{e}) - 2(\vec{a}\vec{c} + \lambda\vec{c}\vec{c})\vec{c} =$$

$$= 2\vec{a} - 2(\vec{a}\vec{c})\vec{c} + \underbrace{2\lambda\vec{e} - 2\lambda\cdot(\underbrace{\vec{c}\vec{c}}_{1})\vec{c}}_{0} = \vec{c}(\vec{a})$$

$$\vec{c} = 2[\vec{a} - (\vec{a}\vec{e})\vec{e}] = 2(\vec{a} - P\vec{a}) = 2(1-P)\vec{a} = Q\vec{a}$$

V $\exists \{\vec{e}^{(k)}\}_{k=1}^{n}$

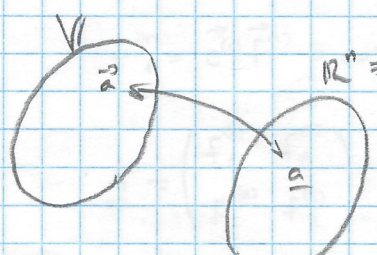
$$\vec{a} = \sum_k a_k \vec{e}^{(k)}$$

$$\vec{a}\vec{e}^{(l)} = \sum_k a_k (\vec{e}^{(k)}\vec{e}^{(l)}) = \sum_k a_k \delta_{kl} = a_l$$

$$a_k = \vec{a}\,\vec{e}^{(k)}$$

$\alpha \in B$

$\vec{e}^{(k)}\vec{e}^{(l)} = \delta_{kl}$



$$\mathbb{R}^n \ni \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = \underline{a}$$

$A \times B \ni (a,b)$ $a \in A$, $b \in B$

$A \times A \ni (a_1, a_2)$

$A \times A \times \ldots \times A \ni \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$

$\vec{c} = \vec{a} + \vec{b} \rightarrow (\underline{a} + \underline{b})_k = a_k + b_k = c_k$

$\lambda\vec{a} \rightarrow (\lambda\underline{a})_k = \lambda a_k$

$$\vec{a}\vec{b} = \left(\sum_k a_k \vec{e}^{(k)}\right)\left(\sum_\ell b_\ell \vec{e}^{(\ell)}\right) = \sum_k \sum_\ell a_k b_\ell \left(\vec{e}^{(k)}\vec{e}^{(\ell)}\right) = \sum_k \sum_\ell a_k b_\ell \delta_{k\ell} = \sum_k a_k b_k = \underline{a}\,\underline{b}$$

$$\vec{a}\times\vec{b} = \left(\sum_k a_k \vec{e}^{(k)}\right)\left(\sum_\ell b_\ell \vec{e}^{(\ell)}\right) = \sum_k \sum_\ell a_k b_\ell \underbrace{\left(\vec{e}^{(k)}\times\vec{e}^{(\ell)}\right)}_{\sum_m \varepsilon_{k\ell m}\vec{e}^{(m)}} = \sum_k \sum_\ell a_k b_\ell \sum_m \varepsilon_{k\ell m}\vec{e}^{(m)} =$$

$$= \sum_m \left(\sum_k \sum_\ell a_k b_\ell \varepsilon_{k\ell m}\right)\vec{e}^{(m)} = \sum_m c_m \vec{e}^{(m)}$$

$$\left(\begin{array}{l} \vec{c} = \vec{a}\times\vec{b} \\ \underline{c} = \underline{a}\times\underline{b} \end{array}\right.$$

$$c_m = \sum_k \sum_\ell \varepsilon_{k\ell m} a_k b_\ell = \sum_k \sum_\ell \varepsilon_{m k \ell} a_k b_\ell$$

$$\hat{C} = \vec{a}\circ\vec{b}$$

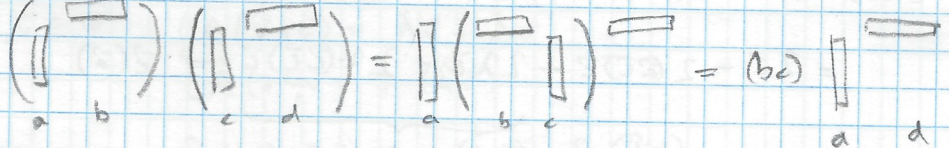
$$\vec{u} = \hat{C}\vec{v} = (\vec{a}\circ\vec{b})\vec{v} = \vec{a}(\vec{b}\vec{v})$$

$$\underline{u} = \underline{\underline{C}}\,\underline{v} = (\underline{a}\circ\underline{b})\underline{v} = \underline{a}(\underline{b}\,\underline{v})$$

$$u_k = a_k(\underline{b}\,\underline{v}) = a_k \sum_\ell b_\ell v_\ell = \sum_\ell \underbrace{(a_k b_\ell)}_{C_{k\ell}} v_\ell \quad \rightarrow \quad (\underline{a}\circ\underline{b})_{k\ell} = a_k b_\ell$$

$$(\vec{a}\circ\vec{b})(\vec{c}\circ\vec{d}) = (\vec{b}\vec{c})(\vec{a}\circ\vec{d})$$

$$(\underline{a}\circ\underline{b})(\underline{c}\circ\underline{d}) = (\underline{b}\,\underline{c})(\underline{a}\circ\underline{d})$$



$$P\vec{v} = (\vec{v}\vec{c})\vec{c} = \vec{c}(\vec{c}\vec{v}) = (\vec{c}\circ\vec{c})\vec{v}$$

$$P = \vec{c}\circ\vec{c}$$

$$Q = \vec{f}\circ\vec{f}$$

$$P^2 = (\vec{c}\circ\vec{c})(\vec{c}\circ\vec{c}) = (\vec{c}\vec{c})(\vec{c}\circ\vec{c}) = \vec{c}\circ\vec{c} = P$$

$$PQ = (\vec{c}\circ\vec{c})(\vec{f}\circ\vec{f}) = \underbrace{(\vec{c}\vec{f})}_{0}(\vec{c}\circ\vec{f}) = 0$$

NONB

$V \Rightarrow \vec{e}^{(k)}$

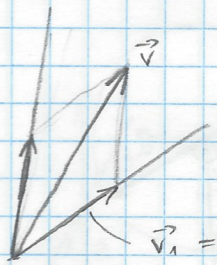
$\vec{E}^{(k)} \quad \vec{E}^{(k)}\cdot\vec{e}^{(\ell)} = \delta_{k\ell}$

$$\boxed{\vec{a} = \sum_k a_k \vec{e}^{(k)}} \quad / \vec{E}^{(k)}$$

$$\vec{a}\vec{E}^{(k)} = \sum_k a_k \vec{e}^{(k)}\vec{E}^{(\ell)} = \sum_k a_k \delta_{k\ell} = a_\ell \rightarrow \boxed{a_k = \vec{a}\vec{E}^{(k)}}$$

$$\vec{a}\vec{b} = \left(\sum_k a_k \vec{e}^{(k)}\right)\left(\sum_l b_l \vec{e}^{(l)}\right) = \sum_k \sum_l a_k b_l \underbrace{\left(\vec{e}^{(k)}\vec{e}^{(l)}\right)}_{g_{kl}} = \sum_k \sum_l g_{kl}\, a_k b_l$$

$$g_{kl} = g_{lk}$$



$$\vec{v}_1 = P^{(1)}\vec{v} = \vec{e}^{(1)}\left(\vec{E}^{(1)}\vec{v}\right) = \left(\vec{e}^{(1)} \circ \vec{E}^{(1)}\right)\vec{v}$$

$$P^{(1)} = \vec{e}^{(1)} \circ \vec{E}^{(1)}$$

$$P^{(1)}P^{(1)} = \left(\vec{e}^{(1)} \circ \vec{E}^{(1)}\right)\left(\vec{e}^{(1)} \circ \vec{E}^{(1)}\right) = \left(\underbrace{\vec{E}^{(1)}\vec{e}^{(1)}}_{1}\right)\left(\vec{e}^{(1)} \circ \vec{E}^{(1)}\right) = P^{(1)}$$

$$P^{(1)}P^{(2)} = \left(\vec{e}^{(1)} \circ \vec{E}^{(1)}\right)\cdot\left(\vec{e}^{(2)} \circ \vec{E}^{(2)}\right) = \left(\underbrace{\vec{E}^{(1)}\vec{e}^{(2)}}_{0}\right)\left(\vec{e}^{(2)} \circ \vec{E}^{(2)}\right) = \hat{O}$$

$\forall\, \vec{v}$

$\exists\, \hat{A}$

$$\vec{u} = \hat{A}\vec{v}$$

$$\underline{u} = \underline{\underline{A}}\,\underline{v}$$

$$u_k = \sum_l A_{kl} v_l$$

$$\| = \square\cdot\|$$

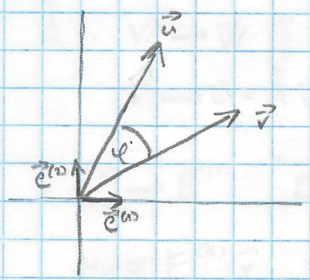
$$u_k = \vec{u}\cdot\vec{E}^{(k)} = \vec{E}^{(k)}\cdot\vec{u} = \vec{E}^{(k)}\left(\hat{A}\vec{v}\right) = \vec{E}^{(k)}\left(\hat{A}\hat{I}\vec{v}\right) = \vec{E}^{(k)}\hat{A}\cdot\sum_l P^{(l)}\vec{v} =$$

$$= \sum_l \vec{E}^{(k)}\hat{A}\left(\vec{e}^{(l)} \circ \vec{E}^{(l)}\right)\cdot\vec{v} = \sum_l \vec{E}^{(k)}\cdot\hat{A}\,\vec{e}^{(l)}\left(\vec{E}^{(l)}\cdot\vec{v}\right) = \sum_l \underbrace{\left(\vec{E}^{(k)}\hat{A}\vec{e}^{(l)}\right)}_{A_{kl}} \cdot v_l$$

$$\boxed{A_{kl} = \vec{E}^{(k)}\hat{A}\vec{e}^{(l)}}$$

$$\vec{u} = \hat{A}\vec{v} = \hat{A}\sum_k v_k \cdot \vec{e}^{(k)} = \sum_k v_k\left(\hat{A}\vec{e}^{(k)}\right) = \sum_k v_k \vec{f}^{(k)}$$

$$u_k = \vec{E}^{(k)}\vec{u} = \vec{E}^{(k)}\left(\sum_l v_l f^{(l)}\right) = \sum_l v_l\left(\vec{E}^{(k)} f^{(l)}\right) = \sum_l v_l f_k^{(l)}$$



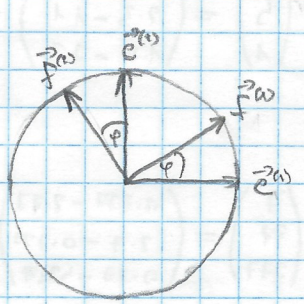
$$\vec{u} = \hat{F}\vec{v}$$

$$\hat{F}\vec{e}^{(1)} = \vec{f}^{(1)}$$

$$f_1^{(1)} = \vec{e}^{(1)}\cdot\vec{f}^{(1)} = \cos\varphi$$

$$f_2^{(1)} = \vec{e}^{(2)}\vec{f}^{(1)} = \sin\varphi$$

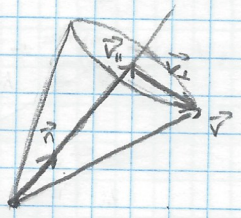
$$f_1^{(2)} = \vec{e}^{(1)}\vec{f}^{(2)} = \cos\left(\tfrac{\pi}{2} + \varphi\right) = \cos\tfrac{\pi}{2}\cdot\cos\varphi - \sin\tfrac{\pi}{2}\cdot\sin\varphi = -\sin\varphi$$



$$\underline{\underline{A}} = \begin{pmatrix} f_1^{(1)} & f_2^{(1)} \\ f_2^{(1)} & f_2^{(2)} \end{pmatrix} \qquad \hat{F} \to \underline{\underline{F}} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longrightarrow \underline{u} = \underline{\underline{F}}\,\underline{v} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\varphi \cdot v_1 - \sin\varphi \cdot v_2 \\ \sin\varphi \cdot v_1 + \cos\varphi \cdot v_2 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

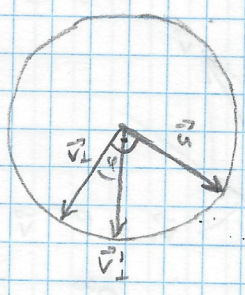
Tétel:



$\vec{n}\vec{n} = 1$

$\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}$ $\qquad \vec{v}_{\parallel} = \vec{n}(\vec{n}\vec{v})$

$\vec{v}' = \vec{v}_{\parallel} + \vec{v}_{\perp}'$



$\vec{u} \perp \vec{v}_{\perp}$

$\vec{u} \perp \vec{n}$

$\vec{v}_{\perp}' = \cos\varphi\, \vec{v}_{\perp} + \sin\varphi \cdot \vec{u}$

$\vec{u} = \vec{n} \times \vec{v}_{\perp}$, $|\vec{u}| = |\vec{n}| \cdot |\vec{v}_{\perp}| \cdot \sin\frac{\pi}{2} = |v_{\perp}|$

$\vec{u} = \vec{n} \times \vec{v}_{\perp} = \vec{n} \times (\vec{v} - \vec{v}_{\parallel}) = \vec{n} \times \vec{v} - \vec{n} \times \vec{v}_{\parallel} = \vec{n} \times \vec{v}$

$\vec{v}' = \hat{F}\vec{v} = \vec{v}_{\parallel} + \vec{v}_{\perp}' = \vec{v}_{\parallel} + \cos\varphi\, \vec{v}_{\perp} + \sin\varphi \cdot \vec{u} = \vec{v}_{\parallel} + \cos\varphi\,(\vec{v} - \vec{v}_{\parallel}) + \sin\varphi \cdot \vec{n} \times \vec{v} =$

$= \cos\varphi\, \vec{v} + (1 - \cos\varphi)\, \vec{v}_{\parallel} + \sin\varphi\, \vec{n} \times \vec{v}$

$$\boxed{\hat{F}\vec{v} = \cos\varphi\, \vec{v} + (1 - \cos\varphi)(\vec{v}\vec{n})\vec{n} + \sin\varphi \cdot \vec{n} \times \vec{v}}$$

$$\underline{u} = \underline{\underline{F}}\,\underline{v} = \cos\varphi\, \underline{v} + (1 - \cos\varphi)(\underline{v}\,\underline{n})\,\underline{n} + \sin\varphi\, \underline{n} \times \underline{v}$$

HF:

(1)

a)

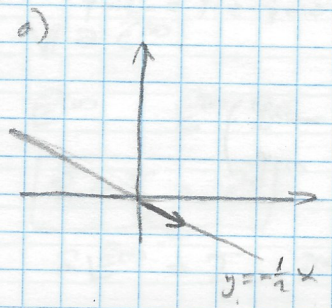
$$\begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \cdot 1 - 2 \cdot 5 \\ 2 \cdot 3 - 1 \cdot 1 \\ 1 \cdot 5 - 7 \cdot 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ -17 \end{pmatrix}$$

b)

$$\begin{pmatrix} 0 \\ 42 \\ 2 \end{pmatrix} \times \begin{pmatrix} 7 \\ 77 \\ 777 \end{pmatrix} = \begin{pmatrix} 42 \cdot 777 - 2 \cdot 77 \\ 2 \cdot 7 - 0 \cdot 777 \\ 0 \cdot 77 - 42 \cdot 7 \end{pmatrix} = \begin{pmatrix} 32490 \\ 14 \\ -294 \end{pmatrix}$$

(2)

a)



$\underline{n} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \frac{1}{\sqrt{5}}$ $\qquad \underline{\underline{P}} = \frac{1}{5} \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix}^{(2\ -1)} = \frac{1}{5} \cdot \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$

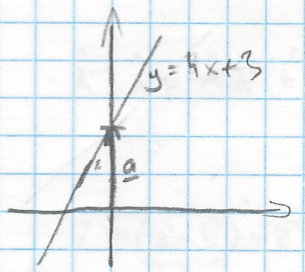
$\underline{v} = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$

$\hookrightarrow \underline{\underline{P}}\,\underline{v} = \frac{1}{5} \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \frac{1}{5} \cdot \begin{pmatrix} 24 \\ -12 \end{pmatrix}$

$$\underline{\underline{T}} = 2\underline{\underline{P}} - \underline{\underline{1}} = \frac{2}{5}\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}$$

$$\underline{\underline{T}}\,\underline{v} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ -4 & -3 \end{pmatrix}\begin{pmatrix} 7 \\ 2 \end{pmatrix} = \frac{1}{5}\cdot\begin{pmatrix} 13 \\ -34 \end{pmatrix}$$

b)



$$\underline{n} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}\cdot\frac{1}{\sqrt{17}} \qquad \underline{\underline{P}} = \frac{1}{17}\begin{pmatrix} 1 \\ 4 \end{pmatrix}^{(1\ 4)} = \frac{1}{17}\begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix}$$

$$\underline{\underline{T}} = 2\underline{\underline{P}} - \underline{\underline{1}} = \frac{2}{17}\begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{1}{17}\left[\begin{pmatrix} 2 & 8 \\ 8 & 32 \end{pmatrix} - \begin{pmatrix} 17 & 0 \\ 0 & 17 \end{pmatrix}\right] = \frac{1}{17}\begin{pmatrix} -15 & 8 \\ 8 & 15 \end{pmatrix}$$

$$\underline{\underline{Q}} = 2(\underline{\underline{1}} - \underline{\underline{P}}) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} - \frac{1}{17}\begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix} = \frac{1}{17}\left[\begin{pmatrix} 34 & 0 \\ 0 & 34 \end{pmatrix} - \begin{pmatrix} 1 & 4 \\ 4 & 16 \end{pmatrix}\right] = \frac{1}{17}\begin{pmatrix} 33 & 4 \\ 4 & 18 \end{pmatrix}$$

$$\underline{v} = \begin{pmatrix} 3 \\ -5 \end{pmatrix} \qquad \underline{v}' = \underline{\underline{T}}\,\underline{v} + \underline{\underline{Q}}\,\underline{a} = \frac{1}{17}\begin{pmatrix} -15 & 8 \\ 8 & 15 \end{pmatrix}\cdot\begin{pmatrix} 3 \\ -5 \end{pmatrix} + \frac{1}{17}\begin{pmatrix} 33 & 4 \\ 4 & 18 \end{pmatrix}\begin{pmatrix} 0 \\ 3 \end{pmatrix} = \frac{1}{17}\left[\begin{pmatrix} -85 \\ -51 \end{pmatrix} + \begin{pmatrix} 12 \\ 54 \end{pmatrix}\right] =$$

$$\underline{a} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \qquad = \frac{1}{17}\begin{pmatrix} -73 \\ 3 \end{pmatrix}$$

2017.10.10.

EA

$\mathbb{V} \ni \vec{v}$ $\qquad \{\vec{e}^{(k)}\} \in \mathbb{V}$

$\mathbb{R}^n \ni \underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad v_n \in \mathbb{R} \qquad \forall \vec{v} \in \mathbb{V} \quad \exists v_k \in \mathbb{R} \quad \vec{v} = \sum_k v_k \vec{e}^{(k)}$

$\vec{v} + \vec{u}$

$\alpha\vec{u}$

$$\vec{v}\vec{u} = \sum_k \sum_l g_{kl} v_k v_l \xrightarrow{\text{ONB}} \sum_k u_k v_k$$

$$g_{kl} = \vec{e}^{(k)}\vec{e}^{(l)}$$

$$\vec{w} = \vec{v} \times \vec{u} \qquad w_k = \sum_l \sum_m \varepsilon_{klm} v_l u_m$$

$$\hat{A} : \mathbb{V} \to \mathbb{V}$$

$$\vec{v} \longmapsto \vec{u} = \hat{A}\vec{v}$$

$$\vec{e}^{(k)} \to \vec{E}^{(l)} \qquad \vec{E}^{(l)}\vec{e}^{(k)} = \delta_{kl}$$

$$v_k = \vec{E}^{(k)}\vec{v}$$

$$\underline{u} = \underline{\underline{A}}\,\underline{v} \qquad u_k = \sum_l A_{kl} v_l \qquad \square = \square\cdot\square$$

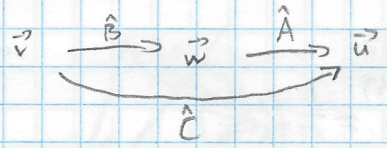
$$A_{kl} = \vec{E}^{(k)}(\hat{A}\vec{e}^{(l)})$$

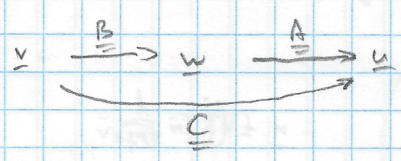
$$\hat{C} = \hat{A} + \hat{B} \qquad \hat{D} = \alpha\hat{A}$$

$$\underline{\underline{C}} = \underline{\underline{A}} + \underline{\underline{B}} \qquad \underline{\underline{D}} = \alpha\underline{\underline{A}}$$

$\hat{F} = \hat{A}\hat{B}$

$\hat{F}\vec{v} = (\hat{A}\hat{B})\vec{v} = \hat{A}(\hat{B}\vec{v})$





$\underline{w} = \underline{\underline{B}}\,\underline{v}$

$w_k = \sum_m B_{km} v_m$

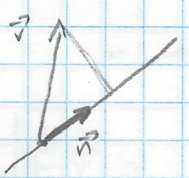
$\underline{u} = \underline{\underline{A}}\,\underline{w}$

$u_k = \sum_\ell A_{k\ell} w_\ell = \sum_\ell A_{k\ell} \left(\sum_m B_{\ell m} v_m\right) = \sum_\ell \sum_m A_{k\ell} B_{\ell m} v_m = \sum_m \left(\sum_\ell A_{k\ell} B_{\ell m}\right) v_m$

$\underline{u} = \underline{\underline{C}}\,\underline{v}$

$u_k = \sum_m C_{km} v_m$

$\Rightarrow C_{km} = \sum_\ell A_{k\ell} B_{\ell m}$



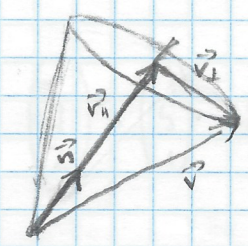
$\hat{P}\vec{v} = (\vec{n}\vec{v})\vec{n} = \vec{n}(\vec{n}\vec{v}) = (\vec{n}\circ\vec{n})\vec{v}$

$\hat{P} = (\vec{n}\circ\vec{n})$ $\qquad$ $\hat{T} = 2\hat{P} - \hat{1}$

$\underline{\underline{P}} = \underline{n} \circ \underline{n}$ $\qquad$ $\hat{1} \to \underline{\underline{1}} \to 1_{k\ell} = \delta_{k\ell}$

$P_{k\ell} = n_k n_\ell$ $\qquad$ $T_{k\ell} = 2 n_k n_\ell - \delta_{k\ell}$

$\underline{\underline{F}} = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$



$\vec{v}' = \hat{F}\vec{v} = \vec{v}\cos\varphi + (\vec{v}\vec{n})\vec{n}\,(1-\cos\varphi) + \vec{n}\times\vec{v}\cdot\sin\varphi$

$\underline{v}' = \underline{v}\cos\varphi + (\underline{v}\circ\underline{n})\underline{n}\,(1-\cos\varphi) + \underline{n}\times\underline{v}\cdot\sin\varphi$

$v_k' = v_k\cdot\cos\varphi + \left(\sum_\ell v_\ell n_\ell\right) n_k\,(1-\cos\varphi) + \sum_\ell \sum_m \varepsilon_{k\ell m} n_\ell v_m \sin\varphi$

$v_k = \sum_\ell v_\ell \delta_{k\ell}$

$v_k' = \sum_\ell v_\ell \delta_{k\ell}\cdot\cos\varphi + \sum_\ell v_\ell n_\ell n_k\,(1-\cos\varphi) + \sum_\ell \sum_m \varepsilon_{k\ell m}\cdot v_\ell n_m \cdot \sin\varphi =$

$= \sum_\ell \Big(\underbrace{\delta_{k\ell}\cos\varphi + n_k n_\ell\cdot(1-\cos\varphi) + \sum_m \varepsilon_{k\ell m} n_m \cdot \sin\varphi}_{F_{k\ell}} \Big) v_\ell$

HF

$\varphi = 30°$, $n = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}$; $\begin{pmatrix}3\\4\\12\end{pmatrix}$

$\underline{\underline{F}}$ kiszámolni és transzponálni $\to \underline{\underline{\tilde{F}}}\cdot\underline{\underline{F}}$

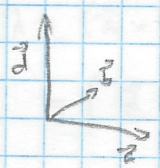
$\vec{u} = \hat{A}\vec{v}$ $\quad \forall \vec{v}$ $\quad \underline{u} = \underline{\underline{A}}\,\underline{v}$

$\vec{v} = \hat{A}^{-1}\vec{u}$ $\quad \underline{v} = \underline{\underline{A}}^{-1}\underline{u}$

Invertálható operátorok → csoportot alkotnak szorzásra nézve

$\hat{A}^{-1}\hat{A} = \hat{A}\hat{A}^{-1} = \hat{1}$ $\quad \underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{A}}^{-1} = \underline{\underline{1}}$

$\vec{u} = (\underbrace{\vec{a}\times\vec{b}}_{\vec{d}})\times\vec{c}$ $\quad \vec{a}\times(\vec{b}\times\vec{c}) = \vec{v}$



$\vec{u} = \alpha\vec{a} + \beta\vec{b}$

$\vec{v} = \gamma\vec{b} + \delta\vec{c}$

$\underline{u} = (\underline{a}\times\underline{b})\times\underline{c} = \underline{d}\times\underline{c}$

$u_k = (\underline{d}\times\underline{c})_k = \sum_\ell \sum_m \varepsilon_{k\ell m} d_\ell c_m$

$\underline{d} = \underline{a}\times\underline{b}$

$d_\ell = (\underline{a}\times\underline{b})_\ell = \sum_s \sum_t \varepsilon_{\ell s t} a_s b_t$

$u_k = \sum_\ell \sum_m \varepsilon_{k\ell m}\left(\sum_s \sum_t \varepsilon_{\ell s t} a_s b_t\right) c_m = \sum_\ell \sum_m \sum_s \sum_t \varepsilon_{k\ell m}\varepsilon_{\ell s t} a_s b_t c_m =$

$= \sum_m \sum_s \sum_t \Big(\sum_\ell \underbrace{\varepsilon_{k\ell m}}_{\varepsilon_{m k \ell}}\varepsilon_{\ell s t}\Big) a_s b_t c_m$

$T_{mkst} = \sum_\ell \varepsilon_{mk\ell}\varepsilon_{\ell st}$

$T_{1223} = \sum_\ell \varepsilon_{12\ell}\varepsilon_{\ell 23} = \underbrace{\varepsilon_{121}}_{0}\varepsilon_{123} + \underbrace{\varepsilon_{122}}_{0}\varepsilon_{223} + \varepsilon_{123}\underbrace{\varepsilon_{323}}_{0}$

$T_{1221} = \sum_\ell \varepsilon_{12\ell}\varepsilon_{\ell 21} = \varepsilon_{121}\varepsilon_{211} + \varepsilon_{122}\varepsilon_{221} + \underbrace{\varepsilon_{123}}_{1}\underbrace{\varepsilon_{321}}_{-1} = -1$

$T_{1212} = \sum_\ell \varepsilon_{12\ell}\varepsilon_{\ell 12} = \varepsilon_{121}\varepsilon_{112} + \varepsilon_{122}\varepsilon_{212} + \varepsilon_{123}\varepsilon_{312} = 1$

$T_{mkst} = \delta_{ms}\delta_{kt} - \delta_{mt}\delta_{ks}$

$\sum_c \varepsilon_{abc}\varepsilon_{cde} = \delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}$

$\Rightarrow u_k = \sum_m \sum_s \sum_t \Big(\sum_\ell \varepsilon_{k\ell m}\varepsilon_{\ell st}\Big) a_s b_t c_m = \sum_m \sum_s \sum_t \left(\delta_{ms}\delta_{kt} - \delta_{mt}\delta_{ks}\right) a_s b_t c_m =$

$= \sum_m \sum_t \left(a_m \delta_{kt} b_t c_m - a_k \delta_{mt} b_t c_m\right) = \sum_m \left(a_m b_k c_m - a_k b_m c_m\right) =$

$= b_k \sum_m a_m c_m - a_k \sum_m b_m c_m = b_k(\underline{a}\,\underline{c}) - a_k(\underline{b}\,\underline{c}) = u_k$

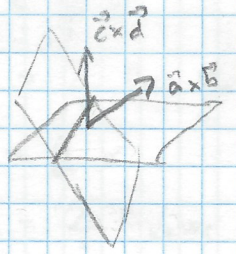
$\underline{u} = (\underline{a}\times\underline{b})\times\underline{c} = \underline{b}(\underline{a}\underline{c}) - \underline{a}(\underline{b}\underline{c})$

$\underline{v} = \underline{a}\times(\underline{b}\times\underline{c}) = \underline{b}(\underline{a}\underline{c}) - \underline{c}(\underline{a}\underline{b})$

KIFEJTÉSI TÉTEL

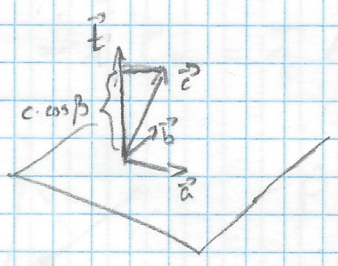
$\underline{w} = (\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = (\underline{a}\times\underline{b})\times\underline{f} = \underline{b}(\underline{a}\underline{f}) - \underline{a}(\underline{b}\underline{f}) = \underline{b}(\underline{a}(\underline{c}\times\underline{d})) - \underline{a}(\underline{b}(\underline{c}\times\underline{d}))$

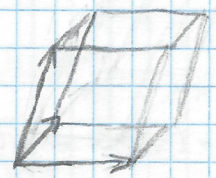
$\underline{w} = \underline{g}\times(\underline{c}\times\underline{d}) = \underline{c}(\underline{g}\underline{d}) - \underline{d}(\underline{c}\underline{g}) = \underline{c}\left((\underline{a}\times\underline{b})\underline{d}\right) - \underline{d}(\underline{c}(\underline{a}\times\underline{b}))$



NÉGY VEKTOR TÉTELE

$\gamma = (\vec{a}\times\vec{b})\,\vec{c} = (\vec{a},\vec{b},\vec{c})$ vegyesszorzat





γ: paralelepipedon térfogata ($\pm$)

$(\vec{a},\vec{b},\vec{c}) = (\vec{c},\vec{a},\vec{b}) = (\vec{b},\vec{c},\vec{a}) = -(\vec{b},\vec{a},\vec{c}) = -(\vec{a},\vec{c},\vec{b}) = -(\vec{c},\vec{b},\vec{a})$

$\gamma = (\vec{a}\times\vec{b})\cdot\vec{c} = (\underline{a}\times\underline{b})\,\underline{c} = \underline{d}\cdot\underline{c} = \sum_m d_m c_m = \sum_m \left(\sum_k\sum_\ell \varepsilon_{mk\ell}\, a_k b_\ell\right) c_m =$

$= \sum_k\sum_\ell\sum_m \varepsilon_{kem}\, a_k b_\ell c_m$

$\vec{a},\vec{b},\vec{c}$ bázis $\iff (\vec{a},\vec{b},\vec{c}) \neq 0$

$\forall\vec{v}\ \ \exists\alpha,\beta,\gamma\in\mathbb{R}\qquad \vec{v} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} \qquad /\cdot(\underline{b}\times\underline{c})$

$\vec{v}\cdot(\vec{b}\times\vec{c}) = \alpha\,\vec{a}\cdot(\vec{b}\times\vec{c}) + \beta\,\vec{b}\cdot(\vec{b}\times\vec{c}) + \gamma\,\vec{c}\cdot(\vec{b}\times\vec{c})$

$(\vec{v},\vec{b},\vec{c}) = \alpha\,(\vec{a},\vec{b},\vec{c})$

$$\alpha = \frac{(\vec{v},\vec{b},\vec{c})}{(\vec{a},\vec{b},\vec{c})}$$

$(\vec{v},\vec{c},\vec{a}) = \alpha\,(\vec{a},\vec{c},\vec{a}) + \beta\,(\vec{b},\vec{c},\vec{a}) + \gamma\,(\vec{c},\vec{c},\vec{a})$

$(\vec{a}\times\vec{c})\cdot\vec{c} = 0$ $\qquad$ $(\vec{c},\vec{c},\vec{a}) = 0$

$$\beta = \frac{(\vec{v},\vec{c},\vec{a})}{(\vec{a},\vec{b},\vec{c})}$$

$(\vec{v},\vec{a},\vec{b}) = \alpha\,(\vec{a},\vec{a},\vec{b}) + \beta\,(\vec{b},\vec{a},\vec{b}) + \gamma\,(\vec{c},\vec{a},\vec{b}) \longrightarrow \gamma = \frac{(\vec{v},\vec{a},\vec{b})}{(\vec{a},\vec{b},\vec{c})}$

$$\alpha = \frac{(v,b,c)}{(a,b,c)}$$

$$\beta = \frac{(a,v,c)}{(a,b,c)}$$

$$\gamma = \frac{(a,b,v)}{(a,b,c)}$$

Cramer - szabály

$$\underline{w} = (\underline{a}\times\underline{b})\times(\underline{c}\times\underline{d}) = (\underline{a}\times\underline{b})\times\underline{f} = \underline{b}(\underline{a}\underline{f}) - \underline{a}(\underline{b}\underline{f}) = \underline{b}\left(\underline{a}(\underline{c}\times\underline{d})\right) - \underline{a}\left(\underline{b}(\underline{c}\times\underline{d})\right) =$$

$$= \underline{b}(\underline{a},\underline{c},\underline{d}) - \underline{a}(\underline{b},\underline{c},\underline{d})$$

$$\vec{\lambda} = (\vec{a}\times\vec{b})(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\vec{f} = (\vec{a},\vec{b},\vec{f}) = (\vec{f},\vec{a},\vec{b}) = \left[(\vec{c}\times\vec{d})\times\vec{a}\right]\vec{b} =$$

$$= \left[\vec{d}(\vec{a}\vec{c}) - \vec{c}(\vec{d}\vec{a})\right]\vec{b} = (\vec{b}\vec{d})(\vec{a}\vec{c}) - (\vec{c}\vec{b})(\vec{d}\vec{a})$$

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$\vec{a} \to \underline{a}$

$\vec{e}^{(k)}$ ONB

$\vec{e}^{(k)}\vec{e}^{(l)} = \delta_{kl}$

$\vec{a}\vec{b} = \underline{a}\underline{b} = \sum_k a_k b_k$

$\vec{c} = \vec{a}\times\vec{b}$

$\underline{c} = \underline{a}\times\underline{b}$

$c_k = \sum_\ell \sum_m \varepsilon_{k\ell m} a_\ell b_m$

$$\gamma = (\vec{a}\times\vec{b})\vec{c} = (\underline{a}\times\underline{b})\underline{c} = \sum_m (\underline{a}\times\underline{b})_m c_m =$$

$$= \sum_m \left(\sum_\ell \sum_\ell \varepsilon_{mk\ell} a_k b_\ell\right) c_m = \sum_k \sum_\ell \sum_m \varepsilon_{k\ell m} a_k b_\ell c_m =$$

$$= \varepsilon_{123} a_1 b_2 c_3 + \varepsilon_{231} a_2 b_3 c_1 + \varepsilon_{312} a_3 b_1 c_2 +$$

$$+ \varepsilon_{132} a_1 b_3 c_2 + \varepsilon_{321} a_3 b_2 c_1 + \varepsilon_{213} a_2 b_1 c_3 + 21.0 =$$

$$= a_1 b_2 c_3 + a_2 b_3 c_1 + c_3 b_1 c_2 - a_1 b_3 c_2 - a_3 b_2 c_1 - a_2 b_1 c_3 =$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$ determináns

$$\begin{pmatrix}3\\2\\-1\end{pmatrix}\begin{pmatrix}5\\4\\0\end{pmatrix}\begin{pmatrix}6\\-2\\1\end{pmatrix}$$

$$\begin{vmatrix} 3 & 5 & 6 \\ 2 & 4 & -2 \\ -1 & 0 & 1 \end{vmatrix} = 12 + 10 + 0 - (-24) - 0 - 10 = 36$$

$\underline{e}^1 \begin{pmatrix}1\\0\\0\end{pmatrix}$ $\underline{e}^2 \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\underline{e}^3 \begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$$\begin{vmatrix} a_1 & b_1 & \underline{i} \\ a_2 & b_2 & \underline{j} \\ a_3 & b_3 & \underline{k} \end{vmatrix} = \underline{i}(a_2b_3 - a_3b_2) + \underline{j}(a_3b_1 - a_1b_3) + \underline{k}(a_1b_2 - a_2b_1)$$

$$\begin{pmatrix}3\\2\\-1\end{pmatrix} \times \begin{pmatrix}5\\1\\0\end{pmatrix} = \begin{pmatrix}1\\-5\\2\end{pmatrix}$$ → ellenőrzés skaláris szorzattal

$$\vec{v} = \alpha\vec{a} + \beta\vec{b} + \gamma\vec{c}$$

$$\alpha = \frac{(\vec{v}, \vec{b}, \vec{c})}{(\vec{a}, \vec{b}, \vec{c})}$$

$$\underline{v} = \begin{pmatrix}5\\-1\\7\end{pmatrix} = \alpha\begin{pmatrix}3\\2\\-1\end{pmatrix} + \beta\begin{pmatrix}5\\1\\0\end{pmatrix} + \gamma\begin{pmatrix}6\\-2\\1\end{pmatrix}$$

$$3\alpha + 5\beta + 6\gamma = 5$$

$$2\alpha + 1\beta - 2\gamma = -1$$

$$-\alpha + \gamma = 7$$

$$\alpha = \frac{(\underline{v}, \underline{b}, \underline{c})}{(\underline{a}, \underline{b}, \underline{c})} = \frac{\begin{vmatrix}5 & 5 & 6\\-1 & 1 & -2\\7 & 0 & 1\end{vmatrix}}{\begin{vmatrix}3 & 5 & 6\\2 & 1 & -2\\-1 & 0 & 1\end{vmatrix}} \qquad \beta = \frac{(\underline{a}, \underline{v}, \underline{c})}{(\underline{a}, \underline{b}, \underline{c})} = \frac{\begin{vmatrix}3 & 5 & 6\\2 & -1 & -2\\-1 & 7 & 1\end{vmatrix}}{\cdots} \qquad \gamma = \frac{\begin{vmatrix}3 & 5 & 5\\2 & 1 & -1\\-1 & 0 & 7\end{vmatrix}}{\cdots}$$

$\underline{a}$ $\underline{b}$ $\underline{c}$

$$\begin{pmatrix}3\\2\\-1\end{pmatrix} \quad \begin{pmatrix}5\\1\\0\end{pmatrix} \quad \begin{pmatrix}6\\-2\\1\end{pmatrix}$$

$\underline{A}$, $\underline{B}$, $\underline{C}$

$\underline{A}\,\underline{b} = 0$ $\quad \underline{B}\,\underline{a} = 0$ $\quad \underline{C}\,\underline{a} = 0$

$\underline{A}\,\underline{c} = 0$ $\quad \underline{B}\,\underline{b} = 1$ $\quad \underline{C}\,\underline{b} = 0$

$\underline{A}\,\underline{a} = 1$ $\quad \underline{B}\,\underline{c} = 0$ $\quad \underline{C}\,\underline{c} = 1$

$\underline{A} = \mu\,\underline{b} \times \underline{c} \quad / \cdot \underline{a}$

$\underline{A}\,\underline{a} = \mu(\underline{b} \times \underline{c}) \cdot \underline{a} = \mu\,(a, b, c) = 1$

$$\underline{A} = \frac{\underline{b} \times \underline{c}}{(a, b, c)} \qquad \underline{B} = \frac{\underline{c} \times \underline{a}}{(a, b, c)} \qquad \underline{C} = \frac{\underline{a} \times \underline{b}}{(a, b, c)}$$

$$\underline{b} \times \underline{c} = \begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix} \times \begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ -36 \end{pmatrix} \qquad \underline{A} = \frac{1}{36} \begin{pmatrix} 4 \\ -5 \\ -36 \end{pmatrix}$$

$\underline{a}, \underline{b} \qquad \underline{a} \times \underline{b} \neq 0$

$\underline{A}, \underline{B}$

$\underline{A}\,\underline{a} = 1 \qquad \underline{B}\,\underline{a} = 0$

$\underline{A}\,\underline{b} = 0 \qquad \underline{B}\,\underline{b} = 1$

$\underline{A} = \mu \underline{a} + \nu \underline{b}$

$\underline{A}\,\underline{a} = \mu \underline{a}^2 + \nu\, \underline{a}\,\underline{b} = 1$

$\underline{A}\,\underline{b} = \mu\, \underline{a}\,\underline{b} + \nu\, \underline{b}^2 = 0$

$\Rightarrow$

$a^2 x + ab\, y = 1$

$ab\, x + b^2 y = 0$

$y = -\frac{(ab)\,x}{b^2}$

$a^2 x - \frac{(ab)^2 x}{b^2} = 1$

$x \left(a^2 - \frac{(ab)^2}{b^2} \right) = 1$

$x \left(a^2 b^2 - (ab)^2 \right) = b^2$

$x = \frac{b^2}{a^2 b^2 - (ab)^2} \qquad y = -\frac{ab}{a^2 b^2 - (ab)^2}$

$$\underline{A} = \frac{b^2\, \underline{a} - (ab)\, \underline{b}}{a^2 b^2 - (ab)^2} = \frac{\underline{b} \times (\underline{a} \times \underline{b})}{|\underline{a} \times \underline{b}|^2}$$

$$F_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$v'_k = \sum_\ell F_{k\ell}\, v_\ell$

$\underline{v} \xrightarrow{F_\phi} \underline{w} \xrightarrow{F_\beta} \underline{u}$

$\underline{w} = \underline{\underline{F}}_\phi\, \underline{v}$

$\underline{u} = \underline{\underline{F}}_\beta\, \underline{w} = F_\beta (F_\phi\, \underline{v}) = (F_\beta F_\phi)\, \underline{v}$

$$F_\beta F_\phi = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} = \begin{pmatrix} (\cos\beta\cos\phi - \sin\beta\sin\phi) & (-\cos\beta\sin\phi - \sin\beta\cos\phi) \\ (\sin\beta\cos\phi + \cos\beta\sin\phi) & (-\sin\beta\sin\phi + \cos\beta\cos\phi) \end{pmatrix} = F_{\phi+\beta}$$

$F(\beta) \cdot F(\phi) = F(\beta + \phi)$

$$\underline{\underline{F}}(t) = e^{\underline{\underline{W}}t} = \underline{\underline{I}} + \underline{\underline{W}}t + \frac{\underline{\underline{W}}^2 t^2}{2!} + \dots \qquad / \quad \frac{d}{dx} e^{cx} = c \cdot e^{cx}$$

$$\frac{d\,\underline{\underline{F}}(t)}{dt} = \underline{\underline{W}} + \underline{\underline{W}}^2 t + \dots$$

$$\left.\frac{d\,\underline{\underline{F}}(t)}{dt}\right|_{t=0} = \underline{\underline{W}}$$

$$\underline{\underline{F}}_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$$\frac{d\underline{\underline{F}}_t}{dt} = \begin{pmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{pmatrix}$$

$$\left.\frac{d\underline{\underline{F}}_t}{dt}\right|_{t=0} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \underline{\underline{J}}$$

$$\underline{\underline{F}}(t) = e^{t\underline{\underline{J}}} = \underline{\underline{I}} + t\underline{\underline{J}} + \frac{t^2\underline{\underline{J}}^2}{2!} + \dots$$

$$\underline{\underline{J}}^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -\underline{\underline{I}}$$

$$\underline{\underline{J}}^3 = \underline{\underline{J}}^1\underline{\underline{J}} = -\underline{\underline{J}}$$

$$\underline{\underline{J}}^4 = \underline{\underline{I}}$$

$$\underline{\underline{J}}^5 = \underline{\underline{J}}$$

$$\vdots$$

$$\underline{\underline{F}}(t) = \underline{\underline{I}} + t\underline{\underline{J}} - \frac{t^2}{2}\underline{\underline{I}} - \frac{t^3}{3!}\cdot\underline{\underline{J}} + \frac{t^4}{4!}\underline{\underline{I}} + \frac{t^5}{5!}\cdot\underline{\underline{J}} - \dots =$$

$$= \underline{\underline{I}}\left(1 - \frac{t^2}{2} + \frac{t^4}{5!} - \frac{t^6}{6!} + \dots\right) + \underline{\underline{J}}\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \dots\right) = \cos t \cdot \underline{\underline{I}} + \sin t \cdot \underline{\underline{J}} =$$

$$= \cos t \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin t \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} = F(t)$$

	1	$\underline{\underline{J}}$
1	1	$\underline{\underline{J}}$
$\underline{\underline{J}}$	$\underline{\underline{J}}$	-1

	1	i
1	1	i
i	i	-1

$a, b \in \mathbb{R}$

$$C(a,b) = a\underline{\underline{I}} + b\underline{\underline{J}} = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} + \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \dots$$

$$\begin{pmatrix} a_1 & -b_1 \\ b_1 & a_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & -b_2 \\ b_2 & a_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2 & -a_1b_2 - b_1a_2 \\ b_1a_2 + a_1b_2 & -b_1b_2 + a_1a_2 \end{pmatrix} = \begin{pmatrix} a_3 & -b_3 \\ b_3 & a_3 \end{pmatrix} = C(a_3, b_3)$$

$$(a_1\underline{\underline{I}} + b_1\underline{\underline{J}}) \cdot (a_2\underline{\underline{I}} + b_2\underline{\underline{J}})$$

$$a_3 = a_1a_2 - b_1b_2$$

$$b_3 = a_1b_2 + a_2b_1$$

$$e^x = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$$

$$F(\alpha) = e^{\alpha\frac{\sigma_3}{2}} = \lim_{n\to\infty}\left(I+\frac{\alpha\frac{\sigma_3}{2}}{n}\right)^n = \lim_{n\to\infty}\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\alpha}{n}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\right]^n = \lim_{n\to\infty}\begin{pmatrix}1&-\frac{\alpha}{n}\\\frac{\alpha}{n}&1\end{pmatrix}^n =$$

$$= \lim_{n\to\infty}\left[\sqrt{1+\frac{\alpha^2}{n^2}}\begin{pmatrix}\frac{1}{\sqrt{1+\frac{\alpha^2}{n^2}}} & \frac{-\frac{\alpha}{n}}{\sqrt{1+\frac{\alpha^2}{n^2}}}\\ \frac{\alpha/n}{\sqrt{1+\alpha^2/n^2}} & \frac{1}{\sqrt{1+\alpha^2/n^2}}\end{pmatrix}\right]^n = \left(\lim_{n\to\infty}\sqrt{1+\frac{\alpha^2}{n^2}}^n\right)\left(\lim_{n\to\infty}(\dots)^n\right) =$$

$$\left(\lim_{n\to\infty}\sqrt{1+\frac{\alpha^2}{n^2}}^n\right) = 1$$

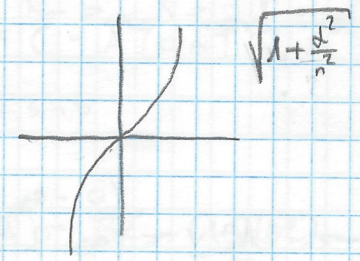
$$= \lim_{n\to\infty}\begin{pmatrix}\cos\varphi_n & -\sin\varphi_n\\ \sin\varphi_n & \cos\varphi_n\end{pmatrix}^n$$

$$\lim_{n\to\infty} F(\varphi_n)^n = \lim_{n\to\infty} F(n\varphi_n) = F(\alpha)$$

$$\operatorname{tg}\varphi_n = \frac{\alpha}{n}$$

$$\operatorname{tg}\varphi_n \sim \varphi_n$$

$$n\varphi_n \sim n\cdot\operatorname{tg}\varphi_n = \alpha$$



$$(\underline{\underline{\tilde{A}}})_{ik} = A_{ek}$$

$$\left.\begin{aligned}&\tilde{\tilde{A}} = A\\ &A^* \quad \text{komplex}\end{aligned}\right\} \tilde{A}^* = A^+ \quad \text{adjungált}$$

$\tilde{S} = S$ szimmetrikus mátrix

$\hat{A} = -A$ antiszimmetrikus mátrix

$$A = \underbrace{\frac{A+\tilde{A}}{2}}_{S} + \underbrace{\frac{A-\tilde{A}}{2}}_{\frac{\hat{A}-\tilde{\hat{A}}}{2} = \frac{\tilde{A}-A}{2}} \Rightarrow$$

Minden mátrix felbontható egy szimm. és egy antiszimm. mátrix összegére

$$\tilde{S} = \frac{\tilde{A}+\tilde{\tilde{A}}}{2} = S$$

$A^+ = A$ hermitikus, Hermite

$U^+ = U^{-1}$ unitér

$U^+U = 1$ unitary

$\tilde{M} = M^{-1}$

$\tilde{M}M = 1$ ortogonális mátrix (valósokra)

$\vec{a} \times \vec{b} = \vec{c}$

$\vec{a} \times \vec{v} = \vec{u}$

$\vec{v} \longmapsto \vec{u}$

$\hat{A}\vec{v} = \vec{u}$ HF. $\hat{A}$ mátrix felírni

HF.

1. $\vec{u} = \vec{a} \times \vec{v}$

$$u_k = (\vec{a} \times \vec{v})_k = \sum_l \sum_m \varepsilon_{klm} a_l v_m = \sum_m \underbrace{\left(\sum_l \varepsilon_{klm} a_l\right)}_{A_{km}} v_m$$

$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = 1$ $\quad \varepsilon_{321} = \varepsilon_{132} = \varepsilon_{213} = -1$

$\varepsilon_{11i} = \varepsilon_{22j} = \varepsilon_{33k} = 0$

$$\underline{\underline{A}}_x = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \longrightarrow \underline{\underline{A}}_x \underline{v} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} a_2 v_3 - a_3 v_2 \\ a_3 v_1 - a_1 v_3 \\ a_1 v_2 - a_2 v_1 \end{pmatrix}$$

2.

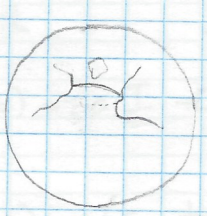
$n = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \varphi = 30^\circ \rightarrow \cos\varphi = \frac{\sqrt{3}}{2}, \quad \sin\varphi = \frac{1}{2}$

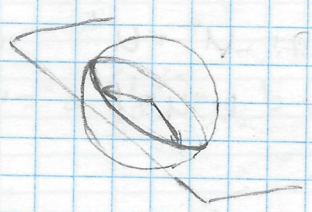
$$\underline{\underline{F}} = \begin{pmatrix} \frac{\sqrt{3}}{2} + \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) & \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) - 1\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{2} & \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) + 1\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{2} \\ \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) + 1\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{2} & \frac{\sqrt{3}}{2} + \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) & \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) - 1\frac{1}{\sqrt{3}}\cdot\frac{1}{2} \\ \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) - 1\frac{1}{\sqrt{3}}\frac{1}{2} & \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) + 1\frac{1}{\sqrt{3}}\cdot\frac{1}{2} & \frac{\sqrt{3}}{2} + \frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{pmatrix}$$

$$\underline{\underline{\tilde{F}}} \cdot \underline{\underline{F}} = \frac{1}{9}\begin{pmatrix} 1+\sqrt{3} & 1 & 1-\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} & 1 \\ 1 & 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix} \cdot \begin{pmatrix} 1+\sqrt{3} & 1-\sqrt{3} & 1 \\ 1 & 1+\sqrt{3} & 1-\sqrt{3} \\ 1-\sqrt{3} & 1 & 1+\sqrt{3} \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

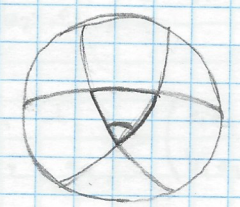
2017.10.12.

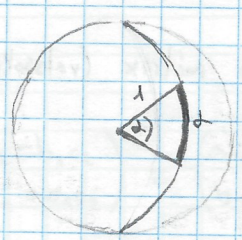
GY

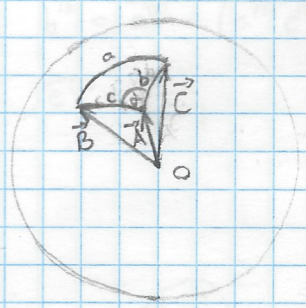




fökör



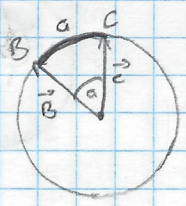


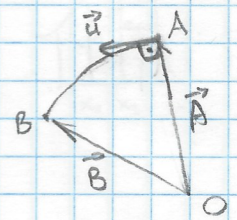


$$\vec{B}\vec{C} = |\vec{B}| \cdot |\vec{C}| \cdot \cos a = \cos a$$

(1 1)

$$\boxed{\begin{aligned} \cos a &= \vec{B}\vec{C} \\ \cos b &= \vec{A}\vec{C} \\ \cos c &= \vec{A}\vec{B} \end{aligned}}$$

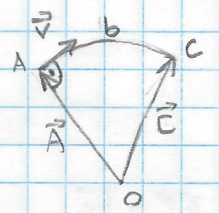




$$\vec{u} = (\vec{A}\times\vec{B})\times\vec{A} = \vec{B}(\vec{A}\vec{A}) - \vec{A}(\vec{B}\vec{A}) = \vec{B} - \vec{A}\underbrace{(\vec{B}\vec{A})}_{\cos c}$$

$$|\vec{u}|^2 = \vec{u}\cdot\vec{u} = \left[\vec{B} - (AB)\vec{A}\right]\left[\vec{B} - (AB)\vec{A}\right] = \vec{B}\vec{B} - (AB)(\vec{A}\vec{B}) - (AB)(\vec{B}\vec{A}) +$$

$$+ (AB)^2(\vec{A}\vec{A}) = 1 - (AB)^2 - (AB)^2 + (AB)^2 = 1 - (AB)^2 = 1 - \cos^2 c = \sin^2 c$$



$$\vec{v} = \vec{C} - \vec{A}(\vec{C}\vec{A})$$

$$|\vec{v}| = \sin b$$

$$\vec{u}\vec{v} = \left[\vec{B} - (AB)\vec{A}\right]\left[\vec{C} - (AC)\vec{A}\right] = \vec{B}\vec{C} - (AB)(\vec{A}\vec{C}) - (AC)(\vec{B}\vec{A}) + (AB)(AC)\vec{A}\vec{A} = (\vec{B}\vec{C}) - (\vec{A}\vec{B})(\vec{A}\vec{C}) =$$

$$= \cos a - \cos c \cdot \cos b$$

$$\vec{u}\vec{v} = \cos a - \cos b \cdot \cos c = |u||v| \cdot \cos d = \sin b \cdot \sin c \cdot \cos d$$

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \cdot \sin c \cdot \cos d \\ \cos b &= \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos \beta \\ \cos c &= \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos \gamma \end{aligned}\right\} \text{ gömbháromszögtani koszinusztétel}$$

$$\cos d = \cos\beta \cdot \cos\gamma - \sin\beta \cdot \sin\gamma \cdot \cos a \quad \rightarrow \text{ poláris háromszögre}$$

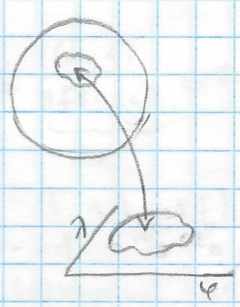
$$\frac{\sin a}{\sin b} = \frac{\sin d}{\sin \beta}$$

$$R \neq 1 \rightarrow \cos\frac{c}{R} = \cos\frac{a}{R} \cdot \cos\frac{b}{R} + \sin\frac{a}{R} \cdot \sin\frac{b}{R} \cdot \cos\gamma$$

$$1 - \frac{1}{2}\left(\frac{c}{R}\right)^2 = \left[1 - \frac{1}{2}\left(\frac{a}{R}\right)^2\right]\left[1 - \frac{1}{2}\left(\frac{b}{R}\right)^2\right] + \frac{a}{R}\cdot\frac{b}{R}\cdot\cos\gamma$$

$$1 - \frac{1}{2}\left(\frac{c}{R}\right)^2 = 1 - \frac{1}{2}\left(\frac{a}{R}\right)^2 - \frac{1}{2}\left(\frac{b}{R}\right)^2 + \frac{ab}{R^2}\cos\gamma$$

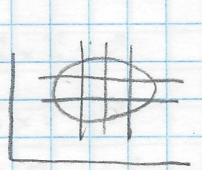
$$c^2 = a^2 + b^2 - 2ab\cdot\cos\gamma \quad \Rightarrow \text{ Kis gömbdarab síkkal közelíthető}$$

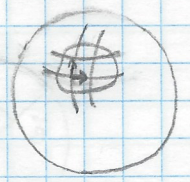


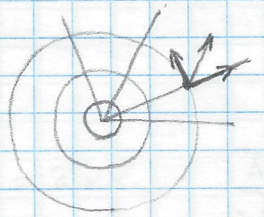
Térképezés ↓ Paraméterezés ↑

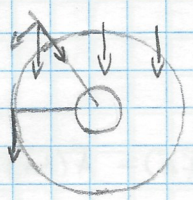
⇓

térképpontok összessége: Atlasz









2017. 10. 12.

EA

$$\vec{a} = \sum_k a_k \vec{e}^{(k)}$$

$$\vec{b} = \sum_k b_k \vec{e}^{(k)}$$

$$\vec{a}\vec{b} = \sum_k \sum_\ell a_k b_\ell (\vec{e}^{(k)} \vec{e}^{(\ell)}) = \sum_k \sum_\ell g_{k\ell} a_k b_\ell$$

$$g_{k\ell} = \vec{e}^{(k)} \vec{e}^{(\ell)} = \vec{e}^{(\ell)} \vec{e}^{(k)} = g_{\ell k}$$

$V \times V \to \mathbb{R}$

$\forall \vec{a}, \vec{b}, \vec{c}$ $(\vec{a} + \vec{b})\vec{c} = \vec{a}\vec{c} + \vec{b}\vec{c}$

$(\lambda \vec{a})\vec{b} = \lambda(\vec{a}\vec{b})$

$\vec{a}\vec{a} = |\vec{a}|^2 \geq 0$

$\vec{a}\vec{a} = 0 \to \vec{a} = 0$

$\vec{a}\vec{b} = \vec{b}\vec{a}$

Spec. $\vec{e}^{(k)} \vec{e}^{(\ell)} = \delta_{k\ell}$

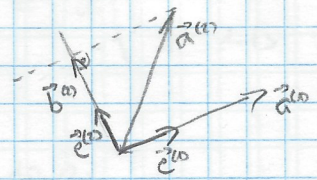
$$\vec{a}\vec{b} = \sum_k \sum_\ell \delta_{k\ell} a_k b_\ell = \sum_k a_k b_k$$

$$\sum_\ell \delta_{k\ell} b_\ell = b_k$$

$\vec{a}^{(k)}$ $k = 1, \dots, n$

$\vec{a}^{(k)}$ $|a^{(k)}| = \sqrt{\vec{a}^{(k)} \vec{a}^{(k)}}$

$$\vec{e}^{(k)} = \frac{1}{|a^{(k)}|} \cdot a^{(k)}$$



$$\vec{b}^{(k)} = \vec{a}^{(k)} + \lambda \vec{e}^{(\ell)}$$

$$\vec{b}^{(k)} \vec{e}^{(\ell)} = \vec{a}^{(k)} \vec{e}^{(\ell)} + \lambda \underbrace{\vec{e}^{(\ell)} \vec{e}^{(\ell)}}_{1} = 0$$

$$\lambda = -\vec{a}^{(k)} \vec{e}^{(\ell)}$$

$$\vec{b}^{(k)} = \vec{a}^{(k)} - (\vec{a}^{(k)} \vec{e}^{(\ell)}) \vec{e}^{(\ell)} = \vec{a}^{(k)} - \vec{e}^{(\ell)} (\vec{e}^{(\ell)} \vec{a}^{(k)}) = \vec{a}^{(k)} - (\vec{e}^{(\ell)} \circ \vec{e}^{(\ell)}) \vec{a}^{(k)} =$$

$$= \left(\hat{1} - \vec{e}^{(\ell)} \circ \vec{e}^{(\ell)}\right) \vec{a}^{(k)}$$

$$\vec{e}^{(k)} = \frac{b^{(k)}}{|b^{(k)}|}$$

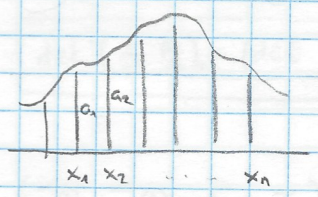
$\vec{a}^{(3)}$ $\vec{b}^{(3)} = \vec{a}^{(3)} + \lambda \vec{e}^{(1)} + \beta \vec{e}^{(2)}$

$$\vec{b}^{(3)} \vec{e}^{(1)} = \vec{a}^{(3)} e^{(1)} + \lambda \underbrace{\vec{e}^{(1)} \vec{e}^{(1)}}_{1} + \beta \underbrace{\vec{e}^{(2)} \vec{e}^{(1)}}_{0} = 0 \quad \to \lambda = -\vec{a}^{(3)} \vec{e}^{(1)} \quad \beta = -\vec{a}^{(3)} \vec{e}^{(2)}$$

$$\vec{b}^{(3)} = \vec{a}^{(3)} - (a^{(3)} e^{(1)})\vec{e}^{(1)} - (a^{(3)} a^{(2)}) e^{(2)} = (\hat{I} - \vec{e}^{(1)} \circ \vec{e}^{(1)} - \vec{e}^{(2)} \circ \vec{e}^{(2)})\, \vec{a}^{(3)}$$

$$\vec{e}^{(3)} = \frac{\vec{b}^{(3)}}{|\vec{b}^{(3)}|}$$

Gram-Schmidt

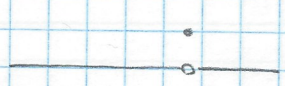


$$\rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \\ f(x_n) \end{pmatrix}$$

$$\sum_k f(x_k)\, g(x_k)$$

$$\downarrow$$

$$(f, g) = \int f(x)\, g(x)\, dx$$



$$f(x) = \begin{cases} 0 & x \neq 1 \\ 5 & x = 1 \end{cases}$$

„Majdnem" azonos $\Longleftrightarrow$ nullmértékű halmazon különbözik

$$\|f\| = \sqrt{(f, f)}$$

$$\cos \text{szög}(f, g) = \frac{(f, g)}{\|f\| \cdot \|g\|}$$

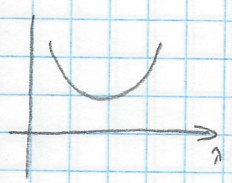
$$F \ni f, g$$

$$h = f + \lambda g \quad \lambda \in \mathbb{R}$$

$$(h, h) \geq 0$$

$$(h, h) = (f + \lambda g, f + \lambda g) = \underbrace{(f, f)}_{A} + \lambda \underbrace{(g, f)}_{B} + \lambda \underbrace{(f, g)}_{B} + \lambda^2 \underbrace{(g, g)}_{C > 0} \geq 0$$

$$C\lambda^2 + 2B\lambda + A \geq 0$$



$$(2B)^2 - 4CA \leq 0$$

$$B^2 - CA \leq 0$$

$$B^2 \leq CA \qquad \sqrt{C} = \sqrt{(g, g)} = \|g\|$$

$$B \leq \sqrt{C}\sqrt{A} \qquad \sqrt{A} = \sqrt{(f, f)} = \|f\|$$

$$|(f, g)| \leq \|f\| \cdot \|g\|$$

$$-1 \leq \underbrace{\frac{(f, g)}{\|f\| \cdot \|g\|}}_{\cos\varphi} \leq 1$$

$$(f, g) = \|f\| \cdot \|g\| \cdot \cos\varphi$$

HF. Hétkomponensű oszlopvektorok normái és szöge kiszámolni

$\underline{v} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$

$\underline{v}\,\underline{v} = 1 \cdot 1 + (2i)(2i) = 1 - 4 = -3$

$\underline{a}\,\underline{b} = \sum_k a_k^* b_k$

$\underline{a}\,\underline{a} = \sum a_k^* a_k = \sum_k |a_k|^2 \geq 0$

$(f, g) = \int f^*(x) \cdot g(x)\, dx$

$(f, f) = \int f^*(x) f(x)\, dx = \int |f|^2(x)\, dx$

$f \in L^2$

Lebesgue

$|(f, g)| \leq \sqrt{(f, f)} \cdot \sqrt{(g, g)}$

$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$

$\underline{a}\,\underline{b} = \sum_n^{\infty} a_k^* b_k$

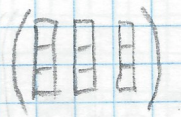
$\underline{a}\,\underline{a} = \sum_k |a_k|^2 < \infty$

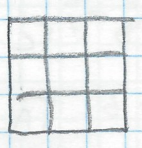
$\boxed{\underline{a}\,\underline{b} = (\underline{b}\,\underline{a})^*}$ $\quad \ell^2$

$\vec{a}, \vec{b}, \vec{c}$ $\qquad (\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a}\,(\vec{b} \times \vec{c})$

$\downarrow$

$\mathbb{R}^3 \ni \underline{a}, \underline{b}, \underline{c}$ $\qquad (\underline{a}, \underline{b}, \underline{c}) = \sum_k \sum_l \sum_m \varepsilon_{klm}\, a_k b_l c_m$



$\underline{\underline{A}} =$  $\qquad \det \underline{\underline{A}} = \begin{vmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{vmatrix}$

$\vec{u} = \hat{A}\,\vec{v}$

$\downarrow$

$\underline{u} = \underline{\underline{A}}\,\underline{v}$

$A_{kl} = \vec{e}^{(k)} \underbrace{(\hat{A}\,\vec{e}^{(l)})}_{\vec{f}^{(l)}} = \vec{e}^{(k)}\,\vec{f}^{(l)} = (\underline{f}^{(l)})_k$

$\hat{A} \downarrow \quad \begin{matrix} \vec{e}^{(1)} & \vec{e}^{(2)} & \vec{e}^{(3)} \\ \vec{f}^{(1)} & \vec{f}^{(2)} & \vec{f}^{(3)} \\ \underline{f}^{(1)} & \underline{f}^{(2)} & \underline{f}^{(3)} \end{matrix}$

$\underline{\underline{A}} = \left[\underline{f}^{(1)}\right] \left[\underline{f}^{(2)}\right] \left[\underline{f}^{(3)}\right]$ Determináns fogalma, jelentése, alakzat nagyítása

$\det \underline{\underline{A}} = (\underline{f}^{(1)}, \underline{f}^{(2)}, \underline{f}^{(3)}) = (\vec{f}^{(1)}, \vec{f}^{(2)}, \vec{f}^{(3)}) = (\hat{A}\vec{e}^{(1)}, \hat{A}\vec{e}^{(2)}, \hat{A}\vec{e}^{(3)})$

$$\boxed{\det(\underline{\underline{A}}\,\underline{\underline{B}}) = (\det \underline{\underline{A}}) \cdot (\det \underline{\underline{B}})}$$

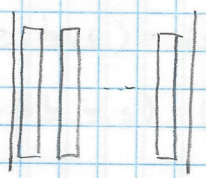
$(\vec{a}^{(1)} + \vec{a}^{(2)}, \vec{b}, \vec{c}) = (a^{(1)}, b, c) + (a^{(2)}, b, c)$

$(\lambda a, b, c) = \lambda (a, b, c)$

V $\dim V = n$

$\vec{a}^{(1)}, \dots \vec{a}^{(n)}$

$V \times V \times \dots V \to \mathbb{R}$



$\Rightarrow$ multilineáris, alternáló függvény

$P_{kl} = e_k e_l \to$ vetítés determinánsa nulla

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$V_3 \ni \vec{a}, \vec{b}, \vec{c}$ $(\vec{a}, \vec{b}, \vec{c}) = (\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

$V \times V \times V \to \mathbb{R}$

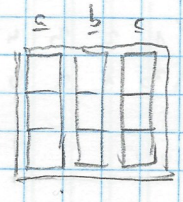
geometriai jelentés

$(\vec{a}, \vec{b}, \vec{c}) = (\vec{b}, \vec{c}, \vec{a}) = (\vec{c}, \vec{a}, \vec{b}) =$

$= -(\vec{b}, \vec{a}, \vec{c}) = -(\vec{c}, \vec{b}, \vec{a}) = -(\vec{a}, \vec{c}, \vec{b})$

$\vec{a} \to \underline{a}$
$\vec{b} \to \underline{b}$
$\vec{c} \to \underline{c}$

$$(\underline{c}, \underline{b}, \underline{c}) = (\underline{a} \times \underline{b}) \cdot \underline{c} = \sum_k \sum_l \sum_m \varepsilon_{klm} a_k b_l c_m$$



$\hat{A} \to \underline{\underline{A}} = \left[f^{(1)}\right] \left[f^{(2)}\right] \left[f^{(3)}\right]$ $\det(\underline{\underline{A}}) = \det(\tilde{\underline{\underline{A}}})$

$\hat{A}\vec{e}^{(1)} = \vec{f}^{(1)}$

$$\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$\det \underline{\underline{A}} = \sum_k \sum_l \sum_m \varepsilon_{klm} \, a_{k1} \cdot a_{l2} \cdot c_{m3}$$

Alternáló: két oszlop felcserélésével előjelet vált (akkor mondjuk, ha az eredmény skalár)

$$(\vec{a} \times \vec{b}) \cdot (\vec{c}_1 + \vec{c}_2) = (\vec{a} \times \vec{b}) \vec{c}_1 + (\vec{a} \times \vec{b}) \vec{c}_2$$

$$(\vec{a} \times \vec{b}) \cdot \lambda \vec{c} = \lambda \left((\vec{a} \times \vec{b}) \cdot \vec{c} \right)$$

$$(\underline{a}, \underline{b}, \underline{c}_1 + \underline{c}_2) = (\underline{a}, \underline{b}, \underline{c}_1) + (\underline{a}, \underline{b}, \underline{c}_2)$$

$$(\underline{a}, \underline{b}, \lambda \underline{c}) = \lambda \, (\underline{a}, \underline{b}, \underline{c})$$

Multilineáris: minden változójára lineáris (bilineáris, trilineáris, ...)

n vektorra értelmezett, multilineáris, alternáló $\longrightarrow$ térfogati forma

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix}$$

$$q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix}$$

$$r = qp = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 2 & 3 & 5 \end{pmatrix}$$

permutációk csoportot alkotnak $\longrightarrow$ S_n szimm. csoport

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 3 & 1 & 4 \end{pmatrix} \qquad 1 \to 2 \to 5 \to 4$$

$(1\ 2\ 5\ 4)\ (3)$ ciklusok

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 5 & 4 \end{pmatrix} \qquad (1\ 3\ 2)(4\ 5)$$

$(1)(2\ 4\ 3)(5)$

1 2 4 5
2 1 4 5
2 4 1 5
2 4 5 1 transzpozíciók $\longrightarrow$ minden permutáció felbontható transzpozíciókra

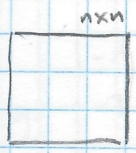
$\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$ $\qquad$ $\begin{pmatrix}1&2&3\\3&2&1\end{pmatrix}$

1 2 3
2 1 3
2 3 1

1 2 3
2 1 3
2 3 1
3 2 1

páratlan permutáció $\qquad$ páros permutáció

$$\varepsilon_{k_1 k_2 \dots k_n} = \begin{cases} 0 & \\ 1 & \text{ps} \\ -1 & \text{pt} \end{cases}$$



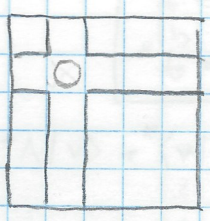
$$\det \underline{\underline{A}} = \sum_{k_1} \sum_{k_2} \dots \sum_{k_n} \varepsilon_{k_1 k_2 \dots k_n} a_{k_1 1} a_{k_2 2} \dots a_{k_n n}$$

$(\underline{a}, \underline{a}, \underline{c}) = 0$

$(\underline{a}+\underline{b}, \underline{b}, \underline{c}) = (\underline{a}, \underline{b}, \underline{c}) + (\underline{b}, \underline{b}, \underline{c}) = (\underline{a}, \underline{b}, \underline{c})$

$(\underline{a}+\beta\underline{b}, \underline{b}, \underline{c}) = (\underline{a}, \underline{b}, \underline{c}) + (\beta\underline{b}, \underline{b}, \underline{c}) = (\underline{a}, \underline{b}, \underline{c}) + \beta(\underline{b}, \underline{b}, \underline{c}) = (\underline{a}, \underline{b}, \underline{c})$

Determináns → aldeterminánsa



Kifejtési tétel: $a_{k1} \cdot A_{k1} + a_{k2} A_{k2} + \dots$

$$\sum_{\ell} (-1)^{k+\ell} a_{k\ell} A_{k\ell} = \det \underline{\underline{A}}$$

2×2-es: $\det \underline{\underline{A}} = \sum_k \sum_\ell \varepsilon_{k\ell} a_k b_\ell = a_1 b_2 - a_2 b_1 = \begin{pmatrix} a_1 \\ a_2 \\ 0 \end{pmatrix} \times \begin{pmatrix} b_1 \\ b_2 \\ 0 \end{pmatrix}$

$\varepsilon_{11} = 0 \quad \varepsilon_{12} = 1$

$\varepsilon_{22} = 0 \quad \varepsilon_{21} = -1$

$$\begin{vmatrix} \operatorname{ch} 100 & \operatorname{sh} 100 \\ \operatorname{sh} 100 & \operatorname{ch} 100 \end{vmatrix} = \operatorname{ch}^2 100 - \operatorname{sh}^2 100 = 1$$

Determinánsa csak négyzetes mátrixnak van

$(\underline{\underline{\tilde{A}}})_{kl} = A_{lk}$

$\underline{\underline{\hat{A}}} = \underline{\underline{A}}$ szimm.

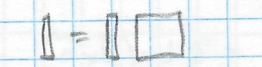
$\underline{\underline{\hat{A}}} = -\underline{\underline{A}}$ antiszimm.

$(\underline{\underline{\widetilde{AB}}}) = \underline{\underline{\tilde{A}}}\,\underline{\underline{\tilde{B}}}$

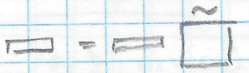
$(AB)_{kl} = \sum_m A_{km} B_{ml}$

$(\widetilde{AB})_{kl} = (AB)_{lk} = \sum_m A_{lm} B_{mk} = \sum_m (\tilde{A})_{ml} (\tilde{B})_{km} = \sum_m (\tilde{B})_{km} (\tilde{A})_{ml} =$

$= (\tilde{B}\tilde{A})_{kl}$

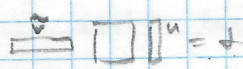


$\underline{b} = \underline{\underline{A}}\,\underline{a}$ $\quad b_k = \sum_l A_{kl} a_l$



$\underline{\tilde{b}} = \underline{\tilde{a}}\,\underline{\underline{\tilde{A}}}$ $\quad (\tilde{b})_k = \sum_l a_l (\tilde{A})_{lk} = \sum_l A_{kl} a_l$

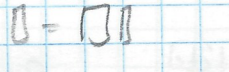
$\underline{\tilde{u}}\,\underline{\underline{A}}\,\underline{v} = \lambda$



$\underline{\tilde{u}}\,\underline{\underline{A}}\,\underline{v} = \underline{\tilde{v}}\,\underline{\underline{A}}\,\underline{u}$

$\underline{\underline{A^+}} = (\underline{\underline{\tilde{A}}})^*$ $\quad (\underline{\underline{AB}})^+ = \underline{\underline{B^+}}\,\underline{\underline{A^+}}$

$\underline{b} = \underline{\underline{A}}\,\underline{a}$



$\underline{b^+} = \underline{a^+}\,\underline{\underline{A^+}}$

$$u^+ A v = \lambda$$

$$v^+ A u = \lambda^*$$

$\langle u|A|v\rangle^* = \langle v|A^+|u\rangle$

$\underline{\tilde{u}} = \underline{\underline{\hat{A}}}\,\underline{\tilde{v}}$

$\underline{\tilde{v}} = \underline{\underline{\hat{B}}}\,\underline{\tilde{u}}$

$\underline{\underline{\hat{B}}} = \underline{\underline{\hat{A}^{-1}}}$ $\quad (AB)^{-1} = B^{-1} A^{-1}$

$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = \underline{\underline{I}}$

$\det(AB) = (\det A)(\det B)$

$\det A^{-1} = \frac{1}{\det A}$

$Sp\,\underline{\underline{A}} = Tr\,\underline{\underline{A}} = \sum_k A_{kk}$

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GY

$Sp(\underbrace{\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}}_{\underline{\underline{F}}}) = Sp(\underbrace{\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{B}}}_{\underline{\underline{G}}})$

$F_{kl} = \sum_p \sum_q A_{kp} B_{pq} C_{ql}$

$G_{kl} = \sum_p \sum_q C_{kp} A_{pq} B_{ql}$

$$\mathrm{Sp}\,\underline{\underline{F}} = \sum_k F_{kk} = \sum_k \sum_p \sum_q A_{kp} B_{pq} C_{qk} = \sum_k \sum_p \sum_q C_{qk} A_{kp} B_{pq} = \sum_q \left(\sum_k \sum_p C_{qk} A_{kp} B_{pq} \right) =$$

$$= \sum_q (CAB)_{qq} = \sum_q G_{qq} = \mathrm{Sp}\,\underline{\underline{G}}$$

$$\begin{vmatrix} 5 & 3 & 4 & 1 & 3 \\ 2 & 1 & 0 & -1 & 2 \\ -1 & -7 & 1 & 2 & -1 \\ 3 & 5 & 3 & 4 & 6 \\ 4 & 2 & 2 & 0 & 2 \end{vmatrix} \overset{\substack{1o-2\times 5o \\ 2o-5o \\ 3o-5o}}{=\!=} \begin{vmatrix} -1 & 0 & 1 & 1 & 3 \\ -2 & -1 & -2 & -1 & 2 \\ 1 & -6 & 2 & 2 & -1 \\ -9 & -1 & -3 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 \end{vmatrix} = 2 \cdot \begin{vmatrix} -1 & 0 & 1 & 1 \\ -2 & -1 & -2 & -1 \\ 1 & -6 & 2 & 2 \\ -9 & -1 & -3 & 4 \end{vmatrix} =$$

$$= 2 \cdot \begin{vmatrix} -1 & 0 & 1 & 1 \\ -2 & -1 & -2 & -1 \\ 13 & 0 & 14 & 8 \\ 7 & 0 & -1 & 5 \end{vmatrix} = 2(-1) \cdot \begin{vmatrix} -1 & 1 & 1 \\ 13 & 14 & 8 \\ -7 & -1 & 5 \end{vmatrix} = -2 \cdot \begin{vmatrix} 0 & 0 & 1 \\ 21 & 6 & 8 \\ -2 & -6 & 5 \end{vmatrix} =$$

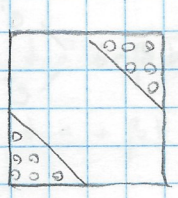
$$= -2 \cdot \begin{vmatrix} 21 & 6 \\ -2 & -6 \end{vmatrix} = -2 \cdot \begin{vmatrix} 19 & 0 \\ -2 & -6 \end{vmatrix} = (-2) \cdot 19 \cdot (-6) = 228$$

$$F_n(a,b) = \begin{vmatrix} a & b & b & \cdots & b \\ b & a & & & \\ b & & a & & \\ \vdots & & & \ddots & \\ b & & & & a \end{vmatrix} = b^n \begin{vmatrix} \frac{a}{b} & 1 & 1 & \cdots & 1 \\ 1 & \frac{a}{b} & & & \\ 1 & & \ddots & & \\ \vdots & & & \ddots & \\ 1 & & & & \frac{a}{b} \end{vmatrix} = b^n f_n\left(x = \tfrac{a}{b}\right)$$

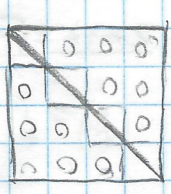
$$f_n(x) = \begin{vmatrix} x & 1 & 1 & \cdots & 1 \\ 1 & x & & & \\ 1 & & x & & \\ \vdots & & & \ddots & \\ 1 & & & & x \end{vmatrix} = \begin{vmatrix} x-1 & 0 & & & 1 \\ 0 & x-1 & & & \\ 0 & 0 & x-1 & & 1 \\ \vdots & \vdots & & \ddots & 1 \\ 1-x & 1-x & \cdots & & x \end{vmatrix} = (x-1)^{n-1} \cdot \begin{vmatrix} 1 & 0 & 0 & \cdots & 1 \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & 1 \\ & & & \ddots & 1 \\ -1 & -1 & -1 & \cdots & x \end{vmatrix} =$$

$$= (x-1)^{n-1} \cdot \begin{vmatrix} 1 & 0 & \cdots & & 1 \\ 0 & 1 & & & 1 \\ \vdots & & 1 & & 1 \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & (x+n-1) \end{vmatrix} = (x-1)^{n-1} \cdot (x+n-1)$$

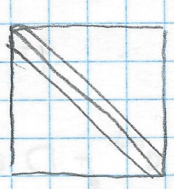
$$F_n(a,b) = b^n \cdot f_n\left(\tfrac{a}{b}\right) = b^n \cdot \left(\tfrac{a}{b} - 1\right)^{n-1} \cdot \left(\tfrac{a}{b} + n - 1\right) = (a-b)^{n-1} \cdot (a + (n-1)b)$$



sávmátrix



diagonálmátrix



kontinuáns mátrix

$$M_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{pmatrix}$$

$$|M - xI| = \begin{vmatrix} 2-x & -1 & & & \\ -1 & 2-x & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2-x \end{vmatrix} = A_n(x) = (2-x)\cdot A_{n-1}(x) - (-1)\cdot \begin{vmatrix} -1 & & \\ 0 & 2-x & -1 \\ \vdots & -1 & \ddots \end{vmatrix} =$$

$$= (2-x)\,A_{n-1} - A_{n-2}$$

$$\boxed{A_n(x) = (2-x)\cdot A_{n-1} - A_{n-2}} \quad \rightarrow \text{homogén, lineáris, másodrendű rekurzió}$$

$$A_n = \alpha A_{n-1} + \beta A_{n-2} + \dots + \mu A_{n-k}$$

TFH $A_n = q^n$

$$q^n = \alpha q^{n-1} + \beta q^{n-2} + \dots + \mu q^{n-k} \qquad / : q^{n-2}$$

$$\boxed{q^k = \alpha \cdot q^{k-1} + \beta \cdot q^{k-2} + \dots + \mu \cdot 1}$$

$q_1, q_2, \dots, q_k \in \mathbb{C}$

$A_n = q_1^n, q_2^n, \dots, q_k^n$

$$A_n = c_1 \cdot q_1^n + c_2 q_2^n + \dots + c_k \cdot q_k^n \qquad c \in \mathbb{C}$$

$2 - x = 2\cdot\cos\alpha$

$x = 2 - 2\cos\alpha = 2\cdot(1-\cos\alpha) = 2\left[1 - \left(\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}\right)\right] = 4\cdot\sin^2\frac{\alpha}{2}$

$A_n(\alpha) = 2\cos\alpha\, A_{n-1} - A_{n-2}$

TFH. $A_n = q^n$

$q^n = 2\cos\alpha \cdot q^{n-1} - q^{n-2}$

$q^2 = 2\cos\alpha \cdot q - 1$

$q^2 - 2\cos\alpha \cdot q + 1 = 0$

$$q = \frac{2\cos\alpha \pm \sqrt{(2\cos\alpha)^2 - 4}}{2} = \frac{2\cos\alpha \pm \sqrt{-4\cdot\sin^2\alpha}}{2} = \cos\alpha \pm i\cdot\sin\alpha \quad \begin{matrix} / \; e^{i\alpha} \\ \backslash \; e^{-i\alpha} \end{matrix}$$

$A_n = C\,e^{in\alpha} + D\,e^{-in\alpha}$

$A_1 = 2\cos\varphi$

$A_2 = \begin{vmatrix} 2\cos\varphi & 1 \\ -1 & 2\cos\varphi \end{vmatrix} = 4\cos^2\varphi - 1 = 2\cos\varphi \cdot A_1 - A_0$

$A_0 = 1 \qquad A_{-1} = 0$

$A_n = C \cdot e^{in\varphi} + D\, e^{in\varphi}$

$A_1 = 2\cos\varphi$

$A_0 = 1$

$A_n = C e^{i\varphi} + D e^{-i\varphi} = 2\cos\varphi$

$A_0 = C + D = 1$

$C\underbrace{(e^{i\varphi} - e^{-i\varphi})}_{2i\cdot\sin\varphi} = 2\cos\varphi - e^{-i\varphi}$ $\qquad e^{i\varphi} + e^{-i\varphi} - e^{-i\varphi}$

$D\underbrace{(e^{i\varphi} - e^{-i\varphi})}_{-2i\cdot\sin\varphi} = \underbrace{2\cos\varphi - e^{i\varphi}}_{e^{-i\varphi}}$

$C = \dfrac{e^{i\varphi}}{2i\cdot\sin\varphi}$ $\qquad D = \dfrac{-e^{-i\varphi}}{2i\cdot\sin\varphi}$

$A_n = \dfrac{e^{i\varphi}}{2i\sin\varphi}\cdot e^{in\varphi} - \dfrac{e^{-i\varphi}}{2i\sin\varphi}\cdot e^{-in\varphi} = \dfrac{e^{i(n+1)\varphi} - e^{-i(n+1)\varphi}}{2i\cdot\sin\varphi} \Rightarrow \boxed{A_n = \dfrac{\sin(n+1)\varphi}{\sin\varphi}}$

$x = 4\cdot\sin^2\dfrac{\varphi}{2}$

$? \; A_n = 0 \qquad (n+1)\varphi = k\pi \qquad k \in \{1, \dots, n\}$

$\varphi_k = \dfrac{k\pi}{n+1}$

$x = 4\cdot\sin^2\left(\dfrac{k\pi/2}{n+1}\right)$

Fizikában:

$\begin{vmatrix} 2-x & -1 & & \\ -1 & 2-x & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 1-x \end{vmatrix}$

$\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^{n-1} \\ 1 & x_3 & x_3^2 & \dots & x_3^{n-2} & x_3^{n-1} \\ \vdots & & & & & \vdots \\ 1 & x_n & \dots & \dots & \dots & x_n^{n-1} \end{vmatrix}$

Van der Monde determináns

2017. 10. 19.

EA

$\vec{v} = \lambda\vec{a} + \beta\vec{b}$

$\underline{v} = \lambda\underline{a} + \beta\underline{b} \qquad \underline{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 5 \\ 7 \end{pmatrix} \quad \underline{v} = \begin{pmatrix} 9 \\ 7 \end{pmatrix}$

$$\begin{pmatrix}9\\7\end{pmatrix} = \alpha\begin{pmatrix}2\\-1\end{pmatrix} + \beta\begin{pmatrix}5\\7\end{pmatrix}$$

$9 = \alpha 2 + \beta 5$

$7 = \alpha\cdot(-1) + \beta 7$

$2\alpha + 5\beta = 9$

$-\alpha + 7\beta = 7$

( )x + ( )y + ... + ( )u = ( )

( )x + ...

( )

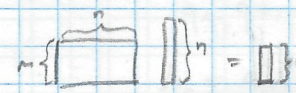
$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n$

$$\boxed{\sum_{k=1}^{n} a_{kl}x_l = b_k} \Rightarrow \boxed{\underline{\underline{A}}\,\underline{x} = \underline{b}}$$



Span $(\vec{a}^{(n)}, \vec{a}^{(n)}, \dots, \vec{a}^{(n)}) < V$

$x_1\underline{a}^{(n)} + x_2\underline{a}^{(n)} + \dots + x_n\underline{a}^{(n)} = \underline{b}$

Van megoldás $\Leftrightarrow$ $\underline{b} \in$ Span $(\underline{a}^{(n)} \dots \underline{a}^{(n)})$

$g(\underline{a}^{(n)} \dots \underline{a}^{(n)}) = \dim$ Span $(\underline{a}^{(n)} \dots \underline{a}^{(n)})$

ran

$g \le \min(m,n)$

Ha $g(\underline{a}^{(n)}, \dots, \underline{a}^{(n)}, \underline{b}) = g(\underline{a}^{(n)}, \dots, \underline{a}^{(n)})$, akkor van megoldás

$\underline{\underline{A}}\,\underline{x} = \underline{0}$ → homogén egyenletrendszer → Mindig van megoldás! → Van-e nemtriviális?

↓

A megoldások lineáris tere alteret

$\underline{x} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$ triviális megoldás

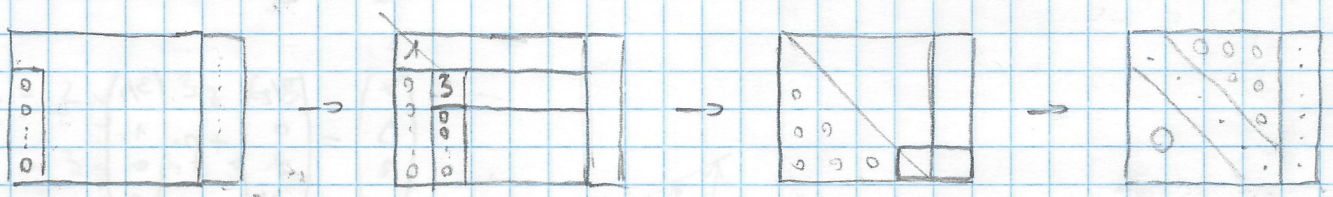
$\underline{\underline{A}}\,\underline{y} = \underline{b}$ $\qquad$ $\underline{\underline{A}}\,\underline{x} = \underline{0}$

$\underline{y}$ $\qquad$ $\underline{x}$

$\underline{z} = \underline{x} + \underline{y}$

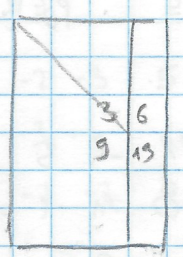
$\underline{\underline{A}}\,\underline{z} = \underline{\underline{A}}(\underline{x}+\underline{y}) = \underline{\underline{A}}\,\underline{x} + \underline{\underline{A}}\,\underline{y} = \underline{0} + \underline{b} = \underline{b}$

Gauss-féle elimináció:

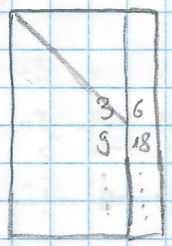


$0\ 0\ 0\ 0\ 5 \mid 10$ $\qquad$ $5 \mid 0 \rightarrow x_n = 0$

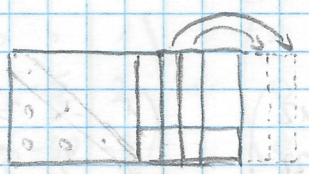
$0 \mid 5 \rightarrow$ nincs mo.



ellentmondás



több egyenlet
de nincs új információ



$5x_4 + 2x_5 + 7x_6 = 1$

$5x_4 = 1 - 2x_5 - 7x_6$

$x_5 = u \qquad x_6 = v$

$5x_4 = 1 - 2u - 7v$

$\Rightarrow$ végtelen sok mo.

2017. 10. 19.
GY

$2x - 3y - 4z + 5u = -13$

$5x - 6y + z - u = 15$

$6x - 9y + z + 2u = 13$

$2x - 3y - 2z - 4u = 9$

$$\left(\begin{array}{cccc|c} 2 & -3 & -4 & 5 & -13 \\ 5 & -6 & 1 & -1 & 15 \\ 6 & -9 & 1 & 2 & 13 \\ 2 & -3 & 2 & -4 & 9 \end{array}\right) \sim \left(\begin{array}{cccc|c} 2 & -3 & -4 & 5 & -13 \\ 0 & 0 & 9 & -11 & 40 \\ 0 & 0 & 13 & -13 & 52 \\ 0 & 0 & 2 & -9 & 22 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 2 & -4 & 5 & -13 & 3 \\ 0 & 9 & -11 & 40 & 0 \\ 0 & 13 & -13 & 52 & 0 \\ 0 & 2 & -9 & 22 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|cc} 2 & -4 & 5 & & -13 & 3 \\ 0 & 1 & -1 & & 4 & 0 \\ 0 & 5 & -11 & & 40 & 0 \\ 0 & 2 & -3 & & 22 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 2 & -4 & 5 & & -13 & 3 \\ 0 & 1 & -1 & & 4 & 0 \\ 0 & 0 & -2 & & 4 & 0 \\ 0 & 0 & -7 & & 14 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 2 & -4 & 5 & & -13 & 3 \\ 0 & 1 & -1 & & 4 & 0 \\ 0 & 0 & 1 & & -2 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|cc} 2 & -4 & 0 & & -3 & 3 \\ 0 & 1 & 0 & & 2 & 0 \\ 0 & 0 & 1 & & -2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|cc} 2 & 0 & 0 & & 5 & 3 \\ 0 & 1 & 0 & & 2 & 0 \\ 0 & 0 & 1 & & -2 & 0 \end{array}\right)$$

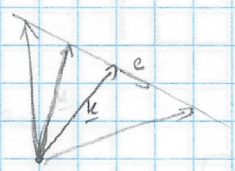
$2x = 5 + 3p$

$y = p$

$z = 2$

$u = -2$

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = \begin{pmatrix} 5/2 \\ 0 \\ 2 \\ -2 \end{pmatrix} + p \cdot \begin{pmatrix} 3/2 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$



$\underline{r} = \underline{k} + p \cdot \underline{e} \qquad p \in \mathbb{R}$

$$\left(\begin{array}{ccc|c} 2 & 3 & 5 & 12 \\ 1 & -4 & 3 & -22 \\ 3 & -1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & -22 \\ 2 & 3 & 5 & 12 \\ 3 & -1 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & -22 \\ 0 & 11 & -1 & 56 \\ 0 & 11 & -11 & 66 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & -22 \\ 0 & 11 & -1 & 56 \\ 0 & 0 & -10 & 10 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & -4 & 3 & -22 \\ 0 & 11 & -1 & 56 \\ 0 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 0 & -19 \\ 0 & 11 & 0 & 55 \\ 0 & 0 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -4 & 0 & -19 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -1 \end{array}\right)$$

$x + 3y + z + u = 0$

$7x - 5y - z + 5u = 0$

$3x + y - z + 2u = 0$

$5x + 7y + z + 4u = 0$

$$\left(\begin{array}{cccc|} 1 & 3 & 1 & 1 \\ 7 & 5 & -1 & 5 \\ 3 & 1 & -1 & 2 \\ 5 & 7 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{cccc|} 1 & 3 & 1 & 1 \\ 0 & -16 & -8 & -2 \\ 0 & -8 & -4 & -1 \\ 0 & -8 & -4 & -1 \end{array}\right) \sim \left(\begin{array}{cccc|} 1 & 3 & 1 & 1 \\ 0 & 8 & 4 & 1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 1 & 3 & -1 & -1 \\ 0 & 8 & -4 & -1 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cc|cc} 8 & 24 & -8 & -8 \\ 0 & 8 & -4 & -1 \end{array}\right) \sim \left(\begin{array}{cc|cc} 8 & 0 & 4 & -5 \\ 0 & 8 & -4 & -1 \end{array}\right)$$

$8x = 4p - 5q$

$8y = -4p - q$

$z = p$

$u = q$

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = p \begin{pmatrix} 1/2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} + q \begin{pmatrix} -5/8 \\ -1/8 \\ 0 \\ 1 \end{pmatrix}$$

$\underline{r} = p\underline{e} + q\underline{f}$

sík

$3x - 2y - 3z = 5$

$x - 3y + 4z = 1$

$7x - 7y - 2z = 0$

$$\left(\begin{array}{ccc|c} 3 & -2 & -3 & 5 \\ 1 & -3 & 4 & 1 \\ 7 & -7 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 3 & -2 & -3 & 5 \\ 7 & -7 & -2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 0 & 7 & -15 & 2 \\ 0 & 14 & -30 & -7 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -3 & 4 & 1 \\ 0 & 7 & -15 & 2 \\ 0 & 0 & 0 & -11 \end{array}\right) \lightning$$

$\underline{v} = \alpha \underline{a} + \beta \underline{b} + \gamma \underline{c}$

$$\alpha = \frac{(\underline{v}, \underline{b}, \underline{c})}{(\underline{a}, \underline{b}, \underline{c})} \qquad \beta = \frac{(\underline{a}, \underline{v}, \underline{c})}{(\underline{a}, \underline{b}, \underline{c})} \qquad \gamma = \frac{(\underline{a}, \underline{b}, \underline{v})}{(\underline{a}, \underline{b}, \underline{c})}$$

Cramer-szabály $\rightarrow$ Ha $(\underline{a}, \underline{b}, \underline{c}) = 0$, akkor nincs megoldás

$$\left(\begin{array}{cccc|} 2 & -1 & -1 & 1 \\ 1 & -1 & -1 & -2 \\ 5 & -1 & -3 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|} 1 & -1 & -1 & -2 \\ 2 & -1 & -1 & 1 \\ 5 & -1 & -3 & 2 \end{array}\right) \sim \left(\begin{array}{cccc|} 1 & -1 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 4 & 2 & 12 \end{array}\right) \sim \left(\begin{array}{cccc|} 1 & -1 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -8 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{cccc|} 1 & -1 & -1 & -2 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & 1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & -1 & 2 \\ 0 & 1 & 1 & -5 \\ 0 & 0 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

$x = -3u$

$y = -u$

$z = -4u$

$u = u$

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = u \begin{pmatrix} -3 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & 3 & \\ 2 & 1 & -1 & -4 & -3 & \\ 2 & -1 & -2 & -5 & -1 & \\ 1 & 0 & 2 & 4 & 1 & \\ 3 & -1 & 0 & -1 & 0 & \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & 3 & \\ 0 & 5 & -3 & -10 & -9 & \\ 0 & 3 & -4 & -11 & -7 & \\ 0 & 2 & 1 & 1 & -2 & \\ 0 & 5 & -3 & -10 & -9 & \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & 3 & \\ 0 & 5 & -3 & -10 & -9 & \\ 0 & 1 & -5 & -12 & -5 & \\ 0 & 2 & 1 & 1 & -2 & \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & 3 & \\ 0 & 1 & -5 & -12 & -5 & \\ 0 & 5 & -3 & -10 & -9 & \\ 0 & 2 & 1 & 1 & -2 & \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & -2 & 1 & 3 & 3 & \\ 0 & 1 & -5 & -12 & -5 & \\ 0 & 0 & 22 & 50 & 16 & \\ 0 & 0 & 11 & 25 & 8 & \end{array}\right) \sim \left(\begin{array}{ccccc|c} 11 & -2 & 1 & 3 & 3 & \\ 0 & 1 & -5 & -12 & -5 & \\ 0 & 0 & 11 & 25 & 8 & \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 1 & -2 & 1 & -3 & -3 \\ 0 & 1 & -5 & 12 & 5 \\ 0 & 0 & 11 & -25 & -8 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 11 & -22 & 11 & -33 & -33 \\ 0 & 11 & -55 & 132 & 55 \\ 0 & 0 & 11 & -25 & -8 \end{array}\right) \sim \left(\begin{array}{ccc|cc} 11 & -22 & 0 & -8 & -8 \\ 0 & 11 & 0 & 7 & 15 \\ 0 & 0 & 11 & -25 & -8 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|cc} 11 & 0 & 0 & 6 & 5 \\ 0 & 11 & 0 & 7 & 15 \\ 0 & 0 & 11 & -25 & -8 \end{array}\right)$$

$$x = \frac{6p}{11} + \frac{5q}{11}$$

$$y = \frac{7p}{11} + \frac{15q}{11}$$

$$z = \frac{-25p}{11} - \frac{8q}{11}$$

$$u = p = 11s$$

$$v = q = 11t$$

$$\begin{pmatrix} x \\ y \\ z \\ u \\ v \end{pmatrix} = s \begin{pmatrix} 6 \\ 7 \\ -25 \\ 11 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ 15 \\ -8 \\ 0 \\ 11 \end{pmatrix}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b} = \sum_k b_k \underline{e}^{(k)}$$

$$\underline{\underline{A}}\,\underline{y}^{(1)} = \underline{e}^{(1)}$$

$$\underline{\underline{A}}\,\underline{y}^{(2)} = \underline{e}^{(2)}$$

$$\vdots$$

$$\underline{\underline{A}}\,\underline{y}^{(n)} = \underline{e}^{(n)}$$

$$\underline{x} = \sum_k b_k \underline{y}^{(k)}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{\underline{A}}\left(\sum_k b_k \underline{y}^{(k)}\right) = \sum_k b_k \left(\underline{\underline{A}}\,\underline{y}^{(k)}\right) = \sum_k b_k \underline{e}^{(k)} = \underline{b}$$

$\Rightarrow$ analóg az inhomogén lin. differenciálegyenletek megoldásával (ld. Regészek)

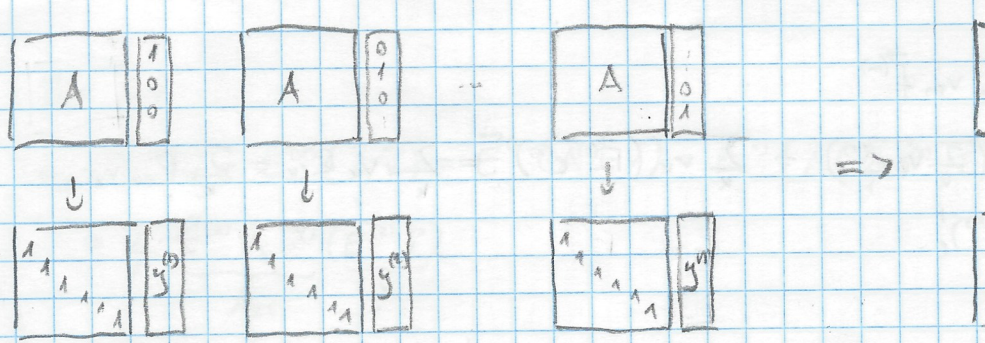
$$x_i = \sum_k b_k y_i^{(k)} = \sum_k Y_{ik} b_k$$

$$Y_{ik} = y_i^{(k)}$$

$$\underline{\underline{A}}\,\underline{x} = \underline{b}$$

$$\underline{x} = \underline{\underline{Y}}\,\underline{b}$$

$$\boxed{\underline{\underline{Y}} = \underline{\underline{A}}^{-1}}$$



$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ -1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 3 & 2 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right) \sim$$

$$\sim \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & -1 & 0 \\ 0 & 3 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|ccc} 6 & 0 & 0 & 6 & -3 & 0 \\ 0 & 6 & 0 & 2 & -2 & 2 \\ 0 & 0 & 6 & 0 & 3 & 0 \end{array}\right)$$

$$A^{-1} = \frac{1}{6} \cdot \begin{pmatrix} 6 & -3 & 0 \\ 2 & -2 & 2 \\ 0 & 3 & 0 \end{pmatrix}$$

$A = (a_{ik})$

$a_{ik} \to B_{ik}$

$$\frac{\tilde{B}}{\det A} = A^{-1}$$

2017. 10. 24.

EA

$\vec{e}^{(k)}$

$\vec{E}^{(k)}$ $\quad \vec{E}^{(i)} \vec{e}^{(k)} = \delta_{ik}$

$\forall \vec{v} \quad \exists v_k \in \mathbb{R}$

$* \quad \vec{v} = \sum_k v_k \vec{e}^{(k)}$

$$\vec{E}^{(i)} \vec{v} = \sum_k v_k \left(\vec{e}^{(k)} \vec{E}^{(i)}\right) = \sum_k v_k \delta_{ik}$$

$** \quad v_k = \vec{E}^{(k)} \vec{v}$

$$\vec{v} = \sum_k \left(\vec{E}^{(k)} \vec{v}\right) \cdot \vec{e}^{(k)} = \sum_k \vec{e}^{(k)} \left(\vec{E}^{(k)} \vec{v}\right) = \sum_k \left(\vec{e}^{(k)} \circ \vec{E}^{(k)}\right) \vec{v} = \left(\sum_k \vec{e}^{(k)} \circ \vec{E}^{(k)}\right) \vec{v} = \hat{I} \vec{v} \quad \forall \vec{v}$$

$\vec{f}^{(k)} \to \vec{F}^{(k)} \qquad \vec{F}^{(i)} \vec{f}^{(k)} = \delta_{ik}$

$$\sum_k \vec{f}^{(k)} \circ \vec{F}^{(k)} = \hat{I}$$

$\forall \vec{v} \quad \exists v'_k \in \mathbb{R} \qquad \vec{v} = \sum_k v'_k \vec{f}^{(k)}$

$$\vec{v} = \sum_k v_k \vec{e}^{(k)} = \sum_m v'_m \vec{f}^{(m)}$$

$$v'_\ell = \vec{F}^{(\ell)} \vec{v} = \vec{F}^{(\ell)} \left(\sum_k v_k \vec{e}^{(k)} \right) = \sum_k v_k \left(\vec{F}^{(\ell)} \vec{e}^{(k)} \right) = \sum_k v_k R_{\ell k} = \sum_k R_{\ell k} v_k$$

$$R_{\ell k} = \vec{F}^{(\ell)} \vec{e}^{(k)} = (e^{(k)})'_\ell$$

$$\underline{v}' = \underline{\underline{R}}\, \underline{v}$$

$$v_p = \vec{E}^{(p)} \vec{v} = \vec{E}^{(p)} \left(\sum_m v'_m f^{(m)} \right) = \sum_m v'_m \underbrace{\left(\vec{E}^{(p)} \vec{f}^{(m)} \right)}_{S_{pm}} = \sum_m S_{pm} v'_m$$

$$\underline{v} = \underline{\underline{S}}\, \underline{v}'$$

$$\underline{v} = \underline{\underline{S}}\, \underline{v}' = \underline{\underline{S}} (\underline{\underline{R}}\, \underline{v}) = (\underline{\underline{S}}\, \underline{\underline{R}})\, \underline{v} \quad \rightarrow \quad \underline{\underline{S}}\, \underline{\underline{R}} = \underline{\underline{I}} \quad \rightarrow \quad \underline{\underline{S}} = \underline{\underline{R}}^{-1}$$

$$(\underline{\underline{S}}\, \underline{\underline{R}})_{k\ell} = \sum_m S_{km} R_{m\ell} = \sum_m \left(\vec{E}^{(k)} \vec{f}^{(m)} \right) \left(\vec{F}^{(m)} \vec{e}^{(\ell)} \right) \overset{*}{=} \sum_m \vec{E}^{(k)} \left(\vec{f}^{(m)} \circ \vec{F}^{(m)} \right) \vec{e}^{(\ell)} =$$

$$= \vec{E}^{(k)} \left(\sum_m \vec{f}^{(m)} \circ \vec{F}^{(m)} \right) \vec{e}^{(\ell)} = \vec{E}^{(k)} \hat{I}\, \vec{e}^{(\ell)} = \vec{E}^{(k)} \vec{e}^{(\ell)} = \delta_{k\ell} = (\underline{\underline{I}})_{k\ell}$$

* $(\vec{a} \circ \vec{b})\, \vec{c} = \vec{a} (\vec{b} \vec{c})$

$\hat{d} \left[(\vec{a} \circ \vec{b})\, \vec{c} \right] = (\hat{d} \vec{a}) (\vec{b} \vec{c})$

Spec. ONB

$\vec{e}^{(k)} \vec{e}^{(\ell)} = \delta_{k\ell}$ $\qquad$ $\vec{f}^{(k)} \vec{f}^{(\ell)} = \delta_{k\ell}$

$\vec{E}^{(k)} = \vec{e}^{(k)}$ $\qquad$ $\vec{F}^{(k)} = \vec{f}^{(k)}$

$R_{\ell k} = \vec{F}^{(\ell)} \vec{e}^{(k)} = \vec{f}^{(\ell)} \vec{e}^{(k)}$

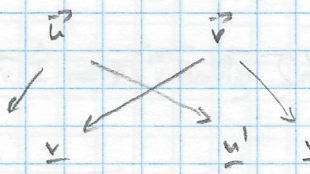
$S_{k\ell} = \vec{E}^{(k)} \vec{f}^{(\ell)} = \vec{e}^{(k)} \vec{f}^{(\ell)}$

$R_{\ell k} = S_{k\ell} \quad \rightarrow \quad \underline{\underline{R}} = \underline{\underline{\tilde{S}}}$

$R = S^{-1}$ $\qquad$ $S = R^{-1}$

ONB $R = S^+$ $\qquad$ $S = \tilde{R}$

$\boxed{R^{-1} = \tilde{R}}$ $\qquad$ ORTOGONÁLIS MÁTRIX



$$\underline{u}\, \underline{v} = \sum_k u_k v_k \qquad \underline{u}'\, \underline{v}' = \sum_k u'_k v'_k = \sum_k \left(\sum_\ell R_{k\ell} u_\ell \right) \left(\sum_m R_{km} v_m \right) = \sum_k \sum_\ell \sum_m R_{k\ell} R_{km} u_\ell v_m =$$

$$= \sum_\ell \sum_m \underbrace{\left(\sum_k \tilde{R}_{\ell k} R_{km} \right)}_{(\underline{\underline{\tilde{R}}}\, \underline{\underline{R}})_{\ell m} = \delta_{\ell m}} u_\ell v_m = \sum_\ell \sum_m \delta_{\ell m} v_m u_\ell = \sum_\ell u_\ell v_\ell$$

$$\vec{u} = \hat{A}\vec{v}$$

$$u_k = \vec{E}^{(k)}\vec{u} = \vec{E}^{(k)}(\hat{A}\vec{v}) = \vec{E}^{(k)}(\hat{A}\hat{I}\vec{v}) = \vec{E}^{(k)}\left[\hat{A}\left(\sum_l \vec{e}^{(l)} \circ \vec{E}^{(l)}\right)\vec{v}\right] = \sum_l \vec{E}^{(k)}\left[\hat{A}\left(\vec{e}^{(l)} \circ \vec{E}^{(l)}\right)\vec{v}\right] =$$

$$= \sum_l \underbrace{\vec{E}^{(k)}\hat{A}\vec{e}^{(l)}}_{A_{kl}}\underbrace{\left(\vec{E}^{(l)}\vec{v}\right)}_{v_l}.$$

$$u_k = \sum A_{kl} v_l \qquad \underline{u} = \underline{\underline{A}}\,\underline{v}$$

$$\vec{u} = \hat{A}\vec{v}$$

$$\underline{u} = \underline{\underline{A}}\,\underline{v} \qquad\qquad \underline{u}' = \underline{\underline{A}}'\,\underline{v}'$$

$$A_{kl} = \vec{E}^{(k)}\hat{A}\vec{e}^{(l)} \qquad A'_{kl} = \vec{F}^{(k)}\hat{A}\vec{f}^{(l)}$$

$$u'_k = \vec{F}^{(k)}\vec{u} = \vec{F}^{(k)}\hat{A}\vec{v} = \vec{F}^{(k)}\hat{A}\sum_l v'_l\vec{f}^{(l)} = \sum_l v'_l\underbrace{\left(\vec{F}^{(k)}\hat{A}\vec{f}^{(l)}\right)}_{A_{kl}}$$

$$\underline{u}' = \underline{\underline{A}}'\,\underline{v}' \qquad \underline{u} = \underline{\underline{A}}\,\underline{v} \qquad\qquad \underline{u}' = \underline{\underline{R}}\,\underline{u} \qquad \underline{v}' = \underline{\underline{R}}\,\underline{v}$$

$$\underline{u} = \underline{\underline{S}}\,\underline{u}' \qquad \underline{v} = \underline{\underline{S}}\,\underline{v}'$$

$$\underline{u}' = \underline{\underline{R}}\,\underline{u} = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{S}}\,\underline{v}' = (\underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{S}})\,\underline{v}'$$

$$\underline{u}' = \underline{\underline{A}}'\,\underline{v}'$$

$$\Rightarrow \quad \underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{S}} = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}$$

$$\underline{\underline{S}}\,\underline{\underline{A}}'\,\underline{\underline{R}} = \underline{\underline{S}}\,(\underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{S}})\,\underline{\underline{R}} = (\underline{\underline{S}}\,\underline{\underline{R}})\,\underline{\underline{A}}\,(\underline{\underline{S}}\,\underline{\underline{R}}) = \underline{\underline{A}}$$

$$A'_{kl} = (R\,A\,S)_{kl} = \sum_p \sum_q R_{kp} A_{pq} S_{ql}$$

ONB $\qquad \underline{\underline{S}} = \underline{\underline{\tilde{R}}}$

$$\underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{\tilde{R}}}$$

$$A'_{kl} = \sum_p \sum_q R_{kp} A_{pq} S_{ql} = \sum_p \sum_q R_{kp} R_{lq} A_{pq}$$

$$T'_{klm} = \sum_p \sum_q \sum_s R_{kp} R_{lq} R_{ms} T_{pqs}$$

$$\vdots$$

$$F'_{klmn}$$

$A_{ik} = a_i b_k$

$\underline{a}' = \underline{\underline{R}}\,\underline{a}$ $\quad a_i' = \sum_p R_{ip} a_p$ $\quad \vec{N} = \vec{r} \times \vec{p}$

$\underline{b}' = \underline{\underline{R}}\,\underline{b}$ $\quad b_k' = \sum_q R_{kq} a_q$ $\quad \underline{N} = \underline{r} \times \underline{p} \rightarrow N_1 = r_2 p_3 - r_3 p_2$

$A_{ik}' = a_i' b_k' = \left(\sum_p \ldots\right)\left(\sum_q \ldots\right) = \sum_p \sum_q R_{ip} R_{kq} a_p b_q$

$t' = t$

$v_i' = \sum_p R_{ip} v_p$

$T_{ik}' = \sum_p \sum_q R_{ip} R_{kq} T_{pq}$ $\rightarrow$ TENZOROK

$B_{ikm}' = \sum_p \sum_q \sum_s R_{ip} R_{kq} R_{ms} B_{pqs}$

$C_{ikmn}' = \sum_p \sum_q \sum_s \sum_t R_{ip} R_{kq} R_{ms} R_{nt} C_{pqst}$

Kovariancia egyenlet: a mennyiségek változnak, de a közöttük lévő kapcsolat nem
(pl. $F' = m'a' = F = ma$)

2017.10.24.

GY

$\vec{u} = \hat{A}\vec{v}$

$\underline{u} = \underline{\underline{A}}\,\underline{v}$ $\qquad$ $\underline{u}' = \underline{\underline{A}}'\,\underline{v}$

$\underline{u}' = \underline{\underline{R}}\,\underline{u}$

$\underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}$

$\det \underline{\underline{A}}' = \det(\underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}) = \det \underline{\underline{R}} \cdot \det \underline{\underline{A}} \cdot \det \underline{\underline{R}}^{-1} = \det \underline{\underline{A}}$

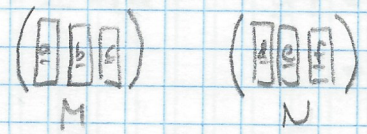
$\det \underline{\underline{R}}^{-1} = \frac{1}{\det \underline{\underline{R}}}$

$Sp\ \underline{\underline{A}} = \sum_k A_{kk}$

$Sp\,(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}) = Sp\,(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{B}})$

$Sp\ \underline{\underline{A}}' = Sp\,(\underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}) = Sp\,(\underline{\underline{R}}^{-1}\,\underline{\underline{R}}\,\underline{\underline{A}}) = Sp\ \underline{\underline{A}}$

$\underline{a}, \underline{b}, \underline{c}$ $\qquad$ $\underline{d}, \underline{e}, \underline{f}$



M $\qquad$ N

$$P = \begin{pmatrix} \underline{a}\underline{d} & \underline{a}\underline{e} & \underline{a}\underline{f} \\ \underline{b}\underline{d} & \underline{b}\underline{e} & \underline{b}\underline{f} \\ \underline{c}\underline{d} & \underline{c}\underline{e} & \underline{c}\underline{f} \end{pmatrix}$$

All. $\det \underline{\underline{M}} \cdot \det \underline{\underline{N}} = \det \underline{\underline{P}}$ (hat vektor tétele)

Biz. $\det \hat{\underline{\underline{A}}} = \det \underline{\underline{A}}$

$$\underline{\underline{\hat{M}}}\,\underline{\underline{N}} = \begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} \begin{pmatrix} \vec{a} & \vec{b} & \vec{c} \end{pmatrix} = \begin{pmatrix} \vec{a}\vec{a} & \vec{a}\vec{b} & \vec{a}\vec{c} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix} = \underline{\underline{P}} \quad \rightarrow \quad \det(\hat{\underline{\underline{M}}}\,\underline{\underline{N}}) = \det \underline{\underline{P}}$$

$$\det \underline{\underline{M}} \cdot \det \underline{\underline{N}} = \det \underline{\underline{P}}$$

$\vec{y} = ?$ $\quad \alpha \in \mathbb{R}$ $\quad \vec{a} \in V_1$

$\boxed{\vec{a}\vec{y} = \alpha}$

- $\vec{a} = \vec{0}$
  - $\alpha = 0$ → $\boxed{\vec{y} \in V_1}$
  - $\alpha \neq 0$ → ↯
- $\vec{a} \neq \vec{0}$
  - $\alpha = 0$ → $\boxed{\vec{y} = \vec{a} \times \vec{z}}$, $\vec{z} \in V_1$
  - $\alpha \neq 0$ → TFH. $\vec{y} = \beta \vec{a}$

$$\vec{a}\vec{y} = \vec{a}\beta\vec{a} = \beta a^2 \overset{!}{=} \alpha$$

$$\beta = \frac{\alpha}{a^2}$$

$$\vec{y} = \frac{\alpha}{a^2}\vec{a}$$

$$\boxed{\vec{y} = \frac{\alpha}{a^2}\vec{a} + \vec{a} \times \vec{z}}$$

$\vec{z} \in V_1$

$\vec{a} \times \vec{y} = \vec{b}$ $\quad \vec{a}, \vec{b} \in V_1$

- $\vec{a} = \vec{0}$
  - $\vec{b} = \vec{0}$ → $\boxed{\vec{y} \in V_1}$
  - $\vec{b} \neq \vec{0}$ → ↯
- $\vec{a} \neq \vec{0}$
  - $\vec{b}\vec{a} \neq 0$ → ↯
  - $\vec{b}\vec{a} = 0$
    - $\vec{b} = \vec{a}$ → $\boxed{\vec{y} = \lambda \vec{a}, \ \lambda \in \mathbb{R}}$
    - $\vec{b} \neq \vec{0}$

$\vec{a}, \vec{b}, \vec{a} \times \vec{b}$ bázis

$\exists \alpha, \beta, \gamma$

$$\vec{y} = \alpha\vec{a} + \beta\vec{b} + \gamma\,\vec{a} \times \vec{b}$$

$$\vec{a} \times \vec{y} = \vec{a} \times (\alpha\vec{a} + \beta\vec{b} + \gamma\,\vec{a} \times \vec{b}) = \alpha \underbrace{\vec{a} \times \vec{a}}_{0} + \beta\,\vec{a} \times \vec{b} +$$

$$+ \gamma\,\vec{a} \times (\vec{a} \times \vec{b}) = 0\vec{a} + (-\gamma a^2)\vec{b} + \beta(\vec{a} \times \vec{b}) \overset{!}{=}$$

$$\vec{a}(\underbrace{\vec{a}\vec{b}}_{0}) - \vec{b}(a^2)$$

$$= 0\vec{a} + 1 \cdot \vec{b} + 0\,\vec{a} \times \vec{b}$$

$0 = 0$

$1 = -\gamma a^2$

$0 = \beta$

$\beta = 0$

$\gamma = -\frac{1}{a^2}$

$\alpha \in \mathbb{R}$

$$\Leftrightarrow \quad \vec{y} = \alpha\vec{a} - \frac{1}{a^2}\,\vec{a} \times \vec{b}$$

$\vec{a} \times \vec{y} = \vec{b}$

$\vec{b} \times \vec{y} = \lambda \vec{a} \qquad \lambda \in \mathbb{R}$

$\vec{a} \neq 0$

$\vec{b} \neq 0$

$\vec{a}\vec{b} = 0$

$\vec{y} = \alpha \vec{a} - \frac{1}{a^2} \vec{a} \times \vec{b}$

$\vec{b} \times \vec{y} = \vec{b} \times (\alpha \vec{a} - \frac{1}{a^2} \vec{a} \times \vec{b}) = \alpha (\vec{b} \times \vec{a}) - \frac{1}{a^2} \underbrace{\vec{b} \times (\vec{a} \times \vec{b})}_{\vec{a}(\vec{b}^2) - \vec{b}(\vec{a}\vec{b})} = -\alpha (\vec{a} \times \vec{b}) - \frac{b^2}{a^2} \cdot \vec{a} \overset{!}{=} \lambda \vec{a}$

$\vec{b}^2 = 0$

$-\frac{b^2}{a^2} = \lambda \qquad \alpha = 0$

$\lambda = -\frac{b^2}{a^2}$

$\vec{y} = -\frac{b^2}{a^2} \vec{a} \times \vec{b}$

$\lambda \neq -\frac{b^2}{a^2}$

↳

$$V_n(x_n, x_1, \dots x_n) = \begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{n-2} & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \dots & x_2^{n-2} & x_2^{n-1} \\ \vdots & & & & & \vdots \\ 1 & x_n & x_n^2 & \dots & x_n^{n-2} & x_n^{n-1} \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^{n-2} & 0 \\ & x_2 & x_2^{n-2} & (x_2-x_1)x_2^{n-2} \\ & & & \\ 1 & x_n & \dots & x_n^{n-2}(x_n-x_1)x_n^{n-2} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-3} & 0 & 0 \\ 1 & x_2 & & x_2^{n-3} & (x_2-x_1)x_2^{n-3} & (x_2-x_1)x_1^{n-2} \\ & & & & & \\ 1 & x_n & & x_n^{n-3} & (x_n-x_1)x_n^{n-3} & (x_n-x_1)x_n^{n-2} \end{vmatrix} = \begin{vmatrix} 1 & x_1 & x_1^2 & 0 & & 0 \\ 1 & x_2 & x_2^2 & (x_2-x_1)x_2^2 & \dots & (x_2-x_1)x_2^{n-2} \\ & & & & & \\ 1 & x_n & x_n^2 & (x_n-x_1)x_n^2 & \dots & (x_n-x_1)x_n^{n-2} \end{vmatrix} =$$

$$= \begin{vmatrix} 1 & x_1 & 0 & \dots \\ 1 & x_2 & (x_2-x_1)x_2 & \\ & & & \\ 1 & x_n & (x_n-x_1)x_n & \dots \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \dots \\ 1 & (x_2-x_1) & (x_2-x_1)x_2 & \\ & & & \\ 1 & (x_n-x_1) & (x_n-x_1)x_n & \dots \end{vmatrix} =$$

$$= (x_2-x_1)(x_3-x_1)\cdot\dots\cdot(x_n-x_1) \cdot \begin{vmatrix} 1 & x_1 & \dots & x_1^{n-2} \\ 1 & x_2 & \dots & x_2^{n-2} \\ & & & \\ 1 & x_n & & x_n^{n-2} \end{vmatrix} = (x_2-x_1)(x_3-x_1)\dots(x_n-x_1) \cdot V_{n-2}(x_2 \dots x_n) =$$

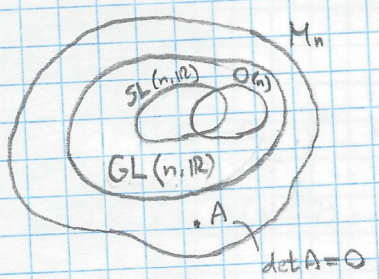
$$= (x_2-x_1)(x_3-x_1)\dots(x_n-x_1)\cdot(x_3-x_2)\cdot(x_4-x_2)\dots(x_n-2)\cdot V_{n-3}(x_3 \dots x_n) = \dots$$

$$V_n(x_1, x_2, \dots, x_n) = \prod_{k>l} (x_k - x_l)$$

- Reguláris mátrix: létezik inverze

Szinguláris mátrix: nincs inverze

$\det \underline{\underline{A}} \neq 0 \quad \exists \underline{\underline{A}}^{-1}$



$GL(n, \mathbb{R})$: általános lineáris csoport

$\det \underline{\underline{A}} = 1 \rightarrow SL(n, \mathbb{R})$: speciális lineáris részcsoport

$\tilde{\underline{\underline{A}}} = \underline{\underline{A}}^{-1} \rightarrow O(n)$: ortogonális csoport

$C = AB$

$\tilde{C} = \tilde{A}\tilde{B} = \tilde{B}\,\tilde{A}$

$C^{-1} = (AB)^{-1} = B^{-1}A^{-1}$

$\underline{\underline{A}}\tilde{\underline{\underline{A}}} = \underline{\underline{I}}$

$(\det \underline{\underline{A}})(\det \tilde{\underline{\underline{A}}}) = \det \underline{\underline{I}}$

$(\det \underline{\underline{A}})^2 = 1$

$\det \underline{\underline{A}} = \pm 1$

$O(n) \cap SL(n, \mathbb{R}) = SO(n)$

$A^+ = \tilde{A}^*$

$A^+ = A^{-1} \Rightarrow U(n)$: unitér csoport

$A^+ A = I$

2017.10.26.

GY

$\hat{A}\vec{u} = \lambda \vec{u}$ sajátértékprobléma

$\underline{\underline{A}}\,\underline{u} = \lambda \underline{u}$

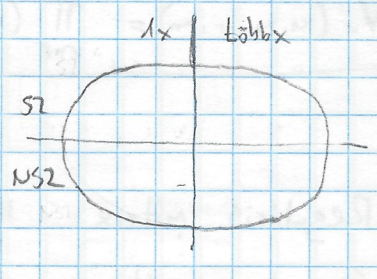
$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{u} = \underline{0}$

$\underline{\underline{C}}(\lambda)\,\underline{u} = \underline{0}$

$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

$$\det \underline{\underline{C}}(\lambda) = 0$$

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots \\ a_{21} & a_{22}-\lambda & \\ \vdots & & \ddots \end{vmatrix} = f(\lambda)$$ Karakteristikus polinom



$f(\lambda) = 0$

$\lambda_1 \dots \lambda_n \in \mathbb{C}$

$\underline{\underline{C}}(\lambda_k)\, \underline{u}^{(k)} = \underline{0}$

sajátérték — sajátvektor

$\hat{A}\vec{u} = \lambda \vec{u}$

$\underline{\underline{A}}\,\underline{u} = \lambda \underline{u}$ $\qquad$ $\underline{\underline{A}}'\,\underline{u}' = \lambda \underline{u}'$

$\underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}$

$$\underline{\underline{C}}'(\lambda) = (\underline{\underline{A}}' - \lambda \underline{\underline{I}}) = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1} - \lambda \underline{\underline{I}} = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1} - \lambda\, \underline{\underline{R}}\,\underline{\underline{I}}\,\underline{\underline{R}}^{-1} = \underline{\underline{R}}\,(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{\underline{R}}^{-1} =$$

$$= \underline{\underline{R}}\,\underline{\underline{C}}(\lambda)\,\underline{\underline{R}}^{-1}$$

$$f'(\lambda) = \det \underline{\underline{C}}'(\lambda) = \det(\underline{\underline{R}}\,\underline{\underline{C}}(\lambda)\,\underline{\underline{R}}^{-1}) = \det \underline{\underline{C}}(\lambda) = f(\lambda)$$

$$\underline{\underline{A}} = \begin{pmatrix} 2 & 0 & -6 \\ 1 & 3 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

$$f(\lambda) = \begin{vmatrix} 2-\lambda & 0 & -6 \\ 1 & 3-\lambda & -2 \\ -1 & 0 & 1-\lambda \end{vmatrix} = (3-\lambda) \cdot \begin{vmatrix} 2-\lambda & -6 \\ -1 & 1-\lambda \end{vmatrix} =$$

$$= (3-\lambda)\left((2-\lambda)(1-\lambda) - 6\right) = (3-\lambda)(\lambda^2 - 3\lambda + 2 - 6) =$$

$$= (3-\lambda)(\lambda^2 - 3\lambda - 4) = 3\lambda^2 - \lambda^3 - 9\lambda + 3\lambda^2 - 12 + 4\lambda =$$

$$= -\lambda^3 + 6\lambda^2 - 5\lambda - 12$$

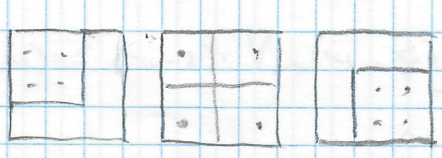
2×2

$$f(\lambda) = \lambda^2 - (\mathrm{Sp}\,A)\lambda + \det A$$

3×3

$$f(\lambda) = \lambda^3 - (\mathrm{Sp}\,A)\lambda^2 + (\text{lzé}\,A)\lambda - \det A$$

lzé = Sp adj $\qquad$ adj $\underline{\underline{A}}$



$Sp\ A = 2+3+1 = 6$

$I_{2\acute{e}}\ A = \begin{vmatrix} 2 & 0 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 2 & -6 \\ -1 & 1 \end{vmatrix} + \begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 6 - 5 + 3 = 5$

$\det A = 23 \cdot 1 - (-6)\,3 \cdot (-1) = -12$

$\lambda^3 - 6\lambda^2 + 5\lambda + 12 = 0$

$f(\lambda) = (\lambda+1)\,g(\lambda)$

$\lambda^3 - 6\lambda^2 + 5\lambda + 12 : (\lambda+1) = \lambda^2 - 7\lambda + 12$

$\lambda^3 + \lambda^2$

$-7\lambda^2 + 5\lambda$

$-7\lambda^2 - 7\lambda$

$12\lambda + 12$

$f(\lambda) = (\lambda+1)(\lambda^2 - 7\lambda + 12)$

$\lambda_1 = \boxed{-1}$ $\qquad \lambda_{2,3} = \dfrac{7 \pm \sqrt{49 - 48}}{2} = \dfrac{7 \pm 1}{2} \nearrow \boxed{5} \searrow \boxed{3}$

$\lambda_1 = -1$

$$\begin{pmatrix} 3 & 0 & -6 \\ 1 & 4 & -2 \\ -1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$-1x + 0y + 2z = 0$

$(-1, 2) \begin{pmatrix} x \\ z \end{pmatrix} = 0 \qquad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\underline{u}^{(1)} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\lambda_2 = 3$

$$\left(\begin{array}{ccc} -1 & 0 & -6 \\ 1 & 0 & -2 \\ 1 & 0 & -2 \end{array}\right) \sim \begin{pmatrix} 0 & 0 & -8 \\ & & \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\underline{u}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda_3 = 5$

$$\begin{pmatrix} -2 & 0 & -6 \\ 1 & -1 & -2 \\ -1 & 0 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$$

$\underline{u}^{(3)} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$

$* \begin{pmatrix} 2 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$2x + y = 0$

$(2\ 1) \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

2017. 10. 26.

EA

$$\underline{\underline{A}} = \begin{pmatrix} 2 & 0 & -6 \\ 1 & 3 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{B}} = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 0 \\ -6 & -2 & 1 \end{pmatrix}$$

$\mathrm{Sp}\, B = 6$

$\mathrm{Inv}\, B = 5$

$\det B = -12$

$\lambda^3 - 6\lambda^2 + 5\lambda + 12$

$\lambda_1 = 1 \quad \underline{u}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \quad \underline{v}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$\lambda_2 = 3 \quad \underline{u}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \underline{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\lambda_3 = 5 \quad \underline{u}^{(3)} = \begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix} \quad \underline{v}^{(3)} = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$\lambda_1 = -1 \quad \begin{pmatrix} 3 & 1 & -1 \\ 0 & 4 & 0 \\ -6 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$\lambda_2 = 3 \quad \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -6 & -2 & -2 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 \\ 8 & 0 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \sim \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\lambda_3 = 5 \quad \begin{pmatrix} -2 & 1 & -1 \\ 0 & -1 & 0 \\ -6 & -2 & -3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$\underline{u}^{(1)} \underline{v}^{(1)} = 0$

$\underline{u}^{(1)} \underline{v}^{(2)} = 0$

$\underline{u}^{(2)} \underline{v}^{(2)} = 0$

$\underline{u}^{(2)} \underline{v}^{(3)} = 0$

$\underline{u}^{(3)} \underline{v}^{(3)} = 0$

$\underline{u}^{(3)} \underline{v}^{(2)} = 0$

$\underline{u}^{(1)} \underline{v}^{(1)} = 5 \rightarrow \underline{u}^{(1)} = \frac{1}{5} \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}$

$\underline{u}^{(2)} \underline{v}^{(2)} = -1 \rightarrow \underline{u}^{(2)} = -1 \cdot \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

$\underline{u}^{(3)} \underline{v}^{(3)} = 5 \rightarrow \underline{u}^{(3)} = \frac{1}{5} \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}$

$\Rightarrow \boxed{\underline{u}^{(k)} \underline{v}^{(k)} = \delta_{kk}}$

bázis – reciprokbázis

$\underline{\underline{A}}\, \underline{u} = \lambda \underline{u}$

$\sum_k A_{ik} u_k = \lambda u_i$

$\underline{\underline{\tilde{A}}}\, \underline{v} = \lambda \underline{v}$

$\sum_k \tilde{A}_{ik} v_k = \lambda v_i$

$\sum_k A_{ki} v_k$

$\sum_k v_k A_{ki} = \lambda v_i$

$\underline{v}^{(k)}$ / $\underline{\underline{A}}\, \underline{u}^{(k)} = \lambda_k \underline{u}^{(k)}$

$\underline{\tilde{v}}^{(l)} \underline{\underline{A}} = \lambda_l \underline{\tilde{v}}^{(l)}$ / $\cdot\, \underline{u}^{(k)}$

$\underline{\tilde{v}}^{(l)} \underline{\underline{A}}\, \underline{u}^{(k)} = \lambda_k \underline{\tilde{v}}^{(l)} \underline{u}^{(k)}$

$= \lambda_l \underline{\tilde{v}}^{(l)} \underline{u}^{(k)}$

$(\lambda_k - \lambda_l)\, \underline{\tilde{v}}^{(l)} \underline{u}^{(k)} = 0$

$\lambda_k = \lambda_l$

$\underline{v}^{(l)} \underline{u}^{(k)} = 0$

$$\underline{\underline{A}} = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$\mathrm{Sp}\, A = 9$

$\det A = 27 - 1 - 1 - 3 - 3 - 3 = 16$

$\lambda^3 - 9\lambda^2 + 24\lambda - 16 = 0$

$\lambda_1 = 1$ $\quad \lambda^3 - 9\lambda^2 + 24\lambda - 16 : (\lambda - 1) = \lambda^2 - 8\lambda + 16 \quad \rightarrow \quad \lambda_{2,3} = \frac{8 \pm \sqrt{64 - 4 \cdot 16}}{2} = 4$

$\lambda^3 - \lambda^2$

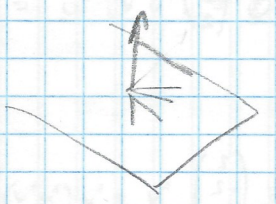
$-8\lambda^2 + 24\lambda$

$-8\lambda^2 + 8\lambda$

$16\lambda - 16$

$\lambda_1 = 1$ $\quad \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 3 & -3 & 0 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\quad \underline{s}^{(1)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda_2 = \lambda_3 = 4$ $\quad \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix}$



$(-1 \; -1 \; -1) \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = 0 \quad \underline{s}^{(2)} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\underline{s}^{(3)} = \underline{s}^{(1)} \times \underline{s}^{(2)} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

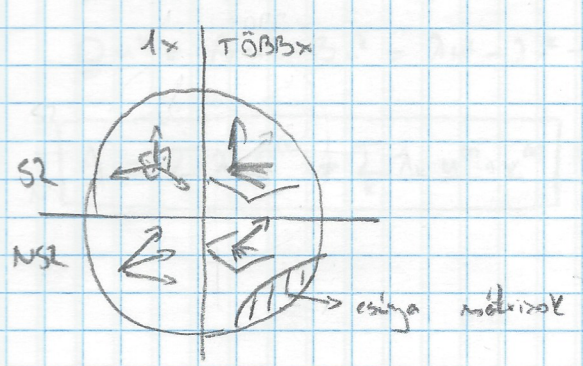
$\underline{u}^{(1)}, \underline{u}^{(2)}$

$\underline{\underline{A}}\, \underline{u}^{(1)} = \lambda \underline{u}^{(1)}$

$\underline{\underline{A}}\, \underline{u}^{(2)} = \lambda \underline{u}^{(2)}$

$\underline{u} = \alpha \underline{u}^{(1)} + \beta \underline{u}^{(2)}$

$\underline{\underline{A}}\, \underline{u} = \underline{\underline{A}} (\alpha \underline{u}^{(1)} + \beta \underline{u}^{(2)}) = \alpha (\underline{\underline{A}} \underline{u}^{(1)}) + \beta (\underline{\underline{A}} \underline{u}^{(2)}) = \alpha \lambda \underline{u}^{(1)} + \beta \lambda \underline{u}^{(2)} = \lambda (\alpha \underline{u}^{(1)} + \beta \underline{u}^{(2)}) = \lambda \underline{u}$



$\underline{\underline{N}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \lambda^2 - (\mathrm{Sp}\, A)\lambda + \det A$

$\mathrm{Sp}\, A = 0 \quad \det A = 0$

$\lambda^2 - 0\lambda + 0 = 0$

$\lambda^2 = 0$

$\lambda_1 = \lambda_2 = 0$

$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad y = 0$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\underline{\underline{C}} = \begin{pmatrix} 2 & -2 & 0 \\ -2 & 1 & -2 \\ 0 & -2 & 0 \end{pmatrix} = \underline{\underline{\tilde{C}}}$$

$\mathrm{Sp}\,\underline{\underline{C}} = 3$

$\mathrm{Izé}\,\underline{\underline{C}} = -2 + 0 - 4 = -6$

$\det \underline{\underline{C}} = -8$

$\lambda_1 = 1$

$\lambda_2 = -2$

$\lambda_3 = 4$

$\lambda^3 - 3\lambda^2 - 6\lambda + 8 = 0$

$\lambda^3 - 3\lambda^2 - 6\lambda + 8 : (\lambda - 1) = \lambda^2 - 2\lambda - 8$

$\lambda_{2,3} = \frac{2 \pm \sqrt{4 + 32}}{2} = \frac{2 \pm 6}{2}$ → -2, 4

$\lambda_1 = 1$ $\quad \begin{pmatrix} 1 & -2 & 0 \\ -2 & 0 & -2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ $\quad \underline{s}^{(1)} = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$

$\lambda_2 = -2$ $\quad \begin{pmatrix} 1 & -2 & 0 \\ -2 & 3 & -2 \\ 0 & -2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ $\quad \underline{s}^{(2)} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$\lambda_3 = 4$ $\quad \begin{pmatrix} -2 & -2 & 0 \\ -2 & -3 & -2 \\ 0 & -2 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ $\quad \underline{s}^{(3)} = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

$$\underline{\underline{B}} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

$\mathrm{Sp}\,\underline{\underline{B}} = 5$

$\det \underline{\underline{B}} = 0$

$\lambda^2 - 5\lambda + 0 = 0$

$\lambda_1 = 0 \quad \lambda_2 = 5$

$\lambda_1 = 0$ $\quad \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ $\quad \underline{s}^{(1)} = \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\lambda_2 = 5$ $\quad \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad \underline{s}^{(2)} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{\underline{A}}\,\underline{u} = \lambda \underline{u}$

$\underline{v}\,\underline{\underline{A}} = \lambda \underline{v}$

$\underline{v}^{(k)} \underline{u}^{(l)} = \delta_{kl}$

$\underline{\underline{P}}^{(k)} = \underline{u}^{(k)} \circ \underline{v}^{(k)}$

$\underline{\underline{P}}^{(k)} \underline{\underline{P}}^{(k)} = (\underline{u}^k \circ \underline{v}^k)(\underline{u}^k \circ \underline{v}^k) = (\underline{v}^k \underline{u}^k)\, \underline{u}^k \circ \underline{v}^k = \underline{u}^k \circ \underline{v}^k = \underline{\underline{P}}^{(k)}$

$\underline{\underline{P}}^1 \underline{\underline{P}}^2 = (\underline{u}^1 \circ \underline{v}^1)(\underline{u}^2 \circ \underline{v}^2) = (\underline{v}^1 \underline{u}^2)(\underline{u}^1 \circ \underline{v}^2) = 0$

$$\boxed{\underline{\underline{P}}^k \underline{\underline{P}}^l = \delta_{kl}\, \underline{\underline{P}}^k \qquad \Sigma\, \underline{\underline{P}}^k = \underline{\underline{I}}}$$

$$\underline{\underline{P}}^1 = u^1 \circ v^1 = \frac{1}{5}\begin{pmatrix}2\\0\\1\end{pmatrix}^{(1\ 0\ 3)} = \frac{1}{5}\begin{pmatrix}2&0&6\\0&0&0\\1&0&3\end{pmatrix}$$

$$\underline{\underline{P}}^2 = u^2 \circ v^2 = \begin{pmatrix}0\\1\\0\end{pmatrix}^{(-1\ 1\ 2)} = \begin{pmatrix}0&0&0\\-1&1&2\\0&0&0\end{pmatrix} = \frac{1}{5}\begin{pmatrix}0&0&0\\-5&5&10\\0&0&0\end{pmatrix}$$

$$\underline{\underline{P}}^3 = u^3 \circ v^3 = \frac{1}{5}\begin{pmatrix}3\\5\\-1\end{pmatrix}^{(1\ 0\ -2)} = \frac{1}{5}\begin{pmatrix}3&0&6\\5&0&-10\\-1&0&2\end{pmatrix}$$

$$\sum \underline{\underline{P}} = \frac{1}{5}\begin{pmatrix}5&0&0\\0&5&0\\0&0&5\end{pmatrix} = \underline{\underline{I}}$$

$$\underline{\underline{B}} = \lambda_1 \underline{\underline{P}}^1 + \lambda_2 \underline{\underline{P}}^2 + \lambda_3 \underline{\underline{P}}^3 = \sum_k \lambda_k \underline{\underline{P}}^k$$

$$\underline{\underline{P}}^{(1)}\, \underline{u}^{(1)} = \left(\underline{u}^{(1)} \circ \underline{v}^{(1)}\right) \underline{u}^{(1)} = \underline{u}^{(1)} \left(\underline{v}^{(1)}\, \underline{u}^{(1)}\right) = \underline{u}^{(1)}$$

$$\underline{\underline{P}}^{(1)}\, \underline{u}^{(2)} = \left(\underline{u}^1 \circ \underline{v}^1\right) \underline{u}^2 = \underline{u}^1 \left(\underline{v}^1\, \underline{u}^2\right) = \underline{0}$$

$$\boxed{\underline{\underline{P}}^k\, \underline{u}^l = \delta_{kl}\, \underline{u}^l}$$

$$\underline{v}^{(1)}\, \underline{\underline{P}}^{(1)} = \underline{v}^1 \left(\underline{u}^1 \circ \underline{v}^1\right) = \left(\underline{v}^1 \underline{u}^1\right) \underline{v}^1 = \underline{v}^1$$

$$\underline{v}^{(1)}\, \underline{\underline{P}}^{(2)} = \underline{0}$$

$$\boxed{\underline{v}^l\, \underline{\underline{P}}^k = \delta_{kl}\, \underline{v}^l}$$

$$\underline{\underline{B}}\, \underline{u}^k = \left(\sum_l \lambda_l \underline{\underline{P}}^l\right) \underline{u}^k = \sum_l \lambda_l \left(\underline{\underline{P}}^l \underline{u}^k\right) = \sum_l \lambda_l \delta_{kl}\, \underline{u}^k = \lambda_k \underline{u}^k$$

$$\underline{v}^k\, \underline{\underline{B}} = \underline{v}^k \left(\sum_l \lambda_l \underline{\underline{P}}^l\right) = \sum_l \lambda_l \left(\underline{v}^k \underline{\underline{P}}^l\right) = \sum_l \lambda_l \delta_{kl}\, \underline{v}^k = \lambda_k \underline{v}^k$$

$$\underline{\underline{D}} = \underline{\underline{A}} - \underline{\underline{B}}$$

$$\underline{\underline{D}}\, \underline{u}^k = \underline{\underline{A}}\, \underline{u}^k - \underline{\underline{B}}\, \underline{u}^k = \lambda \underline{u}^k - \lambda \underline{u}^k = 0$$

$$\boxed{\underline{\underline{A}} = \sum_k \lambda_k \underline{\underline{P}}^{(k)} = \sum_k \lambda_k\, \underline{u}^{(k)} \circ \underline{v}^{(k)}}$$

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EA

$\hat{A}\, \vec{u} = \lambda \vec{u}$

$\underline{\underline{A}}\, \underline{u} = \lambda \underline{u}$

$\square \cdot [] = []$

$\tilde{v}\, \underline{\underline{A}} = \lambda \tilde{v}$

$\square \cdot [] = \square$

$\boxed{\underline{\underline{A}}}\, \underline{v} = \lambda \underline{v}$

KÉTOLDALI SAJÁTÉRTÉK-PROBLÉMA

$$(\underline{\underline{A}} - \lambda \underline{\underline{I}})\,\underline{u} = 0$$

$$\underline{\underline{C}}(\lambda)\,\underline{u} = 0$$

$$f(\lambda) = \det \underline{\underline{C}}(\lambda)$$

$$f(\lambda) = 0 \longrightarrow \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{C}$$

$$f(\lambda) = (\lambda - \lambda_1)^{m_1} (\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_r)^{m_r}$$

$$\sum_r m_r = n$$

↑ multiplicitás

$$\underline{\underline{C}}(\lambda_k)\,\underline{u}^{(k)} = 0$$

$$\hat{A}$$

$$\underline{\underline{A}} \longleftrightarrow \underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}$$

$$f_{\underline{\underline{A}}}(\lambda) = f_{\underline{\underline{A}}'}(\lambda)$$

$$f_{\hat{A}}(\lambda) = f_{\underline{\underline{A}}}(\lambda)$$

$$\tilde{v}^{(k)}\,\underline{u}^{(e)} = \delta_{ke}$$ ortonormált vektorrendszer (nem feltétlenül bázis)

$$P^{(k)} = u^{(k)} \circ \tilde{v}^{(k)}$$

$$P^{(k)} P^{(e)} = \delta_{ke} P^{(e)} \qquad \sum_k \underline{\underline{P}}^{(k)} = \underline{\underline{I}}$$

Minden mátrixnak van min. egy sajátértéke.

Különböző sajátértékekhez tartozó sajátvektorok lineárisan függetlenek

$$\underline{\underline{A}} = \sum_k \lambda_k \underline{\underline{P}}^{(k)}$$ → akkor igaz, ha a sajátvektorok bázist alkotnak (pl. szimmetrikus és antiszimmetrikus mátrixoknál)
↳ egyszerű struktúrájú mátrixok

$$\underline{\underline{A}}^2 = \underline{\underline{A}}\,\underline{\underline{A}} = \left(\sum_k \lambda_k \underline{\underline{P}}^{(k)}\right)\left(\sum_e \lambda_e \underline{\underline{P}}^{(e)}\right) = \sum_k \sum_e \lambda_k \lambda_e \underline{\underline{P}}^{(k)} \underline{\underline{P}}^{(e)} = \sum_k \sum_e \lambda_k \lambda_e \delta_{ke} \underline{\underline{P}}^{(k)} =$$

$$= \sum_k \lambda_k^2 \underline{\underline{P}}^{(k)}$$

$$\underline{\underline{A}}^3 = \left(\sum_k \lambda_k^2 P^{(k)}\right)\left(\sum_k \lambda_k P^{(k)}\right) = \sum_k \sum_e \lambda_k^2 \lambda_e P^{(k)} P^{(e)} = \sum_k \sum_e \lambda_k^2 \lambda_e \delta_{ke} P^{(k)} = \sum_k \lambda_k^3 P^{(k)}$$

$$\underline{\underline{A}}^n = \sum_k \lambda_k^n P^{(k)}$$

$$\underline{\underline{A}}^0 = \underline{\underline{I}} = \sum_k P^{(k)} = \sum_k \lambda_k^0 P^{(k)}$$

$p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots + \alpha_n x^n \qquad \alpha_k \in \mathbb{R}$

$p(\underline{\underline{A}}) = \alpha_0 \underline{\underline{I}} + \alpha_1 \underline{\underline{A}} + \alpha_2 \underline{\underline{A}}^2 + \dots + \alpha_n \underline{\underline{A}}^n = \sum_{r=0}^{n} \alpha_r \underline{\underline{A}}^r = \sum_{r=0}^{n} \alpha_r \left(\sum_k \lambda_k^r \underline{\underline{P}}^{(k)} \right) =$

$= \sum_k \left(\sum_r \alpha_r \lambda_k^r \right) \underline{\underline{P}}^{(k)} = \sum_k p(\lambda_k) \underline{\underline{P}}^{(k)}$

Spec. $p(x) = f(x)$

$f(\underline{\underline{A}}) = \sum_k f(\lambda_k) \underline{\underline{P}}^{(k)} = \underline{\underline{0}}$ Cayley - Hamilton - tétel

$g(x)$ f_V

$$g(\underline{\underline{A}}) = \sum_k g(\lambda_k) \underline{\underline{P}}^{(k)}$$

MÁTRIXFÜGGVÉNYEK ALAPTÉTELE

*Hatványsorral az előzőek szerint

$\underline{\underline{P}}^{(k)} = \underline{u}^{(k)} \circ \underline{v}^{(k)} \qquad \underline{\underline{A}} = \sum_k \lambda_k \underline{\underline{P}}^{(k)} \qquad (\underline{a} \circ \underline{b})_{rs} = a_r b_s$

$A_{rs} = \sum_k \lambda_k P_{rs}^{(k)} = \sum_k \lambda_k u_r^{(k)} v_s^{(k)}$

$\underline{\underline{U}} = \left(\underline{u}^{(1)} \; \underline{u}^{(2)} \; \dots \right) \qquad u_r^{(k)} = U_{rk}$

$\underline{\underline{V}} = \begin{pmatrix} \underline{v}^{(1)} \\ \underline{v}^{(2)} \\ \vdots \end{pmatrix} \qquad v_s^{(k)} = V_{ks} \qquad \Lambda_{kl} = \lambda_k \delta_{kl} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$

$A_{rs} = \sum_k \lambda_k U_{rk} V_{ks} = \sum_k \sum_l \lambda_l \delta_{kl} U_{rk} V_{ks} = \sum_k \sum_l \Lambda_{kl} U_{rk} V_{ls} =$

$= \sum_k \sum_l \Lambda_{kl} U_{rk} V_{ls} = \sum_k \sum_l U_{rk} \Lambda_{kl} V_{ls} = A_{rs}$

$\underline{\underline{A}} = \underline{\underline{U}} \, \underline{\underline{\Lambda}} \, \underline{\underline{V}}$

$\underline{\underline{A}} =$

$\underline{\underline{V}} \, \underline{\underline{U}} = \left(\equiv \right)\left(\| \| \right) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} = \underline{\underline{I}} \quad \Rightarrow \quad \underline{\underline{V}} = \underline{\underline{U}}^{-1} \quad \Rightarrow \quad \boxed{\underline{\underline{A}} = \underline{\underline{U}} \, \underline{\underline{\Lambda}} \, \underline{\underline{U}}^{-1}}$

bázistranszformáció

$V(U \Lambda V) U$

$(VU) \Lambda (VU)$

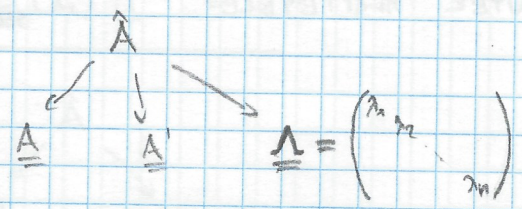
$\boxed{\Lambda = VAU}$ mátrix $\rightarrow$ diagonálmátrix : FŐTENGELYTRANSZFORMÁCIÓ

$$\underline{\underline{A}}^2 = \underline{\underline{A}}\,\underline{\underline{A}} = (\underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}})(\underline{\underline{U}}\,\underline{\underline{\Lambda}}\,\underline{\underline{V}}) = \underline{\underline{U}}\,\underline{\underline{\Lambda}}(\underline{\underline{V}}\underline{\underline{U}})\underline{\underline{\Lambda}}\,\underline{\underline{V}} = \underline{\underline{U}}\,\underline{\underline{\Lambda}}^2\underline{\underline{V}}$$

$$\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n\end{pmatrix}\cdot\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_n\end{pmatrix} = \begin{pmatrix}\lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_n^2\end{pmatrix}$$

$$\underline{\underline{A}}^n = \underline{\underline{U}}\,\underline{\underline{\Lambda}}^n\,\underline{\underline{V}}$$

$$p(\underline{\underline{A}}) = \underline{\underline{U}}\begin{pmatrix}p(\lambda_1) & & & \\ & p(\lambda_2) & & \\ & & \ddots & \\ & & & p(\lambda_n)\end{pmatrix}\underline{\underline{V}} \qquad g(\underline{\underline{A}}) = \underline{\underline{U}}\begin{pmatrix}g(\lambda_1) & & & \\ & g(\lambda_2) & & \\ & & \ddots & \\ & & & g(\lambda_n)\end{pmatrix}\underline{\underline{V}}$$



$$\underline{\underline{\Lambda}}\,\underline{u} = \lambda\,\underline{u}$$

$$\begin{vmatrix}\lambda_1-\lambda & & & \\ & \lambda_2-\lambda & & \\ & & \ddots & \\ & & & \lambda_n-\lambda\end{vmatrix} = (\lambda_1-\lambda)(\lambda_2-\lambda)\cdots(\lambda_n-\lambda)$$

$\lambda_1 = \lambda_1$

$$\begin{pmatrix}0 & & & & \\ & \lambda_2-\lambda_1 & & & \\ & & \lambda_3-\lambda_1 & & \\ & & & \ddots & \\ & & & & \lambda_n-\lambda_1\end{pmatrix}\begin{pmatrix}1\\0\\0\\\vdots\\0\end{pmatrix}$$

$\Rightarrow A_2$ eredetileg választott bázisvektorok a sajátvektorok

Ha $A = \tilde{A}$

$U = \tilde{V} = V^{-1}$ ortogonális mátrixok $\Rightarrow$ ONB $\to$ ONB forgatás

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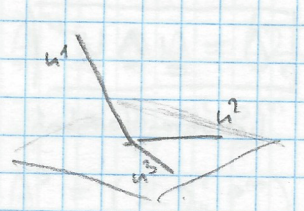

$$A = \lambda_1 P^{(1)} + \lambda_2 P^{(2)} + \lambda_3 P^{(3)} = \lambda_1 P^{(1)} + \lambda_2 (P^{(2)} + P^{(3)})$$

Többszörös sajátérték $\to$ eltérő vektor projektor

$$B = \sin\left(\frac{\pi}{3}A\right)$$

$$A = \sum_k \lambda_k P^{(k)} \qquad B = \sum \sin\left(\frac{\pi}{3}\lambda_k\right) P^{(k)} \qquad \lambda_k = 0,1,2$$

$\sin\frac{\pi}{3}\cdot 1 < \frac{\sqrt{3}}{2}$

$\sin\frac{\pi}{3}\cdot 2 > \frac{\sqrt{3}}{2}$

$B = 0\cdot P^1 + \frac{\sqrt{3}}{2}P^2 + \frac{\sqrt{3}}{2}P^3 = \frac{\sqrt{3}}{2}(P^2+P^3) = \frac{\sqrt{3}}{2}(I-P^1)$

$P^1 + P^2 + P^3 = I$

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GY.

$\underline{u}^1 = \begin{pmatrix}1\\2\end{pmatrix}$ $\underline{u}^2 = \begin{pmatrix}2\\3\end{pmatrix}$

1 -2

$\underline{v}^1 = \begin{pmatrix}-3\\2\end{pmatrix}$ $\underline{v}^2 = \begin{pmatrix}2\\-1\end{pmatrix}$

$\underline{\underline{P}}^{(1)} = u^{(1)} \circ v^{(1)} = \begin{pmatrix}1\\2\end{pmatrix}^{(-3\ 2)} = \begin{pmatrix}-3 & 2\\-6 & 4\end{pmatrix}$ $\underline{\underline{P}}^{(2)} = \begin{pmatrix}2\\3\end{pmatrix}^{(2\ -1)} = \begin{pmatrix}4 & -2\\6 & -3\end{pmatrix}$

$\underline{\underline{P}}^{(1)} + \underline{\underline{P}}^{(2)} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$

$\begin{pmatrix}-3 & 2\\-6 & 4\end{pmatrix}\cdot\begin{pmatrix}-3 & 2\\-6 & 4\end{pmatrix} = \begin{pmatrix}-3 & 2\\-6 & 4\end{pmatrix}$

$\lambda_1 P^{(1)} + \lambda_2 P^{(2)} = 1\cdot P^{(1)} - 2P^{(2)} = \begin{pmatrix}-3 & 2\\-6 & 4\end{pmatrix} - 2\begin{pmatrix}4 & -2\\6 & -3\end{pmatrix}$

$\underline{\underline{A}} = \begin{pmatrix}-11 & 6\\-18 & 10\end{pmatrix}$

$Sp\ \underline{\underline{A}} = -1$ → sajátértékek összege

$\det \underline{\underline{A}} = -2$ → sajátértékek szorzata

} általában is

$\lambda^2 + \lambda - 2 = 0$

$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+4\cdot 2}}{2}$ $\begin{matrix}1\\-2\end{matrix}$

$\lambda_1 = 1$

$\begin{pmatrix}-12 & 6\\-18 & 9\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}$ ← $\underline{u}^1$

$\begin{pmatrix}-12 & -18\\6 & 9\end{pmatrix}\begin{pmatrix}-3\\2\end{pmatrix}$ ← $\underline{v}^1$

$\lambda_2 = -2$

$\begin{pmatrix}-9 & 6\\-18 & 12\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}$ ← $\underline{u}^2$

$\begin{pmatrix}-9 & -18\\6 & 12\end{pmatrix}\begin{pmatrix}2\\-1\end{pmatrix}$ ← $\underline{v}^2$

$B = \sin\left(\frac{\pi}{3}A\right) = \sum \sin\left(\frac{\pi}{3}\lambda_k\right)P^k = \sin\frac{\pi}{3}P^1 + \sin\left(-\frac{\pi}{3}\right)P^2 = \frac{\sqrt{3}}{2}P^1 - \frac{\sqrt{3}}{2}P^2 =$

$= \frac{\sqrt{3}}{2}\left[\begin{pmatrix}-3 & 2\\-6 & 4\end{pmatrix} - \begin{pmatrix}4 & -2\\6 & -3\end{pmatrix}\right] = \frac{\sqrt{3}}{2}\begin{pmatrix}-7 & 4\\-12 & 7\end{pmatrix}$

$\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}$ $\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$Sp = 0$

$\det = 0$

$\lambda^2 = 0$

$\lambda_1 = \lambda_2 = 0$

$y = 0$ $\underline{u}^{(1)} = \begin{pmatrix}1\\0\end{pmatrix}$

$\begin{pmatrix}0 & 0\\x & 0\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$

$x = 0$ $\underline{v}^{(1)} = \begin{pmatrix}0\\1\end{pmatrix}$

$\underline{u}\,\underline{v} = 0$

$\underline{\underline{A}} = \underline{u} \circ \underline{\tilde{v}}$

$\underline{\underline{A}}\,\underline{u} = (\underline{u} \circ \underline{v})\,\underline{u} = \underline{u}\,(\underbrace{\underline{v}\,\underline{u}}_{0}) = \underline{0}$

$\underline{\underline{\tilde{A}}} = \underline{v} \circ \underline{\tilde{u}}$

$\underline{\underline{\tilde{A}}}\,\underline{v} = (\underline{v} \circ \underline{\tilde{u}})\,\underline{v} = \underline{v}\,(\underbrace{\underline{\tilde{u}}\,\underline{v}}_{0}) = 0$

$B(\varepsilon) = \begin{pmatrix} 0 & 1 \\ \varepsilon^2 & 0 \end{pmatrix}$ $\quad Sp\,B = 0$ $\quad \det B = -\varepsilon^2$

$\lambda^2 - 0\lambda + \varepsilon^2 = 0$

$\lambda_1 = \varepsilon \qquad \lambda_2 = -\varepsilon$

$\lambda_1 = \varepsilon$

$\begin{pmatrix} -\varepsilon & 1 \\ \varepsilon^2 & -\varepsilon \end{pmatrix} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} \leftarrow u^{(1)}$

$\begin{pmatrix} -\varepsilon & \varepsilon^2 \\ 1 & -\varepsilon \end{pmatrix} \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} \frac{1}{2\varepsilon} \leftarrow v^1$

$\lambda_2 = -\varepsilon$

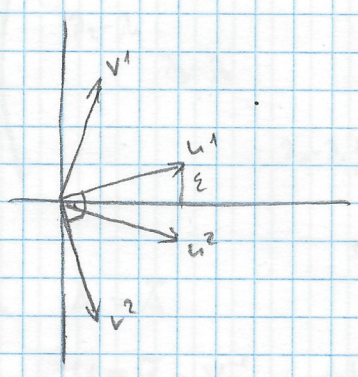
$\begin{pmatrix} \varepsilon & 1 \\ \varepsilon^2 & \varepsilon \end{pmatrix} \begin{pmatrix} 1 \\ -\varepsilon \end{pmatrix} \leftarrow u^2$

$\begin{pmatrix} \varepsilon & \varepsilon^2 \\ 1 & \varepsilon \end{pmatrix} \begin{pmatrix} \varepsilon \\ -1 \end{pmatrix} \frac{1}{2\varepsilon} \leftarrow v^2$

$\underline{\underline{P}}^{(1)} = \underline{u}^1 \circ \underline{v}^1 = \frac{1}{2\varepsilon} \begin{pmatrix} 1 \\ \varepsilon \end{pmatrix} (\varepsilon \; 1) = \frac{1}{2\varepsilon} \begin{pmatrix} \varepsilon & 1 \\ \varepsilon^2 & \varepsilon \end{pmatrix}$

$\underline{\underline{P}}^{(2)} = \underline{u}^2 \circ \underline{v}^2 = \frac{1}{2\varepsilon} \begin{pmatrix} 1 \\ -\varepsilon \end{pmatrix} (\varepsilon \; -1) = \frac{1}{2\varepsilon} \begin{pmatrix} \varepsilon & -1 \\ -\varepsilon^2 & \varepsilon \end{pmatrix}$

$\varepsilon P^1 + (-\varepsilon) P^2 = \frac{1}{2} \begin{pmatrix} \varepsilon & 1 \\ \varepsilon^2 & \varepsilon \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \varepsilon & -1 \\ -\varepsilon^2 & \varepsilon \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon^2 & 0 \end{pmatrix} = B(\varepsilon)$



$\underline{\underline{A}} = \begin{pmatrix} 5 & 6 & 9 \\ 6 & 5 & 9 \\ -6 & -6 & -10 \end{pmatrix}$

$\underline{\underline{B}} = \cos\frac{\pi}{2}\,\underline{\underline{A}}$

$C = (I + \underline{\underline{A}})(I - \underline{\underline{A}})^{-1}$ $\qquad \frac{1+x}{1-x}$

$[A, B] = 0$

$AB = BA$

$\exists C \quad A = f(C)$

$B = f(C)$

$AB = BA$

$A\underline{a} = \lambda \underline{a}$

$a \in V \ni b \quad Bb = \mu b \quad \rightarrow$ Bsz. be!

$Sp\,A = 0$

$12 = -11 + 4 + 4 = -3$

$$\det = \begin{vmatrix} -1 & 1 & 0 \\ 6 & 5 & 3 \\ -6 & -6 & -10 \end{vmatrix} = \begin{vmatrix} -1 & 0 & 0 \\ 6 & 11 & 3 \\ -6 & -12 & -10 \end{vmatrix} = - \begin{vmatrix} 11 & 3 \\ -12 & -10 \end{vmatrix} = 2$$

$\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

$\lambda_1 = -1$

$\lambda^3 - 0\lambda^2 - 3\lambda - 2 : (\lambda + 1) = \lambda^2 - \lambda - 2$

$\lambda^2 - \lambda - 2 = 0$

$\lambda_2 = -1$

$\lambda_3 = 2$

$\lambda_1 = 2$

$\lambda_2 = \lambda_3 = -1$

$\lambda_1 = 2$

$$\begin{pmatrix} 3 & 6 & 3 \\ 6 & 3 & 3 \\ -6 & -6 & -12 \end{pmatrix} \sim \begin{pmatrix} -3 & 3 & 0 \\ 2 & 1 & 3 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \qquad \underline{u}^1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 6 & -6 \\ 6 & 3 & -6 \\ 3 & 3 & -12 \end{pmatrix} \sim \begin{pmatrix} -3 & 3 & 0 \\ 2 & 1 & -2 \\ 3 & 3 & -6 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \qquad \underline{v}^1 = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

$\lambda_2 = -1$

$$\begin{pmatrix} 6 & 6 & 3 \\ 6 & 6 & 3 \\ -6 & -6 & -3 \end{pmatrix} \sim \begin{pmatrix} 2 & 2 & 3 \end{pmatrix} \qquad \underline{u}^2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad \underline{u}^3 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$$

$$\underline{v}^{(2)} = \underline{u}^{(3)} \times \underline{u}^{(1)} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix} \qquad \underline{v}^2 = \begin{pmatrix} -1 \\ -2 \\ -3 \end{pmatrix}$$

$$\underline{v}^{(3)} = \underline{u}^{(1)} \times \underline{u}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \qquad \underline{v}^3 = \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix}$$

$$\underline{\underline{P}}^{(1)} = \underline{u}^{(1)} \circ \underline{v}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 2 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 3 \\ 2 & 2 & 3 \\ -2 & -2 & -3 \end{pmatrix}$$

$$\underline{\underline{P}}^{(2)} = \underline{u}^{(2)} \circ \underline{v}^{(2)} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \begin{pmatrix} -1 & -2 & -3 \end{pmatrix} = \begin{pmatrix} -1 & -2 & -3 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\underline{\underline{P}}^{(3)} = \underline{u}^{(3)} \circ \underline{v}^{(3)} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix} \begin{pmatrix} -1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ -3 & -3 & 6 \\ 2 & 2 & 4 \end{pmatrix}$$

$$Q = P^{(2)} + P^{(3)} = \begin{pmatrix} -1 & -2 & -3 \\ -2 & -1 & -3 \\ 2 & 2 & 4 \end{pmatrix}$$

$$\underline{\underline{A}} = \sum \lambda_k P^k = 2P^{(1)} + (-1)Q = \begin{pmatrix} 4 & 4 & 6 \\ 4 & 4 & 6 \\ -4 & -4 & -6 \end{pmatrix} - \begin{pmatrix} -1 & -2 & -3 \\ -2 & -1 & -3 \\ 2 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 6 & 9 \\ 6 & 5 & 9 \\ -6 & -6 & -10 \end{pmatrix} = \underline{\underline{A}} \quad \checkmark$$

$B = \cos\frac{\pi}{2} A \qquad A = 2P^1 + (-1)Q$

$B = \underbrace{\cos\left(\frac{\pi}{2}2\right)}_{-1} \cdot P^1 + \underbrace{\cos\left(\frac{\pi}{2}(-1)\right)}_{0} Q = -P^1 = \begin{pmatrix} -2 & -2 & -3 \\ -2 & -2 & -3 \\ 2 & 2 & 3 \end{pmatrix}$

$\frac{1-x}{1+x} \qquad (1-A)(1+A)^{-1}$ b

$\frac{1+x}{1-x} \qquad \frac{\lambda+2}{\lambda-2} = -3 \qquad \frac{\lambda-1}{\lambda+1} = 0 \quad \rightarrow \quad \underline{\underline{C}} = -3 \cdot \underline{\underline{P}}^{(1)}$

MINIMÁLEGYENLET: legkisebb fokú egyenlet, amit a mátrix kielégít

$T^2 = 1$

$Tu = \lambda u$

$TTu = \lambda Tu = \lambda\lambda u$

$\lambda^2 = 1$

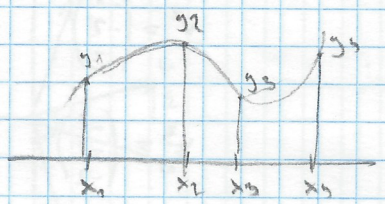
$f(\lambda) = (\lambda-\lambda_1)^{n_1} \cdot (\lambda-\lambda_2)^{n_2} \cdot \ldots \cdot (\lambda-\lambda_r)^{n_r}$

$m(\lambda) = (\lambda-\lambda_1)^{m_1} \cdot (\lambda-\lambda_2)^{m_2} \cdot \ldots \cdot (\lambda-\lambda_r)^{m_r}$

$n_k \leq m_k$

Egyszeres gyökök → egyszerű struktúra

Többszörös gyökök → nem egyszerű struktúra



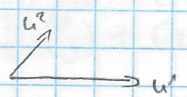
$f(x_1) = y_1$

$ax^3 + bx^2 + cx + d$

$\begin{cases} a x_1^3 + b x_1^2 + c x_1 + d = 1 \\ a x_1^3 + \ldots = 0 \\ \\ \\

2017. 11.09.

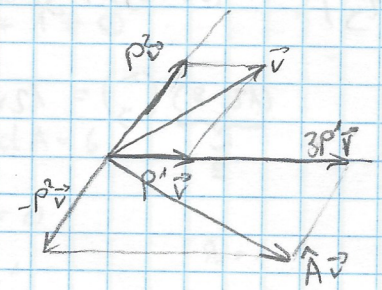
A

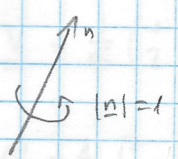


$\lambda_1 = 3$

$\lambda_2 = -1$

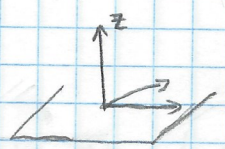
$A = \lambda_1 P^1 + \lambda_2 P^2$





$$F_{ke} = \cos\varphi\, \delta_{ke} + (1-\cos\varphi)\, n_k n_e + \sum_m \varepsilon_{kme}\, n_m \sin\varphi$$

$$RF\vec{R}^t = F' = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\delta_p F = \delta_p F' = 1 + 2\cos\varphi$$

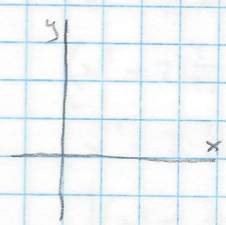
3 irányú vektorok helyben maradnak, → sajátvektor, $\lambda = 1$ sajátértékkel

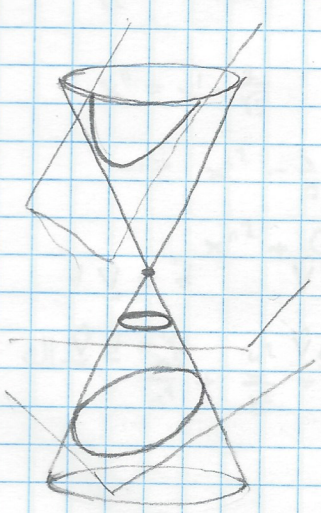
↳ forgatás tengelye megkapható

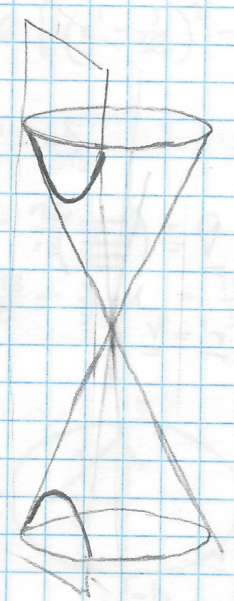
$$5x^2 + 4xy + 2y^2 + 12x + 8y + 1 = 0$$

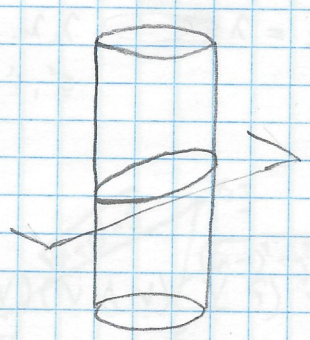
másodrendű görbék (síkban) → kúpszeletek

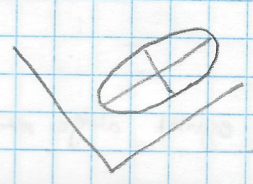
felületek (térben)











→ Kanonikus alakra hozás ⇒ főtengelytranszformáció

$$(x\ y)\begin{pmatrix}5 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = (x\ y)\begin{pmatrix}5x+2y\\ 2x+2y\end{pmatrix} = x(5x+2y)+y(2x+2y) = 5x^2+4xy+2y^2$$

$\underline{r} = \begin{pmatrix}x\\ y\end{pmatrix}$ $\quad \tilde{\underline{r}} = (x\ y)$

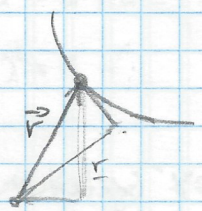
$$\underbrace{(12\ \ 8)}_{\tilde{\underline{b}}}\begin{pmatrix}x\\ y\end{pmatrix} = 12x+8y$$

$$\boxed{\tilde{\underline{r}}\,\underline{\underline{A}}\,\underline{r} + \tilde{\underline{b}}\,\underline{r} + c = 0}$$

$$(x\ y)\begin{pmatrix}5 & 3\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = (x\ y)\begin{pmatrix}5x+3y\\ x+2y\end{pmatrix} = 5x^2+4xy+2y^2$$

$$3x^2+6xy+4y^2 = z^2-2xz$$

$$(x\ y\ z)\begin{pmatrix}3 & 3 & -1\\ 3 & 4 & 0\\ -1 & 0 & -1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}$$



$\underline{v}' = \underline{\underline{R}}\,\underline{v}$ ⇝ ▯ = □ ‖

$\underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}$ ▭ = ▭ □̃ $\quad \tilde{\underline{v}}' = \tilde{\underline{v}}\,\tilde{\underline{\underline{R}}}$

$\underline{\underline{\Lambda}} = \underline{\underline{V}}\,\underline{\underline{A}}\,\underline{\underline{V}}$

$(\equiv)\ A(111)$

$A' = \Lambda = VA\tilde{V}$

$\underline{\underline{A}} \longrightarrow \underline{\underline{A}}\,\underline{s} = \lambda\,\underline{s}$ $\qquad \lambda_1, \lambda_2, \dots, \lambda_n$ $\qquad \underline{s}^1, \underline{s}^2, \dots, \underline{s}^n$ $\qquad \underline{\underline{V}} = \left(\equiv\right)$

$\underline{r}' = \underline{\underline{V}}\,\underline{r}$ $\qquad \tilde{\underline{r}}' = \tilde{\underline{r}}\,\tilde{\underline{\underline{V}}}$ $\qquad \underline{b}' = \underline{\underline{V}}\,\underline{b}$

$\underline{\underline{A}}' = \underline{\underline{V}}\,\underline{\underline{A}}\,\tilde{\underline{\underline{V}}}$ $\qquad \tilde{\underline{b}}' = \underline{b}\,\tilde{\underline{\underline{V}}}$

$$\tilde{\underline{r}}'\,\underline{\underline{A}}'\,\underline{r}' = (\tilde{\underline{r}}\,\tilde{\underline{\underline{V}}})(\underline{\underline{V}}\,\underline{\underline{A}}\,\tilde{\underline{\underline{V}}})(\underline{\underline{V}}\,\underline{r}) =$$

$$= \tilde{\underline{r}}\,(\tilde{\underline{\underline{V}}}\,\underline{\underline{V}})\,\underline{\underline{A}}\,(\tilde{\underline{\underline{V}}}\,\underline{\underline{V}})\,\underline{r} = \tilde{\underline{r}}\,\underline{\underline{A}}\,\underline{r}$$

$$\tilde{\underline{b}}'\,\underline{r}' = (\underline{b}\,\tilde{\underline{\underline{V}}})(\underline{\underline{V}}\,\underline{r}) = \underline{b}\,(\tilde{\underline{\underline{V}}}\,\underline{\underline{V}})\,\underline{r} = \underline{b}\,\underline{r}$$

$$\Rightarrow \quad 0 = \tilde{\underline{r}}\,\underline{\underline{A}}\,\underline{r} + \underline{b}\,\underline{r} + c = \tilde{\underline{r}}'\,\underline{\underline{A}}\,\underline{r}' + \tilde{\underline{b}}'\,\underline{r}' + c' \quad \longrightarrow \text{Az egyenlet alakja nem változik}$$

$5x^2 + 4xy + 7y^2 + 12x + 8y + 1 = 0$

$(x\ y)\begin{pmatrix}5 & 2\\ 2 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + (12,8)\begin{pmatrix}x\\ y\end{pmatrix} + 1 = 0$

$\underline{\underline{A}} = \begin{pmatrix}5 & 2\\ 2 & 2\end{pmatrix}$ $\quad Sp A = 7 \quad \det A = 6$

$\lambda^2 - 7\lambda + 6 = 0$

$\frac{7 \pm \sqrt{49-24}}{2}$ $\quad 1,\ 6$

$\lambda_1 = 1$

$\begin{pmatrix}4 & 2\\ 2 & 1\end{pmatrix}\begin{pmatrix}1\\ -2\end{pmatrix}$ $\quad \underline{s}^{(1)} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\ -2\end{pmatrix}$

$\lambda_2 = 6$

$\begin{pmatrix}-1 & 2\\ 2 & -4\end{pmatrix}\begin{pmatrix}2\\ 1\end{pmatrix}$ $\quad \underline{s}^{(2)} = \frac{1}{\sqrt{5}}\begin{pmatrix}2\\ 1\end{pmatrix}$

$A' = VAV$

$\underline{\underline{V}} = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}$

$\underline{\underline{A}}' = \frac{1}{\sqrt{5}}\begin{pmatrix}1 & -2\\ 2 & 1\end{pmatrix}\begin{pmatrix}5 & 2\\ 2 & 2\end{pmatrix}\frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\ -2 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}$

$\underline{b} = \begin{pmatrix}12\\ 8\end{pmatrix}$ $\quad \underline{b}' = \underline{\underline{V}}\,\underline{b} = \begin{pmatrix}\square\\ \square\end{pmatrix}$

$b_1' = \underline{s}^{(1)}\underline{b} = \frac{1}{\sqrt{5}}(1\ -2)\begin{pmatrix}12\\ 8\end{pmatrix} = -\frac{4}{\sqrt{5}}$

$b_2' = \underline{s}^{(2)}\underline{b} = \frac{1}{\sqrt{5}}(2\ 1)\begin{pmatrix}12\\ 8\end{pmatrix} = \frac{32}{\sqrt{5}}$

$(x'\ y')\begin{pmatrix}1 & 0\\ 0 & 6\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} + \frac{1}{\sqrt{5}}(-4\ 32)\begin{pmatrix}x'\\ y'\end{pmatrix} + 1 = 0$

$x'^2 + 6y'^2 - \frac{4}{\sqrt{5}}x' + \frac{32}{\sqrt{5}}y' + 1 = 0$

$\left(x'^2 - \frac{4}{\sqrt{5}}x'\right) + \left(6y'^2 + \frac{32}{\sqrt{5}}y'\right) + 1 = 0$

$\left(x'^2 - 2\cdot\frac{2}{\sqrt{5}}x'\right) + 6\left(y'^2 + 2\cdot\frac{8}{3\sqrt{5}}y'\right) + 1 = 0$

$\left[x'^2 - 2\cdot\frac{2}{\sqrt{5}}x' + \left(\frac{2}{\sqrt{5}}\right)^2 - \left(\frac{2}{\sqrt{5}}\right)^2\right] + 6\left[y'^2 + 2\frac{8}{3\sqrt{5}}y' + \left(\frac{8}{3\sqrt{5}}\right)^2 - \left(\frac{8}{3\sqrt{5}}\right)^2\right] + 1 = 0$

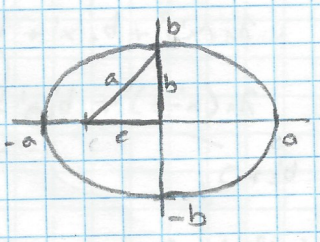
$\left[\left(x' - \frac{2}{\sqrt{5}}\right)^2 - \frac{4}{5}\right] + 6\left[\left(y' - \frac{8}{3\sqrt{5}}\right)^2 - \frac{64}{45}\right] + 1 = 0$

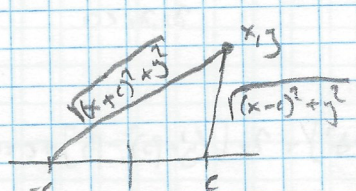
$x'' = x' - \frac{2}{\sqrt{5}}$ $\quad y'' = y' - \frac{8}{3\sqrt{5}}$

$\Rightarrow x''^2 - \frac{4}{5} + 6y''^2 - \frac{128}{15} + 1 = 0$

$x''^2 + 6y''^2 = \frac{128}{15} + \frac{4}{5} - 1 = \frac{25}{3}$

dl. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$





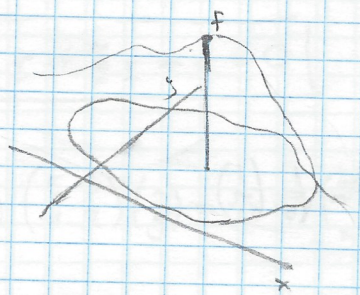
$\sqrt{(x+c)^2+y^2} + \sqrt{(x-c)^2+y^2} = 2a$

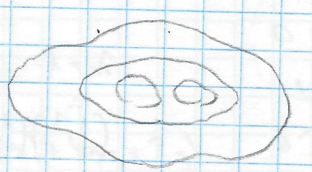
$0 < \varepsilon = \frac{c}{a} < 1$

2017.11.15.

EA

$$f(x,y) = \alpha x^2 + \beta xy + \gamma y^2 + \delta x + \varepsilon y + \varphi = 0$$





$$(x\ y)\begin{pmatrix}\alpha & \beta/2\\ \beta/2 & \gamma\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} + (\delta\ \varepsilon)\begin{pmatrix}x\\ y\end{pmatrix} + \varphi = 0$$

$$\underline{r}\,\underline{\underline{A}}\,\underline{r} + \underline{b}\cdot\underline{r} + c = 0$$

$$\underline{\underline{\tilde{A}}} = \underline{\underline{A}}$$

$$\underline{\underline{A}}\,\underline{s} = \lambda\,\underline{s}$$

$\lambda_1, \lambda_2 \in \mathbb{R}$

$\underline{s}^1, \underline{s}^2 \qquad \underline{s}^k \underline{s}^l = \delta_{kl}$

$$\underline{\underline{V}} = \begin{pmatrix}\underline{s}^1\\ \underline{s}^2\end{pmatrix} \qquad \underline{\underline{U}} = \underline{\underline{\tilde{V}}} = \underline{\underline{V}}^{-1} = (\underline{s}^1\ \underline{s}^2)$$

$$\underline{r}' = \underline{\underline{V}}\,\underline{r} \qquad \underline{b}' = \underline{\underline{V}}\,\underline{b} \qquad \underline{\underline{A}}' = \underline{\underline{V}}\,\underline{\underline{A}}\,\underline{\underline{U}} = \underline{\underline{V}}\,\underline{\underline{A}}\,\underline{\underline{\tilde{V}}} = \Lambda = \begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}$$

$$\rightarrow \underline{r}'\,\underline{\underline{A}}\,\underline{r}' + \underline{b}'\underline{r}' + c = 0$$

$$(x'\ y')\begin{pmatrix}\lambda_1 & \\ & \lambda_2\end{pmatrix}\begin{pmatrix}x'\\ y'\end{pmatrix} + (b_1'\ b_2')\begin{pmatrix}x'\\ y'\end{pmatrix} + c = 0$$

$$\lambda_1 (x')^2 + \lambda_2 (y')^2 + b_1' x' + b_2' y' + c = 0$$

$$\boxed{\underline{b}' = \underline{\underline{V}}\,\underline{b} \qquad b_1' = \underline{s}^1 \underline{b} \qquad b_2' = \underline{s}^2 \underline{b}}$$

$\lambda_1\lambda_2 \neq 0$

- $\lambda_1, \lambda_2 > 0$
- $\lambda_1, \lambda_2 < 0$

$\lambda_1\lambda_2 = 0$

- $\lambda_1 = 0$
- $\lambda_2 = 0$

Mindkét esetben:

$$\lambda_1 [(x'-\delta)^2 - \delta^2] + \lambda_2 [(y'-\beta)^2 - \beta^2] + c = 0$$

$$\lambda_1 x''^2 + \lambda_2 y''^2 = K$$

$$\lambda_1 (x')^2 + b_1' x' + b_2' y' + c = 0$$

$$\lambda_1 (x'-\delta)^2 + b_2' y' + d = 0$$

$\lambda_1 > 0, \lambda_2 > 0$

- $K \neq 0$: $\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = \pm 1$
- $K = 0$: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$

$b_2' \neq 0$: $\lambda_1 (x'')^2 + b_2' y'' = 0$

$b_2' = 0$: $\lambda_1 (x'')^2 + d = 0$

$(x'')^2 = K$

- $K > 0$
- $K = 0$
- $K < 0$

$\lambda_1\lambda_2 \neq 0$

$K \neq 0$ — $\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$ ellipszis $\longrightarrow$ Spec. $a=b$, $\lambda_1 = \lambda_2 \Rightarrow$ kör

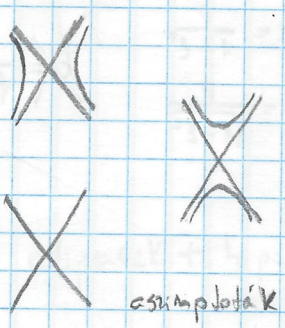
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$ ↯

$K = 0$ — $x = y = 0$ pont

$\lambda_1, \lambda_2 < 0$ — $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$

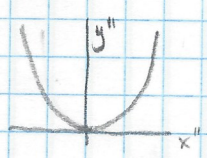


hiperbola

aszimptoták

$\lambda_1\lambda_2 = 0$

$b_2' \neq 0$ — $y'' = n(x'')^2$



$b_2'' = 0$ — $K > 0 \rightarrow x'' = \pm\sqrt{K}$ — párhuzamos egyenesek

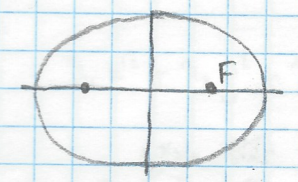
$K = 0 \rightarrow x'' = 0$ — egyenes

$K < 0$ ↯

$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1$ $a > b$

$x' = x'' + \alpha$

$y' = y'' + \beta$



$c^2 = a^2 - b^2$

$F_1''(c, 0)$ $F_2''(-c, 0)$

$F_1'(c+\alpha, \beta)$ $F_2'(-c+\alpha, \beta)$

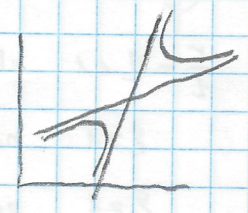
$\underline{r} = \underline{\underline{V}}^{-1}\underline{r}' = \underline{\underline{U}}\,\underline{r}' = (\,|\,|\,)\,\underline{r}'$

$\underline{f}_1 = \left(\boxed{s_1}\ \boxed{s_2}\right)\begin{pmatrix} c+\alpha \\ \beta \end{pmatrix}$

$\underline{f}_2 = \ldots \begin{pmatrix} -c+\alpha \\ \beta \end{pmatrix}$

$y'' = \pm\frac{b}{a}x''$

$(y' - \beta) = \frac{b}{a}(x' - \alpha)$

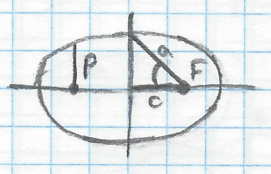


$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} - \\ - \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$x' = (\ )x + (\ )y$

$y' = (\ )x + (\ )y$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad a > b > 0$$



$$a^2 = b^2 + c^2$$

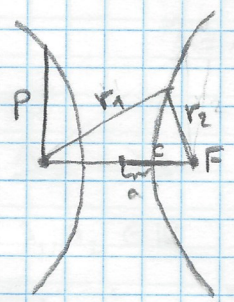
$$\varepsilon = \frac{c}{a} \in [0,1[$$

$$p = \frac{b^2}{a} \qquad a = \frac{p}{1-\varepsilon^2} \qquad b = \frac{p}{\sqrt{1-\varepsilon^2}} \qquad c = \frac{p\varepsilon}{\sqrt{1-\varepsilon^2}}$$

$$V(r) = -\frac{k}{r}$$

$$p = \frac{N^2}{mk}$$

$$\varepsilon^2 = 1 + \frac{2EN^2}{mk^2}$$

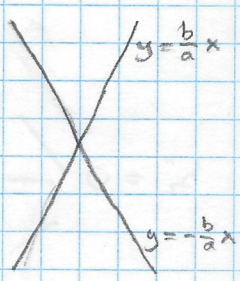


$$r_1 - r_2 = 2a$$

$$r_1 = \sqrt{(x+c)^2 + y^2}$$

$$r_2 = \sqrt{(x-c)^2 + y^2}$$

$$\Rightarrow \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



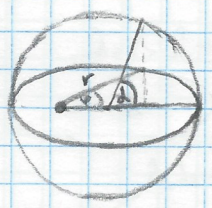
$$c > a$$

$$b^2 = c^2 - a^2$$

$$p = \frac{b^2}{a} \qquad \varepsilon = \frac{c}{a} > 1$$

$$x = a \cdot \mathrm{ch}\, t = x(t)$$

$$y = b \cdot \mathrm{sh}\, t = y(t)$$



$$r = \frac{p}{1 + \varepsilon \cdot \cos\varphi}$$

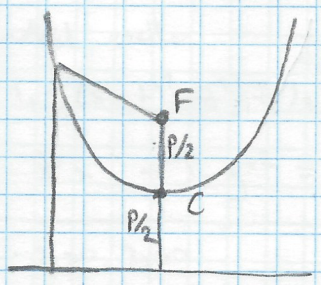
d: excentrikus anomália

$r(t), \varphi(t) \Rightarrow$ minden zárt alakja

$\varphi(d)$

$r(t) \Rightarrow$ előállítható

$t(d)$



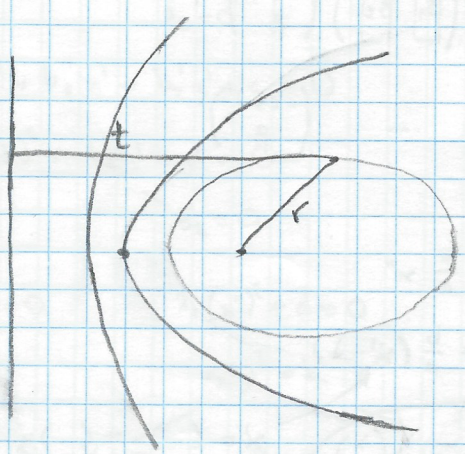
$$2py = x^2$$

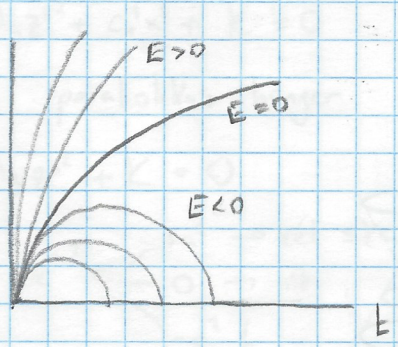
$$\frac{r}{t} = \varepsilon < 1 \quad \text{ellipszis}$$

$$\varepsilon > 1 \quad \text{hiperbola}$$

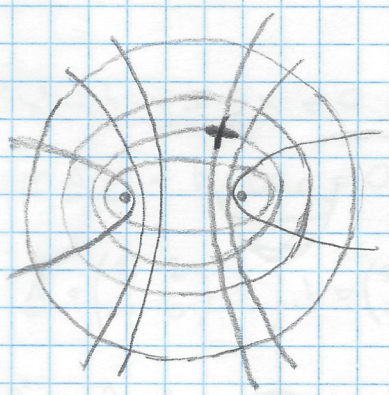
$$\varepsilon = 1 \quad \text{parabola}$$

↳ szeparátrix





$$E = \frac{mv^2}{2} - \frac{GMm}{r}$$



ellipszisek + hiperbolák => koordinátarendszer

---

dim = 3 $\quad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \underline{\underline{A}} = \begin{pmatrix} \# & \# & \# \\ \# & \# & \# \\ \# & \# & \# \end{pmatrix}$

$\underline{r}\,\underline{\underline{A}}\,\underline{r} + \underline{b}\,\underline{r} + c = 0 \qquad \hat{\underline{\underline{A}}} = \underline{\underline{A}}$

$\underline{\underline{A}}\,\underline{s} = \lambda \underline{s} \qquad \lambda_1, \lambda_1, \lambda_3 \in \mathbb{R}$

$\underline{s}^1, \underline{s}^2, \underline{s}^3 \qquad \underline{s}^k \underline{s}^l = \delta_{kl}$

$$\lambda_1 x'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + b_1' x' + b_2' y' + b_3' z' + c = 0$$

$\lambda_1, \lambda_2, \lambda_3 \neq 0$

$+++$ $\quad$ $++-$

$---$ $\quad$ $--+$

$= 0$

$\lambda_1 \neq 0 \quad \lambda_2 \neq 0$

$\lambda_3 = 0$

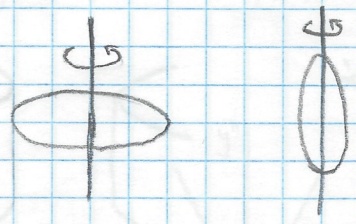
$\lambda_1 = \lambda_2 = 0$

$\lambda_3 \neq 0$

$+++$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$a > b > c$ $\quad$ ellipszoid



forgási ellipszoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0 \quad \text{pont}$$

$= -1$ ↯

$+ + -$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = \begin{cases} 1 \\ 0 \\ -1 \end{cases}$$

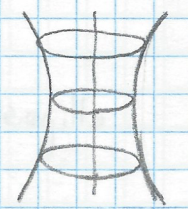
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$

$$z^2 = \frac{x^2}{a^2/c^2} + \frac{y^2}{b^2/c^2}$$

elliptikus kúp

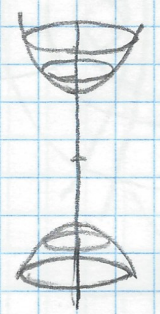
- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{z^2}{c^2}$

egyköpenyű hiperboloid



- $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2} - 1$

Kétköpenyű hiperboloid



$$\lambda_1 x'^2 + \lambda_2 y'^2 + b_1' x' + b_2' y' + b_3' z' + c = 0$$

$x'' = x' - \alpha$

$y'' = y' - \beta$

$$\lambda_1 x''^2 + \lambda_2 y''^2 + \overbrace{b_3' z' + c} = 0$$

$$\lambda_1 x''^2 + \lambda_2 y''^2 + b_3' z'' = 0$$

- $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

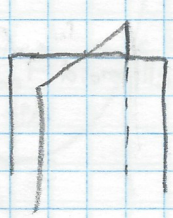
elliptikus paraboloid

- $z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$

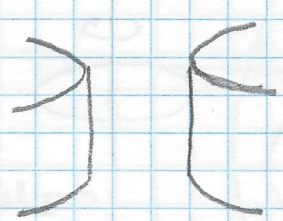
hiperbolikus paraboloid

$b_3' = 0$

$$\lambda_1 x''^2 + \lambda_2 y''^2 + c = 0$$



két metsző sík



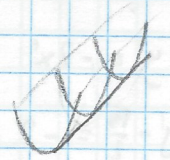
hiperbolikus henger

$$\lambda_3 z''^2 + b_1' x' + b_2' y' + b_3' z' + c = 0$$

$$\lambda_3 z''^2 + \underbrace{b_1' x' + b_2' y'} + K = 0$$

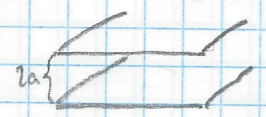
- $\lambda_3 z''^2 + c_1' x'' + K = 0$

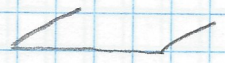
  parabolikus henger



- $\lambda_3 z''^2 + K = 0$

$z''^2 = \begin{cases} +a^2 \to \text{két s.k} \\ 0 \to \text{s.k} \\ -a^2 \end{cases}$





2017. 11. 16.

GY.

$3x^2 - 4xy + 5x = 0$

$\underline{r}\,\underline{\underline{A}}\,\underline{r} + \underline{b}\,\underline{r} + c = 0$

$\underline{\underline{A}} = \begin{pmatrix} 3 & -2 \\ -2 & 0 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \qquad c = 0$

$\lambda^2 - 3\lambda - 4 = 0$

$\lambda_{1,2} = \frac{3 \pm \sqrt{9+16}}{2} = \frac{3 \pm 5}{2} \begin{cases} 4 \\ -1 \end{cases}$

$\lambda_1 = 4$

$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} \qquad \underline{s}^{(1)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\lambda_2 = -1$

$\begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad \underline{s}^{(2)} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ / $\underline{s}^1, \underline{s}^2$ jobbrendszer

$b_1' = \underline{s}^{(1)}\,\underline{b} = \frac{1}{\sqrt{5}} (2 \;\; -1) \begin{pmatrix} 5 \\ 0 \end{pmatrix} = 2\sqrt{5}$

$b_2' = \underline{s}^{(2)}\,\underline{b} = \frac{1}{\sqrt{5}} (1 \;\; 2) \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \sqrt{5}$

$4x'^2 - y'^2 + 2\sqrt{5}x' + \sqrt{5}y' = 0$

$4\left(x'^2 + \frac{\sqrt{5}}{2}x'\right) - \left(y'^2 - \sqrt{5}y'\right) = 0$

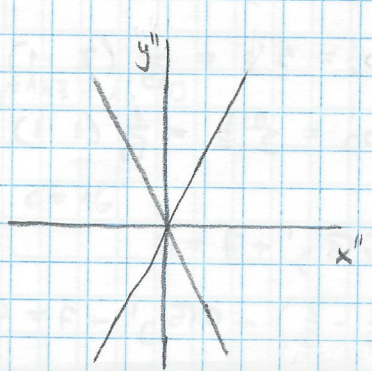
$4\left[\left(x' + \frac{\sqrt{5}}{4}\right)^2 - \frac{5}{16}\right] - \left[\left(y' - \frac{\sqrt{5}}{2}\right)^2 - \frac{5}{4}\right] = 0$ / $x'' = x' + \frac{\sqrt{5}}{4} \qquad y'' = y' - \frac{\sqrt{5}}{2}$

$4x''^2 - \frac{5}{4} - y''^2 + \frac{5}{4} = 0$

$4x''^2 - y''^2 = 0$

$y''^2 = 4x''^2$

$\boxed{\begin{aligned} y'' &= 2x'' \\ y'' &= -2x'' \end{aligned}}$



$y'' = 2x''$

$y' - \frac{\sqrt{5}}{2} = 2\left(x' + \frac{\sqrt{5}}{5}\right)$

$y' = 2x' + \frac{\sqrt{5}}{2} + \frac{\sqrt{5}}{2}$

$y' = 2x' + \sqrt{5}$

$y'' = -2x'$

$y' - \frac{\sqrt{5}}{2} = -2\left(x' + \frac{\sqrt{5}}{5}\right)$

$y' = -2x'$

$\underline{r}' = \underline{\underline{V}}\,\underline{r} = \left(\underline{\underline{\quad}}\right)\underline{r}$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$x' = \frac{1}{\sqrt{5}}(2x - y)$

$y' = \frac{1}{\sqrt{5}}(x + 2y)$

$\frac{1}{\sqrt{5}}(x+2y) = 2\frac{1}{\sqrt{5}}(2x-y) + \sqrt{5}$

$x + 2y = 2(2x - y) + 5$

$x + 2y = 4x - 2y + 5$

$4y = 3x + 5$

$\boxed{y = \frac{3}{4}x + \frac{5}{4}}$

$\frac{1}{\sqrt{5}}(x+2y) = -2\frac{1}{\sqrt{5}}(2x-y)$

$x + 2y = -2(2x - y)$

$x + 2y = -4x + 2y$

$5x = 0$

$\boxed{x = 0}$

$4x^2 - 4xy + y^2 - 2x - 14y + 7 = 0$

$\underline{\underline{A}} = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$ $\quad Sp\,\underline{\underline{A}} = 5$ $\quad det\,\underline{\underline{A}} = 0$

$\lambda^2 - 5\lambda + 0 = 0$

$\lambda_1 = 0$

$\lambda_2 = 5$

$\begin{pmatrix} -1 & -2 \\ -2 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ $\qquad s' = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$s^2 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$b = \begin{pmatrix} -2 \\ -14 \end{pmatrix}$ $\qquad b_1' = \underline{\underline{S}}^{(1)}\underline{b} = \frac{1}{\sqrt{5}}(2 \;\; -1)\begin{pmatrix} -2 \\ -14 \end{pmatrix} = 2\sqrt{5}$

$b_2' = \underline{\underline{S}}^{(2)}\underline{b} = \frac{1}{\sqrt{5}}(1 \;\; 2)\begin{pmatrix} -2 \\ -14 \end{pmatrix} = \frac{-30}{\sqrt{5}} = -6\sqrt{5}$

$5x'^2 + 2\sqrt{5}x' - 6\sqrt{5}y' + 7 = 0$

$5\left(x'^2 + \frac{2}{\sqrt{5}}x' + \frac{1}{5} - \frac{1}{5}\right) - 6\sqrt{5}y' + 7 = 0$

$5\left(x'+\frac{1}{\sqrt{5}}\right)^2 - 1 - 6\sqrt{5}y' + 7 = 0$

$5\left(x'+\frac{1}{\sqrt{5}}\right) - 6\sqrt{5}y + 6 = 0$

$5\left(x'+\frac{1}{\sqrt{5}}\right) - 6\sqrt{5}\left(y'-\frac{1}{\sqrt{5}}\right) = 0$

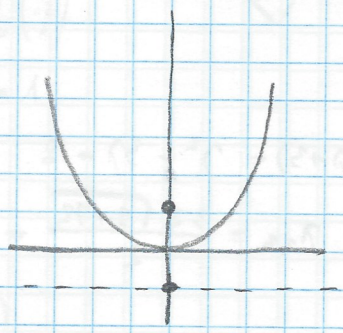
$x'' = x' + \frac{1}{\sqrt{5}}$

$y'' = y' - \frac{1}{\sqrt{5}}$

$5x''^2 - 6\sqrt{5}y'' = 0$

$x''^2 = \frac{6}{\sqrt{5}}y''$

$p = \frac{3}{\sqrt{5}}$



$\underline{f}'' = \begin{pmatrix} 0 \\ \frac{3\sqrt{5}}{10} \end{pmatrix}$

$\underline{r}'' = \underline{r}' + \begin{pmatrix} \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{pmatrix}$

$\underline{r}' = \underline{r}'' - \begin{pmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$

$\underline{f}' = \underline{f}'' - (\ ) = \begin{pmatrix} 0 \\ \frac{3\sqrt{5}}{10} \end{pmatrix} - \begin{pmatrix} 1/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix} = \begin{pmatrix} -1/\sqrt{5} \\ \sqrt{5}/2 \end{pmatrix}$

$y'' = y' - \frac{1}{\sqrt{5}} = -\frac{3}{2\sqrt{5}}$

$y' = -\frac{1}{2\sqrt{5}}$

$\underline{r}' = \underline{\underline{V}}\,\underline{r}$

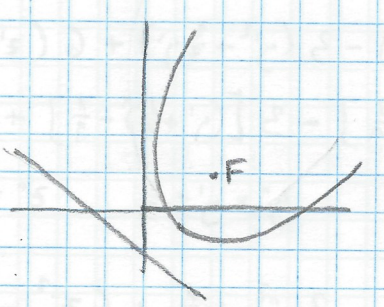
$\underline{r} = \underline{\underline{\tilde{V}}}\,\underline{r}' = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\underline{r}'$

$\underline{f} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1/\sqrt{5} \\ \sqrt{5}/2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -\frac{1}{5} \\ 1/2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 \\ 12 \end{pmatrix}$

$\begin{pmatrix} x' \\ y' \end{pmatrix} = V\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$y' = \frac{1}{\sqrt{5}}(x+2y) = -\frac{1}{2\sqrt{5}}$

$\boxed{x+2y = -\frac{1}{2}}$



Nem függvény

↓

paraméteresen kell

$x'' = t$

$y'' = \frac{\sqrt{5}}{6}t$

$x^2 + 2y^2 + z^2 + 4xy - 8xz - 4yz - 16x - 4y + 16z + 16 = 0$

$\underline{\underline{A}} = \begin{pmatrix} 1 & 2 & -4 \\ 2 & -2 & -2 \\ -4 & -2 & 1 \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} -16 \\ 4 \\ 16 \end{pmatrix} \qquad c = 16$

$Sp \underline{\underline{A}} = 0$

$I_{2c} = -6 - 15 - 6 = +27$

$$\det = \begin{vmatrix} 1 & 3 & -3 \\ 2 & 0 & 0 \\ -4 & -6 & -3 \end{vmatrix} = (-2) \cdot \begin{vmatrix} 3 & -3 \\ -6 & -3 \end{vmatrix} = (-2) \cdot (-9 - 18) = 54$$

$\lambda^3 - 27\lambda - 54 = 0$

$\lambda_1 = -3$

$\lambda^3 + 0 \cdot \lambda^2 - 27\lambda - 54 : (\lambda + 3) = \lambda^2 - 3\lambda - 18$

$$\lambda_{2,3} = \frac{3 \pm \sqrt{9 + 72}}{2} = \frac{3 \pm 9}{2} \begin{matrix} \nearrow 6 \\ \searrow -3 \end{matrix}$$

$\lambda_1 = \lambda_2 = -3$

$\lambda_3 = 6$

$$\begin{pmatrix} 4 & 2 & -4 \\ 2 & 1 & -2 \\ -4 & -2 & 4 \end{pmatrix} \qquad 2x + y - 2z = 0$$

$$(2 \; 1 \; -2) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$\underline{s}^3 = \frac{1}{3}\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ → merőlegesek: $\underline{s}^2 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ $\underline{s}^1 = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

$$\underline{\underline{V}} = \frac{1}{3}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$$

$$b_1' = \underline{s}^{(1)} \underline{b} = \frac{1}{3}(1 \; 2 \; 2)\begin{pmatrix} -15 \\ -6 \\ 15 \end{pmatrix} = 2$$

$$b_2' = \underline{s}^{(2)} \underline{b} = \frac{1}{3}(2 \; -2 \; 1)\begin{pmatrix} -15 \\ -6 \\ 15 \end{pmatrix} = -2$$

$$b_3' = \underline{s}^{(3)} \underline{b} = -20$$

$-3x'^2 - 3y'^2 + 6z'^2 + 2x' - 2y' - 20z' + 16 = 0$

$-3\left(x'^2 - \frac{2}{3}x'\right) - 3\left(y'^2 + \frac{2}{3}y'\right) + 6\left(z'^2 - \frac{10}{3}z'\right) + 16 = 0$

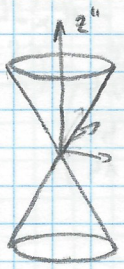
$-3\left[\left(x' - \frac{1}{3}\right)^2 - \frac{1}{9}\right] - 3\left[\left(y' + \frac{1}{3}\right)^2 - \frac{1}{9}\right] + 6\left[\left(z' - \frac{5}{3}\right)^2 - \frac{25}{9}\right] + 16 = 0$

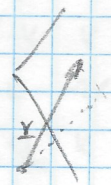
$x'' = x' - \frac{1}{3} \qquad y'' = y' + \frac{1}{3} \qquad z'' = z' + \frac{5}{3}$

$-3x''^2 + \frac{1}{3} - 3y''^2 + \frac{1}{3} + 6z''^2 - \frac{50}{3} + 16 = 0$

$-3x''^2 - 3y''^2 + 6z''^2 = 0$

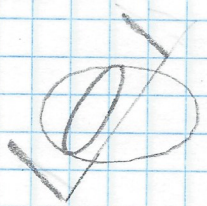
$x''^2 + y''^2 = 2z''^2$ kúp





$\vec{r}\,\vec{n} = \text{const.}$

$\underline{r}\,\underline{n} = c$



$x^2 + \frac{y^2}{5} + \frac{z^2}{9} = 1$

$2x - y - 2z = 3$ $\qquad \underline{n} = \frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix}$

$(2\ -1\ 2)\begin{pmatrix}x\\y\\z\end{pmatrix} = 3$ $\qquad \underline{n}\,\underline{r} = 1$

$\underline{n} = \frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix} \xrightarrow{F} \begin{pmatrix}0\\0\\1\end{pmatrix} = \underline{e}$

$\underline{f} = \underline{e} \times \underline{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} \times \frac{1}{3}\begin{pmatrix}2\\-1\\2\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\2\\0\end{pmatrix}$

$\underline{f} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\\0\end{pmatrix}$

$\cos\varphi = \underline{n}\cdot\underline{e} = \frac{2}{3}$ $\qquad \sin\varphi = \sqrt{1-\left(\frac{2}{3}\right)^2} = \frac{\sqrt{5}}{3}$

$F_{kl} = \cos\varphi\,\delta_{kl} + f_k f_l (1-\cos\varphi) + \sum_m \varepsilon_{klm}\, n_m \sin\varphi$

$F_{11} = \cos\varphi + f_1 f_1 (1-\cos\varphi) = \frac{2}{3} + \frac{1}{5}\cdot\frac{1}{3} = \frac{11}{15}$

$F_{22} = \cos\varphi + f_2 f_2 (1-\cos\varphi) = \frac{2}{3} + \frac{2}{5}\cdot\frac{1}{3} = \frac{12}{15}$

$F_{33} = \cos\varphi + f_3 f_3 (1-\cos\varphi) = \frac{2}{3}$

$F_{12} = \frac{2}{5}\,\frac{1}{3} + \ldots = \frac{2}{15}$

$F_{21} = \frac{2}{5}\,\frac{1}{3} + \ldots = \frac{2}{15}$

$F_{13} = \frac{2}{\sqrt{5}}\cdot\frac{\sqrt{5}}{3} = \frac{2}{3}$

$F_{31} = -\frac{2}{3}$

$F_{23} = -\frac{1}{\sqrt{5}}\cdot\frac{\sqrt{5}}{3} = -\frac{1}{3}$

$F_{32} = \frac{1}{3}$

$$\underline{\underline{F}} = \begin{pmatrix} \frac{11}{15} & \frac{2}{15} & \frac{2}{3} \\ \frac{2}{15} & \frac{12}{15} & -\frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{pmatrix}$$

$\underline{r}' = \underline{\underline{F}}\,\underline{r}$

$\underline{\underline{\tilde{r}}}' = \tilde{r}\,\underline{\underline{\tilde{F}}}$

$A' = F A \tilde{F}$

$$A' = (F)\begin{pmatrix} 1 & & \\ & \frac{1}{i} & \\ & & \frac{1}{3} \end{pmatrix}$$

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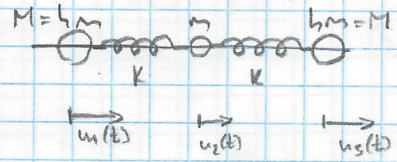
GY

$$A = \sum_k \lambda_k P^{(k)}$$

$$f(A) = \sum_k f(\lambda_k) P^{(k)}$$

$$A^{-1} = \sum_k \frac{1}{\lambda_k} P^{(k)}$$

$$A A^{-1} = \left(\sum_k \lambda_k P^{(k)}\right)\left(\sum_l \frac{1}{\lambda_l} P^{(l)}\right) = \sum_k \sum_l \frac{\lambda_k}{\lambda_l} P^{(k)} P^{(l)} = \sum_k \sum_l \frac{\lambda_k}{\lambda_l} \delta_{kl} P^{(k)} = \sum_k P^{(k)} = I$$



$$\begin{cases} M\ddot{u}_1 = K\cdot(u_2 - u_1) \\ m\ddot{u}_2 = K\cdot(u_1 - u_2) + K(u_3 - u_2) \\ M\ddot{u}_3 = K\cdot(u_2 - u_3) \end{cases}$$

$$u_k = a_k \cdot \cos\omega t$$

$$\ddot{u}_k = -\omega^2 \cdot a_k \cdot \cos\omega t$$

$$K(a_2 - a_1) = -M\omega^2 a_1$$

$$K(a_1 - a_2) + K(a_3 - a_2) = -m\omega^2 a_2$$

$$K(a_2 - a_3) = -M\omega^2 a_3$$

$$-Ka_1 + Ka_2 + 0\,a_3 = -\omega^2(Ma_1 + 0\cdot a_2 + 0\cdot a_3)$$

$$Ka_1 - 2Ka_2 + Ka_3 = -\omega^2(0\cdot a_1 + m a_2 + 0\cdot a_3)$$

$$0\,a_1 + Ka_2 - Ka_3 = -\omega^2(0\cdot a_1 + 0\cdot a_2 + Ma_3)$$

$$\begin{pmatrix} -k & k & 0 \\ k & -2k & k \\ 0 & k & -k \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = -\omega^2 \begin{pmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

$$-\underline{\underline{K}}\,\underline{a} = -\omega^2\,\underline{\underline{M}}\,\underline{a}$$

$$\underline{\underline{M}}\,\ddot{\underline{u}} = -\underline{\underline{K}}\,\underline{u}$$

$$\ddot{u}_1 = -\frac{k}{M}u_1 + \frac{k}{M}u_2$$

$$\ddot{u}_2 = \frac{k}{M}u_1 - \frac{2k}{M}u_2 + \frac{k}{M}u_3$$

$$\ddot{u}_3 = \frac{k}{M}u_2 - \frac{k}{M}u_3$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}'' = -\begin{pmatrix} \frac{k}{M} & -\frac{k}{M} & 0 \\ -\frac{k}{M} & \frac{2k}{M} & -\frac{k}{M} \\ 0 & -\frac{k}{M} & \frac{k}{M} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$$

$$\ddot{\underline{u}}(t) = -\underline{\underline{A}}\,\underline{u}(t)$$

TFH $\quad \underline{u}(t) = \underline{a}\cdot\cos\omega t$

$$\ddot{\underline{u}}(t) = -\omega^2\,\underline{a}\cdot\cos\omega t$$

$$-\omega^2\underline{a}\cos\omega t = -\underline{\underline{A}}\,\underline{a}\cos\omega t$$

$$\underline{\underline{A}}\,\underline{a} = \omega^2\underline{a}$$

$\omega_1, \omega_2, \omega_3 \rightarrow \underline{a}_1, \underline{a}_2, \underline{a}_3$ sajátrezgések

$$\frac{M}{M} = c \qquad \omega_0^2 = \frac{k}{M}$$

$$\underline{\underline{A}} = \frac{k}{M}\begin{pmatrix} 1 & -1 & 0 \\ -c & 2c & -c \\ 0 & -1 & 1 \end{pmatrix} = \omega_0^2\cdot\underline{\underline{B}}$$

$$\omega_0^2\,\underline{\underline{B}}\,\underline{a} = \omega^2\,\underline{a}$$

$$\underline{\underline{B}}\,\underline{a} = \left(\frac{\omega^2}{\omega_0^2}\right)\cdot\underline{a}$$

$$\downarrow$$

$$\lambda$$

$$\downarrow$$

$$\omega = \omega_0\cdot\sqrt{\lambda}$$

$\frac{M}{m} = 4$

$$\underline{\underline{B}} = \begin{pmatrix} 1 & -1 & 0 \\ -4 & 8 & -4 \\ 0 & -1 & 1 \end{pmatrix}$$

$Sp\,\underline{\underline{B}} = 10$

$I_{2\acute{e}} = 4 + 1 + 4 = 9$

$\det = 0$

$\lambda^3 - 10\lambda^2 + 9\lambda - 0 = 0$

$\lambda = 0 \qquad \lambda^2 - 10\lambda + 9 = 0$

$$\lambda_{1,2} = \frac{10 \pm \sqrt{100 - 36}}{2} = \frac{10 \pm 8}{6} \begin{matrix} \nearrow 1 \\ \searrow 9 \end{matrix}$$

$\lambda_1 = 0 \qquad \lambda_2 = 1 \qquad \lambda_3 = 9$

$\omega_1 = 0 \qquad \omega_2 = \omega_0 \qquad \omega_3 = 3\omega_0$

$\lambda_1 = 0$ $\begin{pmatrix} 1 & -1 & 0 \\ -4 & 8 & -4 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ zérusmódus (eltolás: szimmetria) $\rightarrow\rightarrow\rightarrow$

$\lambda_2 = 1$ $\begin{pmatrix} 0 & -1 & 0 \\ -4 & 7 & -4 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$\lambda_3 = 9$ $\begin{pmatrix} -8 & -1 & 0 \\ -4 & -1 & -4 \\ 0 & -1 & -8 \end{pmatrix} \begin{pmatrix} 1 \\ -8 \\ 1 \end{pmatrix}$

$\dot{x}_1(t) = \alpha x_1 + \beta x_2 + \gamma x_3 + \dots$

$\dot{x}_2(t) = \mu x_1 + \nu x_2$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}^{\cdot} = \begin{pmatrix} \alpha & \beta & \gamma & \dots \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$$

$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x}$ | $\dot{x} = \alpha x \qquad t=0 \quad x = x_0$

$x = C \cdot e^{\alpha t} \qquad x_0 = C \cdot e^0 = C \rightarrow x(t) = x_0 \cdot e^{\alpha t}$

$\Rightarrow$ $\boxed{\underline{x}(t) = e^{\underline{\underline{A}}t} \cdot \underline{x}_0}$

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \underline{\underline{I}} + \underline{\underline{A}}t + \frac{1}{2}\underline{\underline{A}}^2 t^2 + \frac{1}{3!}\underline{\underline{A}}^3 t^3 + \dots$$

$$\dot{\underline{x}} = \left(\underline{\underline{A}} + \underline{\underline{A}}^2 t + \frac{t^2}{2}\underline{\underline{A}}^3 + \dots\right)\underline{x}_0 = \underline{\underline{A}}\underbrace{\left(\underline{\underline{I}} + \underline{\underline{A}}t + \frac{\underline{\underline{A}}^2 t^2}{2} + \dots\right)}_{e^{At}}\underline{x}_0$$

$$\underline{\underline{A}}\,\underline{u} = \lambda_u$$

$$\hat{\underline{\underline{A}}}\,\underline{v} = \lambda_v$$

$$\underline{\underline{A}} = \sum \lambda_k \underline{\underline{P}}^{(k)}$$

$$e^{\underline{\underline{A}}t} = \sum e^{\lambda_k t}\,\underline{\underline{P}}^{(k)}$$

$$\underline{x}(t) = \sum_k e^{\lambda_k t}\,\underline{\underline{P}}^{(k)}\,\underline{x}_0$$

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EA

$S \to V$

$V \to S$

$V \to V$

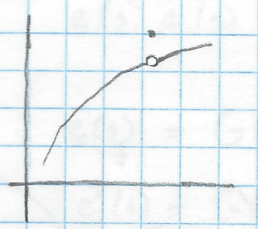
Pontok $\leftrightarrow V \to V'$ mező (pl. elektromágneses)

$\mathbb{R} \supset I \longrightarrow P_3 \longleftrightarrow V_3$

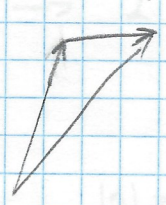
t $\quad$ P (pont) $\quad$ $\vec{r}$

$\vec{r}(t)$

Görbe: $I \to P_3$ folytonos hozzárendelés



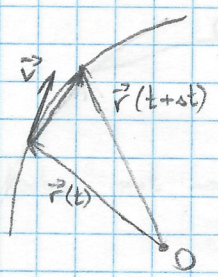
$|y-a| < \varepsilon$

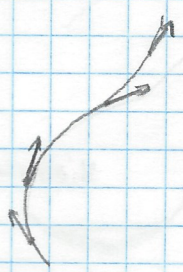


$|\vec{y}-\vec{a}| < \varepsilon$

$$\lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \dot{\vec{r}}(t) = \vec{v}(t)$$

$$\dot{\vec{v}}(t) = \vec{a}(t)$$

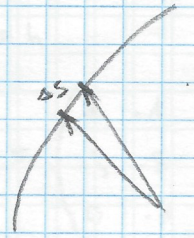




$v = |\vec{v}|$

$\vec{e}(t) = \frac{\vec{v}}{v}$

$|\vec{e}| = 1$ érintő egységvektor $\longrightarrow$ paraméterezéstől független



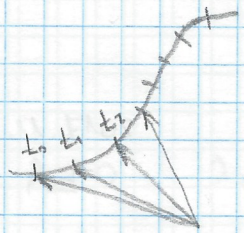
$$\vec{r}(t+\Delta t) - \vec{r}(t) = \Delta\vec{r}$$

$$\Delta s = |\Delta\vec{r}|$$

$$\frac{\Delta\vec{r}}{\Delta t} \approx \vec{v}$$

$$\Delta\vec{r} \approx \vec{v}\,\Delta t$$

$$\Delta s = |\Delta\vec{r}| = |\vec{v}\cdot\Delta t|$$

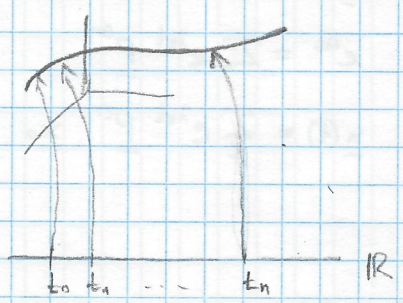


$$\frac{\vec{r}(t_{k+1}) - \vec{r}(t_k)}{t_{k+1} - t_k}$$

$$\Delta\vec{r}_k = \vec{v}(t_k)\,\Delta t_k$$

$$\Delta s_k = |\Delta\vec{r}_k| = |\vec{v}(t_k)|\,\Delta t_k$$

$$s \approx \sum \Delta s_k = \sum |v_k|\,\Delta t_k \longrightarrow \boxed{\int |\vec{v}(t)|\,dt}$$



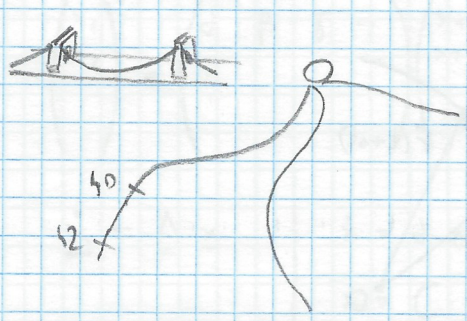
$$\mathbb{R} \supset I \longleftrightarrow P_3 \longrightarrow V_3 \longrightarrow \mathbb{R}^3$$

$$\vec{r}(t) \longrightarrow \underline{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

$$\underline{\dot{r}}(t) = \lim_{\Delta t \to 0} \frac{\underline{r}(t+\Delta t) - \underline{r}(t)}{\Delta t} = \begin{pmatrix} \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \\ \lim \\ \lim \end{pmatrix} = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \underline{v}$$

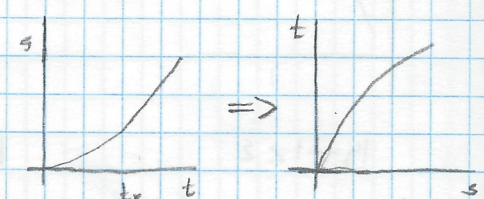
$$|\underline{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2 + \dot{z}(t)^2}$$

$$s = \int |v|\,dt = \int_{t_{kezdet}}^{t_{vég}} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\,dt$$



$$C = \int_a^b f(x)\,dx$$

$$C(t) = \int_a^t f(x)\,dx$$



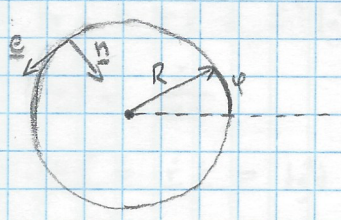
$$s(t_{vég}) = \int_{t_k}^{t_v} \sqrt{\dots}\,dt$$

$$\vec{r}(t) = \vec{r}(t(s)) = \vec{r}(s)$$

ívhosszparaméterezés

$$\vec{r}'(s) = \lim \frac{\Delta\vec{r}}{\Delta s}$$

$$|\vec{r}'| = \lim \frac{|\Delta\vec{r}|}{\Delta s} = \lim \frac{\Delta s}{\Delta s} = 1$$



$\varphi = \omega t$ $\quad \underline{r}(t) = \begin{pmatrix} R\cos\omega t \\ R\sin\omega t \end{pmatrix}$

$$\underline{v} = \dot{\underline{r}} = \begin{pmatrix} -R\omega\sin\omega t \\ R\omega\cos\omega t \end{pmatrix} = R\omega \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}$$

$$|\underline{v}|^2 = \underline{v}\cdot\underline{v} = R^2\omega^2 \qquad |\underline{v}| = R\omega$$

$$\underline{e} = \frac{\underline{v}}{v} = \begin{pmatrix} -\sin\omega t \\ \cos\omega t \end{pmatrix}$$

$$s(t) = \int_0^t dt\, |\underline{v}(t)| = \int_0^t dt\, R\omega = R\omega t$$

$$t = \frac{1}{R\omega}\, s$$

$$\underline{r}(s) = \begin{pmatrix} R\cdot\cos\frac{s}{R} \\ R\cdot\sin\frac{s}{R} \end{pmatrix}$$

$$\underline{r}'(s) = \begin{pmatrix} R\left(-\sin\frac{s}{R}\right)\cdot\frac{1}{R} \\ R\cdot\cos\frac{s}{R}\cdot\frac{1}{R} \end{pmatrix} = \begin{pmatrix} -\sin\frac{s}{R} \\ \cos\frac{s}{R} \end{pmatrix} = \underline{e}(s)$$

$$\underline{e}'(s) = \begin{pmatrix} -\frac{1}{R}\cdot\cos\frac{s}{R} \\ -\frac{1}{R}\cdot\sin\frac{s}{R} \end{pmatrix} = -\frac{1}{R}\begin{pmatrix} \cos\frac{s}{R} \\ \sin\frac{s}{R} \end{pmatrix}$$

$$\underline{e}'\cdot\underline{e} = 0$$

$$\frac{d}{dt}\left[\underline{a}(t)\,\underline{b}(t)\right] = \dot{\underline{a}}\,\underline{b} + \underline{a}\,\dot{\underline{b}}$$

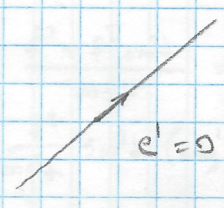
$$\frac{d}{dt}\left(\underline{a}\times\underline{b}\right) = \dot{\underline{a}}\times\underline{b} + \underline{a}\times\dot{\underline{b}}$$

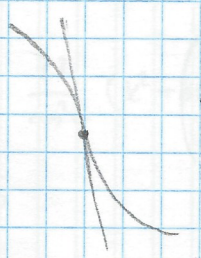
$$\underline{e}(s)\,\underline{e}(s) = 1 \quad \longrightarrow \quad 2\,\underline{e}(s)\cdot\underline{e}'(s) = 0$$

$$|\underline{e}'(s)| = G(s)$$

$$\underline{e}'(s) = G(s)\,\underline{n}(s) \qquad |\underline{n}| = 1$$

G: görbület $\quad$ $\underline{n}$: főnormális



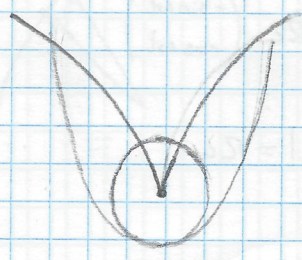


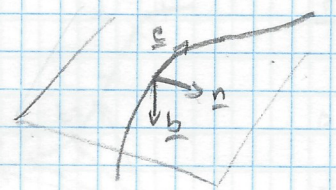
inflexiós pont → görbület nem számítható

simulókör

$$R = \frac{1}{G}$$

evolúta: a simulókörök középpontjainak halmaza





binormális: $\underline{b} = \underline{e} \times \underline{n}$

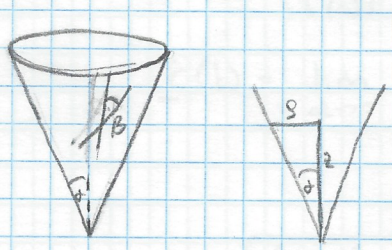
$|b'(s)| = T(s)$ torzió

$\underline{e}, \underline{n}, \underline{b}$ bázis $\longrightarrow$ kísérő triéder

GÖRBÉK ALAPTÉTELE: $G(s)$, $T(s)$ meghatározza a görbét

2017. 11. 21.

GY



$$\frac{\rho}{z} = \operatorname{tg}\alpha$$

$$x^2 + y^2 = \rho^2$$

$$\Rightarrow x^2 + y^2 = z^2 \cdot \operatorname{tg}^2\alpha$$

$z = f(\varphi)$ görbét keressem

$$x = \rho \cdot \cos\varphi = z \cdot \operatorname{tg}\alpha \cdot \cos\varphi = f(\varphi) \cdot \operatorname{tg}\alpha \cdot \cos\varphi$$

$$y = \rho \cdot \sin\varphi = z \cdot \operatorname{tg}\alpha \cdot \sin\varphi = f(\varphi) \cdot \operatorname{tg}\alpha \cdot \sin\varphi$$

$$\underline{r}(\varphi) = \begin{pmatrix} x(\varphi) \\ y(\varphi) \\ z(\varphi) \end{pmatrix} = \begin{pmatrix} f(\varphi) \cdot \operatorname{tg}\alpha \cos\varphi \\ f(\varphi) \operatorname{tg}\alpha \cdot \sin\varphi \\ f(\varphi) \end{pmatrix}$$

$$\dot{\underline{r}}(\varphi) = \begin{pmatrix} \operatorname{tg}\alpha \cdot (\dot{f}(\varphi)\cos\varphi - f(\varphi)\cdot\sin\varphi) \\ \operatorname{tg}\alpha \cdot (\dot{f}(\varphi)\cdot\sin\varphi + f(\varphi)\cdot\cos\varphi) \\ \dot{f}(\varphi) \end{pmatrix}$$

$$\underline{r}\,\dot{\underline{r}} = |\underline{r}|\,|\dot{\underline{r}}| \cdot \cos\beta$$

$$\underline{r}\,\dot{\underline{r}} = f \cdot \operatorname{tg}^2\alpha \cdot \cos\varphi \cdot (\dot{f}\cdot\cos\varphi - f\cdot\sin\varphi) + f \cdot \operatorname{tg}^2\alpha \sin\varphi (\dot{f}\sin\varphi + f\cdot\cos\varphi) + f\cdot\dot{f} =$$

$$= f\dot{f}\left(\operatorname{tg}^2\alpha\cos^2\varphi + \operatorname{tg}^2\alpha\sin^2\varphi + 1\right) = f\dot{f}\left(\operatorname{tg}^2\alpha + 1\right) = \frac{f\dot{f}}{\cos^2\alpha}$$

$$|\underline{r}|^2 = \underline{r}\,\underline{r} = f^2 \operatorname{tg}^2\delta \cos^2\varphi + f^2 \operatorname{tg}^2\delta \sin^2\varphi + f^2 = f^2(\operatorname{tg}^2\delta + 1) = \frac{f^2}{\cos^2\delta}$$

$$|\underline{r}| = \frac{f}{\cos\delta}$$

$$|\dot{\underline{r}}|^2 = \dot{\underline{r}}\,\dot{\underline{r}} = \operatorname{tg}^2\delta\,(\dot{f}\cos\varphi - f\sin\varphi)^2 + \operatorname{tg}^2\delta\,(\dot{f}\sin\varphi + f\cos\varphi) + \dot{f}^2 =$$

$$= \operatorname{tg}^2\delta\left[\dot{f}^2\cos^2\varphi - 2\dot{f}f\cos\varphi\cdot\sin\varphi + f^2\sin^2\varphi\right] + \operatorname{tg}^2\delta\left[\dot{f}^2\sin^2\varphi + 2\dot{f}f\sin\varphi\cos\varphi + f^2\cos^2\varphi\right] + \dot{f}^2 =$$

$$= \operatorname{tg}^2\delta\,(\dot{f}^2 + f^2) + \dot{f}^2 = \dot{f}^2(\operatorname{tg}^2\delta + 1) + f^2\operatorname{tg}^2\delta = \frac{\dot{f}^2 + f^2\sin^2\delta}{\cos^2\delta}$$

$$\underline{r}\,\dot{\underline{r}} = |\underline{r}|\,|\dot{\underline{r}}|\cos\beta$$

$$\frac{f\dot{f}}{\cos^2\delta} = \frac{f}{\cos\delta}\cdot\frac{\sqrt{\dot{f}^2 + f^2\sin^2\delta}}{\cos\delta}\cdot\cos\beta$$

$$\dot{f} = \sqrt{\dot{f}^2 + f^2\cdot\sin^2\delta}\cdot\cos\beta$$

$$\dot{f}^2 = \dot{f}^2\cos^2\beta + f^2\cdot\sin^2\delta\cdot\cos^2\beta$$

$$\dot{f}^2\sin^2\beta = f^2\cdot\sin^2\delta\cos^2\beta$$

$$\dot{f}\sin\beta = f\cdot\sin\delta\cdot\cos\beta$$

$$\dot{f} = \sin\delta\cdot\operatorname{ctg}\beta\cdot f$$

$$k = \sin\delta\cdot\operatorname{ctg}\beta$$

$$\dot{f} = kf$$

$$\frac{df}{d\varphi} = kf$$

$$\int\frac{df}{f} = k\,d\varphi$$

$$\ln f = k(\varphi - \varphi_0)$$

$$f(\varphi) = e^{k(\varphi-\varphi_0)} = C\cdot e^{k\varphi}$$

$$\underline{r}(\varphi) = \begin{pmatrix} e^{k\varphi}\cdot\operatorname{tg}\delta\cdot\cos\varphi \\ e^{k\varphi}\cdot\operatorname{tg}\delta\cdot\sin\varphi \\ e^{k\varphi}\end{pmatrix} \qquad \dot{\underline{r}}(\varphi) = \begin{pmatrix} \operatorname{tg}\delta\,(k e^{k\varphi}\cdot\cos\varphi - e^{k\varphi}\sin\varphi) \\ \operatorname{tg}\delta\,(k e^{k\varphi}\cdot\sin - e^{k\varphi}\cdot\cos\varphi) \\ k\cdot e^{k\varphi}\end{pmatrix} = e^{k\varphi}\begin{pmatrix} \operatorname{tg}\delta\,(k\cos\varphi - \sin\varphi) \\ \operatorname{tg}\delta\,(k\cdot\sin\varphi - \cos\varphi) \\ k\end{pmatrix}$$

$$\dot{\underline{r}}^2 = \frac{\dot{f}^2 + f^2\sin^2\delta}{\cos^2\delta} = \frac{k^2f^2 + f^2\cdot\sin^2\delta}{\cos^2\delta} = \frac{f^2}{\cos^2\delta}\,(k^2 + \sin^2\delta) = \frac{f^2}{\cos^2\delta}\,(\sin^2\delta\cdot\operatorname{ctg}^2\beta + \sin^2\delta) =$$

$$= \frac{f^2}{\cos^2\delta}\sin^2\delta\,(\operatorname{ctg}^2\beta + 1) = f^2\frac{\operatorname{tg}^2\delta}{\sin^2\beta}$$

$$|\dot{\underline{r}}| = f\cdot\frac{\operatorname{tg}\delta}{\sin\beta} = e^{k\varphi}\cdot\frac{\operatorname{tg}\delta}{\sin\beta}$$

$$s = \int|\dot{\underline{r}}|\,d\varphi = \int e^{k\varphi}\cdot\frac{\operatorname{tg}\delta}{\sin\beta}\,d\varphi = \left[\frac{1}{k}\,e^{k\varphi}\,\frac{\operatorname{tg}\delta}{\sin\beta}\right]_{-\infty}^{\varphi_0} = \frac{1}{\sin\delta\cdot\operatorname{ctg}\beta}\cdot e^{k\varphi}\cdot\frac{\operatorname{tg}\delta}{\sin\beta}$$

$$s(\varphi) = \frac{1}{\cos\alpha \cdot \cos\beta} \cdot e^{k\varphi}$$ logaritmikus spirál (kúpra tekerve)

$\underline{r}(s) \longrightarrow \underline{r}'(s) = \underline{e}(s) \longrightarrow \underline{e}'(s) = G(s)\,\underline{n}(s) \longrightarrow \underline{b}(s) = \underline{e}(s) \times \underline{n}(s) \longrightarrow |\underline{b}'(s)| = T(s)$

$\underline{r}(t(s))$ $\quad \underline{r}(t)$ $\quad \underline{v} = \dot{\underline{r}}$, $\quad \underline{a} = \dot{\underline{v}} = \ddot{\underline{r}}$; $\quad \underline{c} = \dot{\underline{a}} = \ddot{\underline{v}} = \dddot{\underline{r}}$

$$\underline{e} = \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{dt}\frac{dt}{ds} = \dot{\underline{r}}(t)\cdot t'(s) = \underline{v}\cdot\frac{1}{v}$$

$$\frac{ds}{dt} = |\underline{v}| \longrightarrow \frac{dt}{ds} = \frac{1}{|\underline{v}|}$$

$$\underline{e} = \frac{\underline{v}}{v} \qquad \underline{e}' = \frac{d\underline{e}(t(s))}{ds} = \frac{d\underline{e}}{dt}\cdot\frac{dt}{ds} = \frac{1}{v}\cdot\frac{d\underline{e}}{dt} = \frac{1}{v}\cdot\left(\frac{\underline{v}}{v}\right)^{\cdot} = \frac{1}{v}\cdot\left(\underline{v}\cdot\frac{1}{v}\right)^{\cdot} =$$

$$= \frac{1}{v}\left[\dot{\underline{v}}\frac{1}{v} + \underline{v}\left(\frac{1}{v}\right)^{\cdot}\right] = \frac{1}{v}\left[\underline{a}\cdot\frac{1}{v} - \underline{v}\cdot\frac{|v|^{\cdot}}{v^2}\right]$$

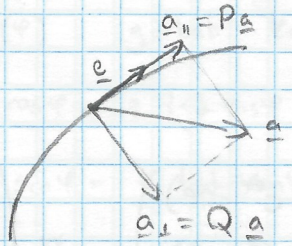
$$|v|^{\cdot} = \left(\sqrt{|v|^2}\right)^{\cdot} = \left(\sqrt{\underline{v}\,\underline{v}}\right)^{\cdot} = \frac{1}{2}\frac{1}{\sqrt{\underline{v}\,\underline{v}}}(\underline{v}\,\underline{v})^{\cdot} = \frac{1}{2}\frac{1}{v}(2\underline{v}\,\dot{\underline{v}}) = \frac{\underline{v}\,\underline{a}}{v}$$

$$\underline{e}' = \frac{1}{v}\left[\underline{a}\frac{1}{v} - \underline{v}\frac{1}{v^2}\frac{(\underline{v}\,\underline{a})}{v}\right] = \frac{1}{v^3}\left[v^2\underline{a} - \underline{v}(\underline{v}\,\underline{a})\right] = \frac{1}{v^3}\,\underline{v}\times(\underline{a}\times\underline{v})$$

$$G = |\underline{e}'| = \frac{1}{v^3}\left|\underline{v}\times(\underline{a}\times\underline{v})\right| = \frac{1}{v^3}\left|\underline{d}\times\underline{v}\right| = \frac{1}{v^3}|\underline{d}|\cdot|\underline{v}| = \frac{|\underline{v}\times\underline{a}|}{|v|^3} = \frac{|\dot{\underline{r}}\times\ddot{\underline{r}}|}{|\dot{r}|^3}$$

$$\underline{e}' = G\,\underline{n} = \frac{1}{v^3}\left[v^2\underline{a} - v(\underline{v}\,\underline{a})\right] = \frac{1}{v^2}\left[\underline{a} - \frac{1}{v^2}\underline{v}(\underline{v}\,\underline{a})\right] = \frac{1}{v^2}\left[\underline{a} - \underline{e}(\underline{e}\,\underline{a})\right] =$$

$$= \frac{1}{v^2}\left(\underline{\underline{I}} - \underline{e}\circ\underline{e}\right)\underline{a} = \frac{1}{v^2}\,\underline{\underline{Q}}\,\underline{a}$$



$$|\underline{e}'| = G = \frac{1}{v^2}|\underline{\underline{Q}}\,\underline{a}| = \frac{1}{v^2}|\underline{a}_{\perp}|$$

$$\| \quad \frac{1}{R}$$

$$|\underline{a}_{\perp}| = \frac{v^2}{R} \qquad |\underline{a}_{\parallel}| = |\underline{v}|^{\cdot} = \frac{\underline{v}\,\underline{a}}{v} = \underline{e}\,\underline{a}$$

$$\underline{e}' = \frac{1}{v^3}\cdot\underline{v}\times(\underline{a}\times\underline{v}) = \frac{\underline{d}\times\underline{v}}{v^4} \qquad G = \frac{|\underline{d}|}{v^3}$$

$$\underline{n} = \frac{\underline{e}'}{|\underline{e}'|} = \frac{\underline{e}'}{G} = \frac{\underline{d}\times\underline{v}}{v^4}\cdot\frac{v^3}{|\underline{d}|} = \frac{\underline{d}\times\underline{v}}{v\,d}$$

$$\underline{b} = \underline{e}\times\underline{n} = \underline{e}\times\frac{\underline{d}\times\underline{v}}{dv} = \frac{1}{dv}\,\underline{e}\times(\underline{d}\times\underline{v}) = \frac{1}{dv}\left[\underline{d}\underbrace{(\underline{e}\,\underline{v})}_{v} - \underline{v}\underbrace{(\underline{d}\,\underline{e})}_{0}\right] = \frac{\underline{d}}{d}$$

$$\underline{d}\,\underline{e} = \frac{d\underline{v}}{v} = \frac{(\underline{v}\times\underline{a})\cdot\underline{v}}{v} = 0$$

$$\underline{b}' = \frac{d}{ds}\,\underline{b}(t(s)) = \frac{d\underline{b}}{dt}\frac{dt}{ds} = \frac{1}{v}\cdot\dot{\underline{b}} = \frac{1}{v}\left(\frac{\underline{d}}{d}\right)^{\cdot} = \frac{1}{v}\left[\dot{\underline{d}}\cdot\frac{1}{d} + \underline{d}\cdot\left(\frac{1}{d}\right)^{\cdot}\right] = \frac{1}{v}\left[\dot{\underline{d}}\frac{1}{d} - \underline{d}\cdot\frac{|\underline{d}|^{\cdot}}{d^2}\right] =$$

$$= \frac{1}{v}\left[\dot{\underline{d}}\frac{1}{d} - \underline{d}\cdot\frac{1}{d^2}\cdot\frac{d\dot{d}}{d}\right] = \frac{1}{vd^3}\left[d^2\dot{\underline{d}} - \underline{d}(\underline{d}\dot{\underline{d}})\right] = \frac{1}{vd^3}\cdot(\underline{d}\times\dot{\underline{d}})\times\underline{d}$$

$$T = |\underline{b}'| = \frac{1}{vd^3}\cdot|(\underline{d}\times\dot{\underline{d}})\times\underline{d}| = \frac{1}{vd^3}\cdot|\underline{d}\times\dot{\underline{d}}|\cdot|\underline{d}| = \frac{1}{vd}|\underline{d}\times\dot{\underline{d}}|$$

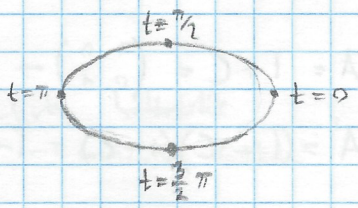
$$\underline{d} = \underline{v}\times\underline{a}$$

$$\dot{\underline{d}} = \dot{\underline{v}}\times\underline{a} + \underline{v}\times\dot{\underline{a}} = \underbrace{\underline{a}\times\underline{a}}_{0} + \underline{v}\times\dot{\underline{a}} = \underline{v}\times\dot{\underline{a}} = \underline{v}\times\underline{c}$$

$$T = \frac{1}{vd^2}\cdot\left|\underline{v}(dc) - c\underbrace{(\underline{v}\underline{d})}_{0}\right| = \frac{1}{vd^2}|\underline{v}(\underline{d}\underline{c})| = \frac{|\underline{d}\underline{c}|}{d^2} = \frac{(\underline{v}\times\underline{a})\cdot\underline{c}}{d^2} = \frac{(\underline{v},\underline{a},\underline{c})}{|\underline{v}\times\underline{a}|^2}$$

$$T = \frac{(\underline{r},\dot{\underline{r}},\ddot{\underline{r}})}{|\dot{\underline{r}}\times\ddot{\underline{r}}|^2}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



$$x = a\cdot\cos t$$

$$y = b\cdot\sin t$$

$$\underline{r}(t) = \begin{pmatrix} a\cdot\cos t \\ b\cdot\sin t \\ 0 \end{pmatrix} \qquad \dot{\underline{r}}(t) = \begin{pmatrix} -a\cdot\sin t \\ b\cdot\cos t \\ 0 \end{pmatrix} \qquad \ddot{\underline{r}}(t) = \begin{pmatrix} -a\cdot\cos t \\ -b\cdot\sin t \\ 0 \end{pmatrix}$$

$$\dot{\underline{r}}\times\ddot{\underline{r}} = \begin{pmatrix} -a\sin t \\ b\cdot\cos t \\ 0 \end{pmatrix}\times\begin{pmatrix} -a\cdot\cos t \\ -b\cdot\sin t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ab\cdot\sin^2 t + ab\cos^2 t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ab \end{pmatrix}$$

$$|\dot{\underline{r}}\times\ddot{\underline{r}}| = ab$$

$$|\dot{\underline{r}}|^2 = a^2\sin^2 t + b^2\cos^2 t$$

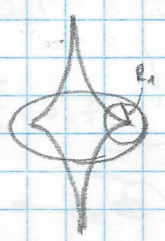
$$G = \frac{|\dot{\underline{r}}\times\ddot{\underline{r}}|}{|\dot{\underline{r}}|^3} = \frac{ab}{(a^2\sin^2 t + b^2\cos^2 t)^{3/2}}$$

$$a^2\sin^2 t + b^2\cos^2 t = a^2(1-\cos^2 t) + b^2\cdot\cos^2 t = a^2\left(1 - \cos^2 t + \frac{b^2}{a^2}\cos^2 t\right) =$$

$$= a^2\left(1 - \frac{a^2-b^2}{a^2}\cos^2 t\right) = a^2\left(1 - \frac{\varepsilon^2}{a^2}\cdot\cos^2 t\right) = a^2(1-\varepsilon^2\cos^2 t)$$

$$G = \frac{ab}{a^3(1-\varepsilon^2\cos^2 t)^{3/2}} = \frac{b/a^2}{(1-\varepsilon^2\cos^2 t)^{3/2}}$$

$$R(t) = \frac{1}{G} = \frac{a^2}{b}(1-\varepsilon^2\cos^2 t)^{3/2}$$



$$R_1 = \frac{a^2}{b}$$

$$R_2 = \frac{a^2}{b}(1-\varepsilon^2)^{3/2}$$

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$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$Sp = 2$

$\det = 1$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda_1 = \lambda_2 = 1$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad v' = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$B = e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

$$A^2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

$$A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

$$A = 1 + C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = (1 + C)(1 + C) = 1 + C + C + 0 = 1 + 2C$$

$$B = e^A = \sum_{n=0}^{\infty} \frac{1}{n!}(1 + nC) = \underbrace{\left(\sum \frac{1}{n!}\right)}_{e} I + \underbrace{\left(\sum \frac{n}{n!}\right)}_{\sum \frac{1}{(n-1)!} = e-1} C = eI + (e-1)C$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \\ & 0 \end{pmatrix}^2 = \begin{pmatrix} & & 1 \end{pmatrix}$$

„Csúcsa" mátrixokban → nilpotens mátrix → betűvégei eltűnnek

Jordan-féle normálalak

$\underline{r}(t) \to \underline{r}(s)$

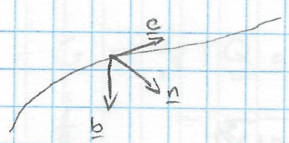
$\underline{r}'(s) = \underline{e}(s)$

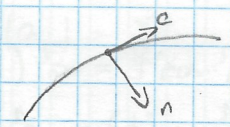
$\underline{e}'(s) = G(s)\,\underline{n}(s)$

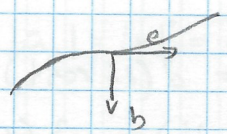
$\underline{b}(s) = \underline{e}(s) \times \underline{n}(s)$

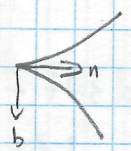
$|\underline{b}'| = T(s)$

$\underline{b}' = -T \cdot \underline{n}$









$\underline{\omega} = -G\underline{n} + T\underline{e}$ Darboux-vektor

$\underline{e}' = \omega \times \underline{e}$

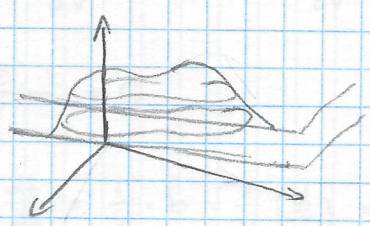
$\underline{n}' = \omega \times \underline{n}$

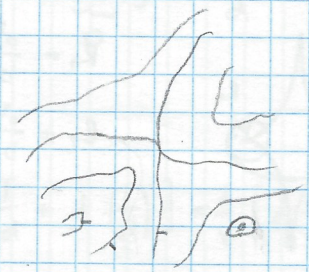
$\underline{b}' = \omega \times \underline{b}$

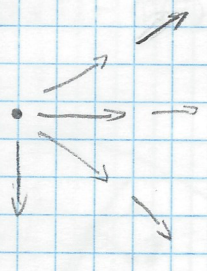
$\underline{v}(s)$

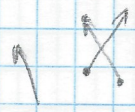
$s(\underline{v})$

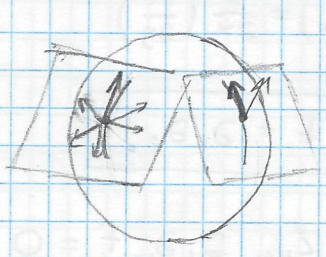
$\underline{w}(\underline{v})$



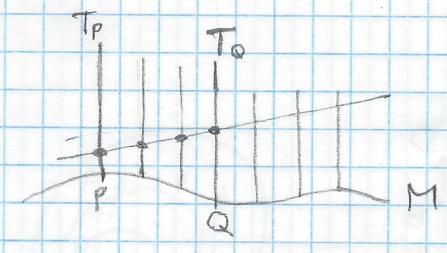


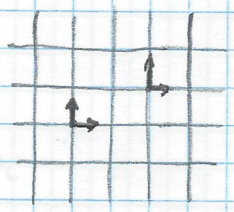




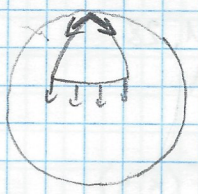


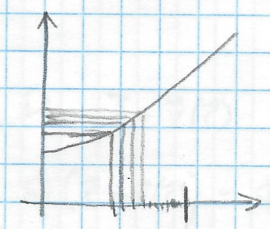
érintősíkok





lokális bázisvektorok





függvény határértéke
→ sorozatokkal vizsgálva

Vektorokon értelmezett fv-ek → reprezentálás → többváltozós függvények

$u = f(x,y)$ / $f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$

$\lim_{\delta \to 0} \frac{f(x+\delta, y) - f(x,y)}{\delta} = \frac{\partial f}{\partial x}$ parciális derivált

$\lim_{\varepsilon \to 0} \frac{f(x, y+\varepsilon) - f(x,y)}{\varepsilon} = \frac{\partial f}{\partial y}$

$f = x^2 \cdot \sin\frac{x}{y}$

$\frac{\partial f}{\partial x} = 2x \cdot \sin\frac{x}{y} + x^2 \cdot \cos\frac{x}{y} \cdot \frac{1}{y}$

$\frac{\partial f}{\partial y} = x^2 \cdot \cos\frac{x}{y} \cdot \left(-\frac{x}{y^2}\right)$

$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$

$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$

$\frac{\partial f}{\partial x} = \partial_x f$

$\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \partial_x f$

$\frac{\partial f(x)}{\partial x_k} = \partial_k f$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\boxed{\partial_\ell \partial_k f = \partial_k \partial_\ell f}$$ Young-tétel

$$\sum_\ell \sum_m \varepsilon_{klm}\, \partial_l \partial_m f = 0$$

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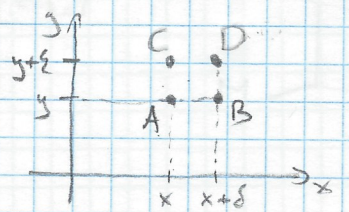
EA

$f(x,y)$

$$\frac{\partial f}{\partial x} = \lim_{\delta \to 0} \frac{f(x+\delta, y) - f(x,y)}{\delta} = \frac{\partial f}{\partial x}(x,y)$$



$$\frac{\partial}{\partial x}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial x^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$ YOUNG-TÉTEL

$\frac{\partial f}{\partial x} = g(x,y)$ $\quad$ $\frac{\partial f}{\partial y} = h(x,y)$

$$\lim_{\varepsilon \to 0} \frac{g(x,y+\varepsilon) - g(x,y)}{\varepsilon} \stackrel{?}{=} \lim_{\delta \to 0} \frac{h(x+\delta, y) - h(x,y)}{\delta}$$

$$\lim_{\varepsilon \to 0} \frac{\lim_{\delta \to 0} \frac{f(x+\delta, y+\varepsilon) - f(x, y+\varepsilon)}{\delta} - \lim_{\delta \to 0} \frac{f(x+\delta, y) - f(x,y)}{\delta}}{\varepsilon} \stackrel{?}{=} \lim_{\delta \to 0} \frac{\lim_{\varepsilon \to 0} \frac{f(x+\delta, y+\varepsilon) - f(x+\delta, y)}{\varepsilon} - \lim_{\varepsilon \to 0} \frac{f(x, y+\varepsilon) - f(x,y)}{\varepsilon}}{\delta}$$

$$\lim_{\varepsilon \to 0} \frac{\lim_{\delta \to 0} \frac{f(D) - f(C)}{\delta} - \lim_{\delta \to 0} \frac{f(B) - f(A)}{\delta}}{\varepsilon} \stackrel{?}{=} \lim_{\delta \to 0} \frac{\lim_{\varepsilon \to 0} \frac{f(D) - f(B)}{\varepsilon} - \lim_{\varepsilon \to 0} \frac{f(C) - f(A)}{\varepsilon}}{\delta}$$

Ha $\lim_\delta$ és $\lim_\varepsilon$ felcserélhető $\longrightarrow$ teljesül (gyakorlatban mindig!)

$$f(x,y) = x^3 - 3x^2 \cdot \sin(xy)$$

$$\frac{\partial f}{\partial x} = 3x^2 - 6x \cdot \sin xy - 3x^2 y \cdot \cos(xy) = g(x,y)$$

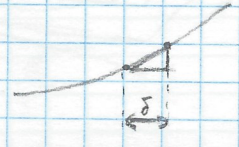
$$\frac{\partial f}{\partial y} = -3x^3 \cdot \cos(xy) = h(x,y)$$

$$\frac{\partial g}{\partial y} = -3x^2 \left(\cos(xy) - \sin(xy) \cdot x\right) = -3x^2 \cdot \cos(xy) + 3x^3 y \sin(xy)$$

$$\frac{\partial h}{\partial x} = -3x^2 \cdot \cos(xy) + 3x^3 \cdot y \cdot \sin(xy)$$

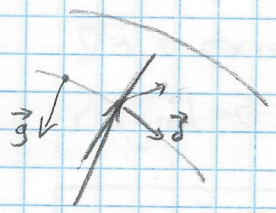
$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta}$$

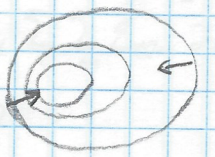
$$f(x+\delta) - f(x) \approx f'(x)\delta$$

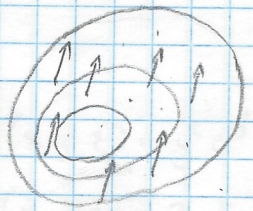


$$\phi(\vec{r}+\vec{\delta}) - \phi(\vec{r}) \approx (\vec{g}(\vec{r}) \cdot \vec{\delta}$$

GRADIENS







$$\vec{\delta} = \delta \cdot \vec{n}$$

$$|\vec{n}| = 1$$

$$\phi(\vec{r} + \delta\vec{n}) - \phi(\vec{r}) = \vec{g}(\vec{r}) \cdot \vec{n} \cdot \delta$$

$$\lim_{\delta \to 0} \frac{\phi(\vec{r}+\delta\vec{n}) - \phi(\vec{r})}{\delta} = \vec{g}(\vec{r}) \cdot \vec{n}$$ IRÁNYMENTI DERIVÁLT

Ha $\vec{\delta} = \vec{v}(\vec{r})$ mindenkinek más (vektormező)

$$\phi(\vec{r} + \vec{v}(\vec{\delta})) - \phi(\vec{r}) \approx \vec{g}(\vec{r})\, \vec{v}(\vec{r})$$ VEKTORMEZŐ MENTI DERIVÁLT

Mindenki x irányba mozdul el → $\underline{n} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\phi(\underline{r} + \delta\underline{n}) - \phi(\underline{r}) \approx \underline{g}(\underline{r}) \cdot \delta\underline{n} = \delta \underline{g}(\underline{r}) \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \delta g_1(\underline{r})$$

$$\frac{\partial \phi}{\partial x} = \lim_{\delta \to 0} \frac{\phi(x+\delta, y) - \phi(x,y)}{\delta} = g_1(x)$$

$$\phi(\vec{r}) \to \phi(\underline{r}) = \phi(x_k)$$

$$\frac{\partial \phi}{\partial x_k} = \left(\underline{g}(\underline{x})\right)_k$$

$\phi(\underline{r})$

$\vec{g}(\vec{r}) = \text{grad}\ \phi = \vec{\nabla}\phi$ NABLA

$g_k = (\underline{\nabla}\phi)_k = \frac{\partial \phi}{\partial x_k} = \partial_k \phi$

$(\underline{\nabla})_k = \partial_k$

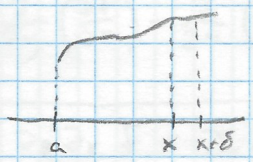
$$\underline{\nabla}(\phi\Psi) = (\underline{\nabla}\phi)\Psi + \phi(\underline{\nabla}\Psi)$$

$f'(x) = g(x)$

? $f(x) \leftarrow g(x)$

$f(x) = \int g(x)\, dx$

$\Sigma \to \int_a^b g(x)\, dx = f(b) - f(a)$



$$\int_a^x f(x)\,dx = F(x)$$

$$F(x+\delta) - F(x) = \int_a^{x+\delta} f - \int_a^x f = \int_x^{x+\delta} f\,dx \approx f(x)\cdot\delta$$

$$\underline{v}(\underline{r}) = \underline{\nabla}\,\phi(\underline{r})$$

$$v_k(\underline{r}) = \partial_k \phi$$

$$\partial_l v_k = \partial_l(\partial_k \phi) = \partial_k(\partial_l \phi) = \partial_k v_l$$

dim = 2 $\quad \phi(x,y)$

$$v_1 = v_x(x,y) = \frac{\partial \phi}{\partial x}$$

$$v_2 = v_y(x,y) = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\frac{\partial \phi}{\partial y} = \frac{\partial v_y}{\partial x}$$ → Ha teljesül: $\underline{v}(x,y)$ lehet gradiens

ÁLL: Ez elégséges feltétel.

dim = 3 $\quad \phi(x,y,z)$

$$v_1 = v_x = \frac{\partial \phi}{\partial x} \qquad \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} \qquad \partial_y v_x = \partial_x v_y \qquad \partial_2 v_1 = \partial_1 v_2$$

$$v_2 = v_y = \frac{\partial \phi}{\partial y} \qquad \frac{\partial v_x}{\partial z} = \frac{\partial v_z}{\partial x} \qquad \partial_z v_x = \partial_x v_z \qquad \partial_3 v_1 = \partial_1 v_3$$

$$v_2 = v_z = \frac{\partial \phi}{\partial z} \qquad \frac{\partial v_y}{\partial z} = \frac{\partial v_z}{\partial y} \qquad \partial_z v_y = \partial_y v_z \qquad \partial_3 v_2 = \partial_2 v_3$$

$$\Rightarrow \quad \begin{aligned} \partial_2 v_3 - \partial_3 v_2 &= 0 \\ \partial_3 v_1 - \partial_1 v_3 &= 0 \\ \partial_1 v_2 - \partial_2 v_1 &= 0 \end{aligned} \quad \Rightarrow \quad \underline{\nabla} \times \underline{v} = \underline{0}$$

$$\underline{w}(\underline{r}) = \text{rot}\,\underline{v} = \underline{\nabla} \times \underline{v}$$

$$\boxed{\underline{v} = \underline{\nabla}\phi \longrightarrow \text{rot}\,\underline{v} = 0}$$ ELSŐ GRADIENSTÉTEL

$$\underline{w} = \text{rot}\,\underline{v} = \underline{\nabla} \times \underline{v}$$

$$w_k = \sum_l \sum_m \varepsilon_{klm}\,\partial_l v_m$$

$$(\text{rot grad}\,\phi)_k = (\underline{\nabla} \times (\underline{\nabla}\phi))_k = \sum_l \sum_m \varepsilon_{klm}\,\partial_l \partial_m \phi = 0$$

szimm. antiszimm. Young-t.

$$\begin{aligned} A_{lm} &= -A_{ml} \\ S_{lm} &= S_{ml} \end{aligned} \longrightarrow \sum_m \sum_l S_{ml} A_{ml} = 0$$

$$\Psi(v) = \underline{\nabla} \cdot \underline{w} = \text{div}\,\underline{w}$$

TFH $\quad \underline{w} = \underline{\nabla} \times \underline{v}$

$$w_k = \sum_k \sum_\ell \varepsilon_{k\ell m} \partial_\ell v_m$$

$$\underline{\nabla} \underline{w} = \text{div rot } \underline{v} = \sum_k \partial_k \sum_\ell \sum_m \varepsilon_{k\ell m} \partial_\ell v_m = \sum_k \sum_\ell \sum_m \varepsilon_{k\ell m} \partial_k \partial_\ell v_m = 0$$

rot grad = 0 $\qquad$ div rot = 0

$$\boxed{\underline{w}(r) = \underline{\nabla} \times \underline{v} \quad \rightarrow \nabla \cdot \underline{w} = 0}$$

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GY

$$v_x(x,y,z) = y^2 e^{xy^2} \cdot \sin x + e^{xy^2} \cos x + 2xz$$

$$v_y(x,y,z) = 2xy\, e^{xy^2} \cdot \sin x + \cos yz - yz \cdot \sin yz$$

? $\quad \phi \quad \underline{v} = \underline{\nabla} \phi \qquad v_k = \partial_k \phi$

$v_z = ?$

$\phi = ?$

$$v_x = \frac{\partial \phi}{\partial x} \qquad v_y = \frac{\partial \phi}{\partial y}$$

$$\rightarrow \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} = \frac{\partial^2 \phi}{\partial x \partial y}$$

$$\frac{\partial v_x}{\partial y} = 2y e^{xy^2} \cdot \sin x + y^2 2yx e^{xy^2} \sin x + 2xy \cdot \cos x \cdot e^{xy^2}$$

$$\frac{\partial v_y}{\partial x} = 2y e^{xy^2} \cdot \sin x + 2xy y^2 e^{xy^2} \sin x + 2y e^{xy^2} \cos x \quad //$$

$$\frac{\partial v_z}{\partial x} = \frac{\partial v_x}{\partial z} = 2x$$

$$v_z = x^2 + f(y,z)$$

$$\frac{\partial v_z}{\partial y} = \frac{\partial v_y}{\partial z} = \underbrace{-y \cdot \sin yz - y \cdot \sin yz}_{-2y \cdot \sin yz} - y^2 z \cos yz \overset{!}{=} \frac{\partial v_z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$$

$$f(y,z) = -y^2 \cdot \sin yz + \psi(z)$$

$$v_z = x^2 - y^2 \cdot \sin yz + \psi(z)$$

$$v_x = \frac{\partial \phi}{\partial x}$$

$$\phi(x,y,z) = e^{xy^2} \cdot \sin x + x^2 z + F(y,z)$$

$$v_y = \frac{\partial \phi}{\partial y} = 2xy\, e^{xy^2} \cdot \sin x + \frac{\partial F(y,z)}{\partial y} = 2xy\, e^{xy^2} \cdot \sin x + \cos yz - yz \cdot \sin yz$$

$$\frac{\partial F(y,z)}{\partial y} = \cos yz - yz \cdot \sin yz \qquad F(y,z) = y \cdot \cos yz + G(z)$$

$$\phi = e^{xy} \cdot \sin x + x^2 z + y \cdot \cos yz + G(z)$$

$$\frac{\partial \phi}{\partial z} = x^2 - y^2 \cdot \sin yz + G'(z) = v_z = x^2 - y^2 \cdot \sin yz + \varphi(z)$$

$$v_x = \frac{-y}{x^2+y^2}$$

$$v_y = \frac{x}{x^2+y^2}$$

$$v_z = 0$$

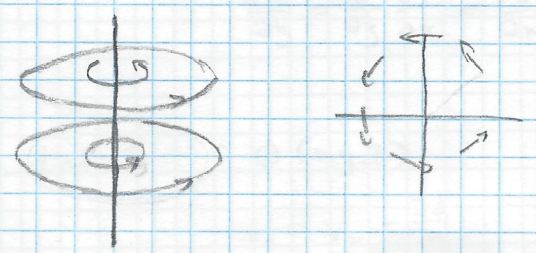
$$\underline{w} = \underline{\nabla} \times \underline{v} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = \begin{pmatrix} \partial_y v_z - \partial_z v_y \\ \partial_z v_x - \partial_x v_z \\ \partial_x v_y - \partial_y v_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ? \end{pmatrix}$$

$$w_z = \partial_x v_y - \partial_y v_x = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) =$$

$$= \frac{\frac{\partial x}{\partial x}(x^2+y^2) - x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} - \frac{\frac{\partial}{\partial y}(-y)(x^2+y^2) - (-y)\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} =$$

$$= \frac{x^2+y^2 - x\,2x}{(x^2+y^2)^2} - \frac{(-1)\cdot(x^2+y^2) - (-y)\,2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{-y^2-x^2}{(x^2+y^2)^2} = 0$$

$\operatorname{rot} \underline{B} = \underline{j}$



$$x = r \cdot \cos\varphi \qquad r = \sqrt{x^2+y^2}$$

$$y = r \cdot \sin\varphi \qquad \varphi = \operatorname{arctg}\left(\frac{y}{x}\right)$$

$$u_x = \frac{\partial \varphi}{\partial x} = \frac{\partial}{\partial x} \operatorname{arctg}\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{y}{-x^2} = \frac{-y}{x^2+y^2}$$

$$u_y = \frac{\partial \varphi}{\partial y} = \frac{\partial}{\partial y} \cdot \operatorname{arctg}\frac{y}{x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2}$$

$$\underline{u} = \underline{\nabla} \varphi$$

$\underline{r} \quad x_k \qquad |\underline{r}| = r$

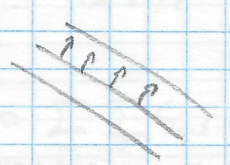
$$\phi = \underline{a}\,\underline{r} = \sum_k a_k x_k$$

$$\underline{v} = \underline{\nabla} \phi$$

$$v_k = \frac{\partial}{\partial x_k}\phi = \frac{\partial}{\partial x_k} \sum_\ell a_\ell x_\ell = \sum_\ell a_\ell \frac{\partial x_\ell}{\partial x_k} = \sum_\ell a_\ell \delta_{\ell k} = a_k$$

$$\underline{v} = \underline{\nabla}\phi = \underline{a}$$

$\operatorname{rot} \underline{r} = 0$



Pl. fény és atom kölcsönhatása

$$\underline{v} = \underline{\omega} \times \underline{r}$$

$$v_k = \sum_\ell \sum_m \varepsilon_{k\ell m}\, \omega_\ell x_m$$

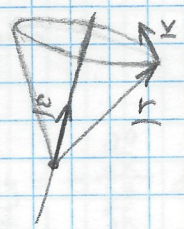
$$\underline{w} = \underline{\nabla} \times \underline{v}$$

$$w_k = \sum_\ell \sum_m \varepsilon_{k\ell m}\, \partial_\ell v_m = \sum_\ell \sum_m \varepsilon_{k\ell m}\, \partial_\ell \left(\sum_p \sum_q \varepsilon_{mpq}\, \omega_p x_q \right) = \sum_\ell \sum_m \sum_p \sum_q \varepsilon_{k\ell m} \varepsilon_{mpq}\, \omega_p \partial_\ell x_q =$$

$$= \sum_\ell \sum_p \sum_q \left(\sum_m \varepsilon_{k\ell m} \varepsilon_{mpq} \right) \omega_p \delta_{\ell q} = \sum_\ell \sum_p \left(\delta_{kp}\delta_{\ell\ell} - \delta_{k\ell}\delta_{\ell p} \right) \omega_p =$$

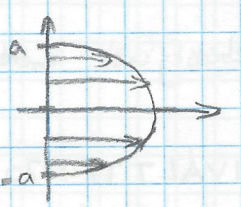
$\delta_{kp}\delta_{\ell q} - \delta_{kq}\delta_{\ell p}$

$$= \sum_p \left(\delta_{kp} \cdot 3 - \delta_{kp} \right) \omega_p = \sum_p 2\,\delta_{kp}\omega_p = 2\omega_k$$



$\frac{1}{2}$ rot → forgás

$v_x = a^2 - y^2 \qquad -a < y < a$



$v_y = 0$

$v_z = 0$

$$\underline{w} = \frac{1}{2}\,\mathrm{rot}\,\underline{v} = \frac{1}{2}\,\underline{\nabla} \times \underline{v} = \frac{1}{2} \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} v_x \\ 0 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 \\ \partial_z v_x \\ -\partial_y v_x \end{pmatrix}$$

$$w_z = -\frac{1}{2} \frac{\partial}{\partial y} \left(a^2 - y^2 \right) = y$$

! Einstein-féle némaindex konvenció → szumma elhagyható

$$\underline{v} = \underline{\omega} \times \underline{r}$$

$$v_k = \varepsilon_{k\ell m}\, \omega_\ell x_m$$

$$\underline{w} = \mathrm{rot}\,\underline{v}$$

$$w_k = (\underline{\nabla} \times \underline{v})_k = \varepsilon_{k\ell m}\, \partial_\ell v_m = \varepsilon_{k\ell m}\, \partial_\ell \left(\varepsilon_{mpq}\, \omega_p x_q \right) = \varepsilon_{k\ell m} \varepsilon_{mpq}\, \omega_p \partial_\ell x_q =$$

$$= \left(\delta_{kp}\delta_{\ell q} - \delta_{kq}\delta_{\ell p} \right) \omega_p\, \delta_{\ell q} = \left(\delta_{kp}\delta_{\ell\ell} - \delta_{k\ell}\delta_{\ell p} \right) \omega_p = \left(\delta_{kp} 3 - \delta_{kp} \right) \omega_p =$$

$$= 2\,\delta_{kp}\,\omega_p = 2\omega_k$$

↓

v_k

$(\underline{v} + \underline{u})_k = v_k + u_k$

$(d\underline{u})_k = d u_k$

$\underline{u}\,\underline{v} = u_k v_k$

$(\underline{u} \times \underline{v})_k = \varepsilon_{k\ell m}\, u_\ell v_m$

$(\underline{u} \circ \underline{v})_{kl} = u_k v_l$

$\underline{\underline{C}} = \underline{\underline{A}}\,\underline{\underline{B}} \rightarrow C_{kl} = A_{km} B_{ml}$

$\underline{u} = \underline{\underline{A}}\,\underline{v} \rightarrow u_k = A_{kl} v_l$

$\hat{u} = \nabla \underline{\underline{A}} \quad u_k = v_l A_{lk}$

$(\nabla)_l = \partial_l = \frac{\partial}{\partial x_l}$

$v_k = (\nabla \phi)_k = \partial_k \phi$

$\underline{v}\,\underline{\nabla} = v_k \partial_k$

$u_k = v_l \partial_l w_k$

$\underline{w} = \underline{\nabla} \times \underline{v} \rightarrow w_k = \varepsilon_{klm} \partial_l v_m$

$(\underline{\nabla} \circ \underline{v})_{kl} = \partial_k v_l$

$\Psi = \underline{\nabla}\,\underline{v} = \partial_k v_k$

$\underline{v}\,\underline{\nabla}$

$(\underline{v}\,\underline{\nabla})\,\phi = \underline{v}\,(\nabla \phi) = \underline{v} \cdot \mathrm{grad}\,\phi \rightarrow$ vektormező menti derivált

$\underline{u} = (\underline{v}\,\underline{\nabla})\,\underline{w}(r)$

$(\underline{v}\,\underline{\nabla})\,\underline{v} \rightarrow$ „folyó sebességének mérése" → hidrodinamikában, NEM LINEÁRIS

$(\underline{\nabla} \circ \underline{v})_{kl} = \partial_k v_l$ DERIVÁLTTENZOR

$\underline{v} = \frac{\underline{a} \times (\underline{b} \times \underline{r})}{r^3} (\underline{e}\,\underline{r})^2$

$\partial_k x_l = \delta_{kl}$

$\partial_k r$

$\partial_k e_l$

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GY

$f(x+a) = f(x) + a \cdot f'(x) + a^2 \cdot \frac{f''(x)}{2} + a^3 \cdot \frac{f'''(x)}{3!}$

$\phi(\vec{r}+\vec{\delta}) \approx \phi(\vec{r}) + \vec{g}(\vec{r}) \cdot \vec{\delta}$

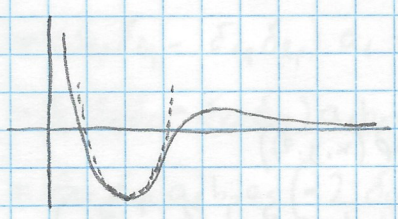
$\phi(\underline{r}+\underline{\delta}) \approx \phi(\underline{r}) + \underline{g}(\underline{r}) \cdot \underline{\delta}$

$\phi(x+\delta_1, y+\delta_2, z+\delta_3) \approx \phi(x,y,z) + \frac{\partial \phi}{\partial x} \delta_1 + \frac{\partial \phi}{\partial y} \delta_2 + \frac{\partial \phi}{\partial z} \delta_3$

$\phi(\underline{r}+\underline{\delta}) \approx \phi(\underline{r}) + \sum_k \frac{\partial \phi}{\partial x_k} \delta_k + \frac{1}{2} \sum_k \sum_l \frac{\partial^2 \phi}{\partial x_k \partial x_l} \delta_k \delta_l + \frac{1}{3!} \sum_k \sum_l \sum_m \frac{\partial^3 \phi}{\partial x_k \partial x_l \partial x_m} \delta_k \delta_l \delta_m + \dots$

$\approx \phi(\underline{r}) + \partial_k \phi\, \delta_k + \frac{1}{2} \partial_k \partial_l \phi\, \delta_k \delta_l + \frac{1}{3!} \partial_k \partial_l \partial_m \phi\, \delta_k \delta_l \delta_m + \dots$

$V(r)$

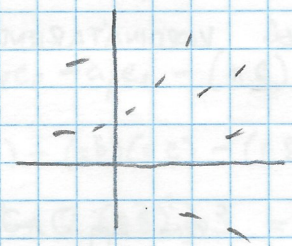


Egyensúlyi helyzet körül: parabolával közelítve → harmonikus oszcillátor

$y(x)$

$y' = F(x,y)$

iránymező

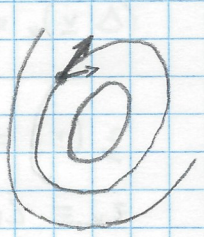


$G(x,y) = 0$ (pl. ellipszis)

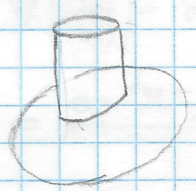
$\phi(x,y) = G(x,y)$

$d\phi = \underline{g}\, d\underline{r} = \frac{\partial\phi}{\partial x}\cdot dx + \frac{\partial\phi}{\partial y}\cdot dy$

Teljes differenciál



$$dy = -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}}\,dx \quad \rightarrow \quad \frac{dy}{dx} = -\frac{\frac{\partial\phi}{\partial x}}{\frac{\partial\phi}{\partial y}} = F(x,y)$$



$G(x,y,z) = 0$

$H(x,y,z) = 0$

$y(x)$
$z(x)$ ⇒ görbén integrálás → testek nehézségmezője

$\vec{r} \rightarrow \underline{r} \rightarrow x_k$

$|\underline{r}| = r$

$\underline{e} = \frac{\underline{r}}{r} \quad |\underline{e}| = 1$

$\nabla_k = \frac{\partial}{\partial x_k} = \partial_k$

$\sum a_k b_k \Rightarrow a_k b_k$

$(a \times b)_k = \varepsilon_{klm} a_l b_m$

$(\underline{\underline{A}}\,\underline{\underline{B}})_{kl} = A_{km} B_{ml}$

$(\underline{\underline{A}}\,\underline{u})_k = A_{kl}\, c_l$

$(\nabla\phi)_k = \partial_k \phi$

$\nabla \underline{v} = \partial_k v_k$

$(\nabla \times \underline{v})_k = \varepsilon_{klm}\, \partial_l v_m$

$(\nabla \circ \underline{v})_{kl} = \partial_k v_l$

$\phi \rightarrow \text{grad}\,\phi = \underline{v}$

- $\text{div}\,\underline{v} = \text{divgrad}\,\phi = \nabla(\nabla\phi) = (\nabla\nabla)\phi = \nabla^2\phi = \Delta\phi$ Laplace
- $\text{rot grad}\,\phi = 0$

$\underline{v} \rightarrow \nabla\underline{v} = \psi \rightarrow \text{grad div}\,\underline{v} = \nabla(\nabla\underline{v}) = \underline{u}$

$\nabla \times \underline{v} = \underline{w} \rightarrow \text{div rot}\,\underline{v} = 0$

$\text{rot rot}\,\underline{v} = \nabla \times (\nabla \times \underline{v})$

$\Delta\phi = 0$ Laplace-egyenlet

$\Delta\phi = \varrho$ Poisson-egyenlet

$\underline{v} = \text{grad}\,\phi \qquad v_k = \partial_k \phi$

$\psi = \nabla v = \nabla(\nabla\phi) = \partial_k v_k = \partial_k \partial_k \phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

Polárkoordináták: követett fv $\longrightarrow \phi\,(r(x,y,z),\ \vartheta(x,y,z),\ \varphi(x,y,z))$

$$\Delta = \frac{\partial^2}{\partial r} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial^2}{\partial\vartheta^2} + \text{ctg}\,\vartheta\,\frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\cdot\frac{\partial^2}{\partial\varphi^2}\right)$$

$(\Delta\,\underline{v})_k = \Delta(v_k) = \partial_l \partial_l v_k \Rightarrow$ CSAK DERÉKSZÖGŰ KOORDINÁTARENDSZERBEN

$\underline{z}(\underline{r}) = \underline{\nabla} \times (\underline{\nabla} \times \underline{v})$

$z_k = \varepsilon_{klm}\,\partial_l\,(\underline{\nabla}\times\underline{v})_m = \varepsilon_{klm}\,\partial_l(\varepsilon_{mpq}\,\partial_p v_q) = \varepsilon_{klm}\varepsilon_{mpq}\,\partial_l\partial_p v_q =$

$= (\delta_{kp}\delta_{lq} - \delta_{kq}\delta_{lp})\,\partial_l\partial_p v_q = \partial_l\partial_k v_l - \partial_l\partial_l v_k = \partial_k(\partial_l v_l) - \partial_l(\partial_l v_k) =$

$= (\underline{\nabla}(\underline{\nabla}\,\underline{v}))_k - (\Delta\,\underline{v})_k$

$\boxed{\text{rot rot}\,\underline{v} = \text{grad div}\,\underline{v} - \Delta\,\underline{v}}$

$\text{grad div} = \text{rot rot} + \Delta$

$(\underline{v}\,\underline{\nabla})_k = v_k\,\partial_k$

$\text{div}\left(\frac{\underline{a}\times(\underline{b}\times(\underline{a}\underline{\nabla})\underline{c})}{r}\right) = \partial_k(\quad)_k = \partial_k\left(\frac{1}{r}(\quad)_k\right) = \partial_k\left(\frac{1}{r}\varepsilon_{klm}\,a_l\,(b\times\ldots)_m\right) =$

$= \partial_k\left(\frac{1}{r}\varepsilon_{klm}\,a_l\,\varepsilon_{mpq}\,b_p\,((\underline{a}\underline{\nabla})\underline{c})_q\right) = \partial_k\left(\frac{1}{r}\varepsilon_{klm}\,a_l\,\varepsilon_{mpq}\,b_p\,a_s\,\partial_s c_q\right) =$

$= \varepsilon_{klm}\varepsilon_{mpq}\,a_l\,b_p\,a_s\cdot\partial_k\left(\frac{1}{r}\partial_s c_q\right)$

⊛ $\partial_k x_l = \delta_{kl}$

$\partial_k r = \partial_k\sqrt{\underline{r}\,\underline{r}} = \frac{1}{2\sqrt{\underline{r}\,\underline{r}}}\,\partial_k(x_l x_l) = \frac{1}{2r}\cdot 2x_l\underbrace{(\partial_k x_l)}_{\delta_{kl}} = \frac{x_k}{r} = c_k$

$\partial_k c_l = \partial_k\left(\frac{x_l}{r}\right) = \partial_k\left(x_l\frac{1}{r}\right) = (\partial_k x_l)\frac{1}{r} + x_l\,\partial_k\left(\frac{1}{r}\right) = \delta_{kl}\frac{1}{r} + x_l\cdot\left(-\frac{1}{r^2}\right)\cdot\partial_k r =$

$= \delta_{kl}\cdot\frac{1}{r} - \frac{x_l}{r^2}\frac{x_k}{r} = \frac{\delta_{kl} - c_k c_l}{r}$

$\partial_k c_k = \frac{\delta_{kk} - c_k c_k}{r} = \frac{3-1}{r} = \frac{2}{r}$

$c_k\partial_k c_l = c_k\frac{\delta_{kl} - c_k c_l}{r} = \frac{c_k\delta_{kl} - c_k c_k c_l}{r} = \frac{c_l - c_l}{r} = 0$

$(\underline{c}\,\underline{\nabla})\,c_l = 0$

$c_l\,\partial_k c_l = c_l\,\frac{\delta_{kl} - c_{kl}}{r} = 0$

$\partial_k\,\frac{c_l c_l}{2}$

$$\operatorname{div}(\dots) = (\delta_{\kappa\rho}\delta_{\varepsilon\alpha} - \delta_{\varepsilon\alpha}\delta_{\varepsilon\rho})\, a_\varepsilon b_\rho a_\sigma \partial_\kappa \left(\frac{1}{r}\,\frac{\delta_{\sigma\eta} - e_\sigma e_\eta}{r}\right) = (\delta_{\kappa\rho}\delta_{\varepsilon\eta} - \delta_{\kappa\eta}\delta_{\varepsilon\rho})\, a_\varepsilon b_\rho a_\sigma \partial_\kappa \left(\frac{\delta_{\sigma\eta}}{r^2} - \frac{x_\sigma x_\eta}{r^4}\right) =$$

$$= (\delta_{\kappa\rho}\delta_{\varepsilon\eta} - \delta_{\kappa\eta}\delta_{\varepsilon\rho})\, a_\varepsilon b_\rho a_\sigma \left[\delta_{\sigma\eta}\left(-\frac{2}{r^3}\right) e_\kappa - \delta_{\kappa\sigma}\frac{x_\eta}{r^4} - \delta_{\kappa\eta}\frac{x_\sigma}{r^4} + \frac{4}{r^5} x_\sigma x_\eta e_\kappa\right] =$$

$$= \frac{1}{r^3}(\delta_{\kappa\rho}\delta_{\varepsilon\eta} - \delta_{\kappa\eta}\delta_{\varepsilon\rho})\, a_\varepsilon b_\rho a_\sigma \left[-2\delta_{\sigma\eta} e_\kappa - \delta_{\kappa\sigma} e_\eta - \delta_{\kappa\eta} e_\sigma + 4 e_\sigma e_\eta e_\kappa\right] =$$

$$= \frac{1}{r^3}\delta_{\kappa\rho}\, a_\varepsilon b_\rho a_\sigma \left(\underbrace{-2\delta_{\sigma\varepsilon} e_\kappa}_{-2e_\kappa} - \delta_{\kappa\sigma} e_\varepsilon - \underbrace{\delta_{\kappa\varepsilon} e_\sigma}_{-e_\sigma} + 4 e_\sigma e_\varepsilon e_\kappa\right) - \frac{1}{r^3}\delta_{\varepsilon\eta}\, a_\varepsilon b_\rho a_\sigma \left(-2\delta_{\kappa\varepsilon} e_\kappa - \delta_{\kappa\sigma} e_\kappa - \right.$$

$$\left. - \underbrace{\delta_{\kappa\kappa} e_\sigma}_{-3e_\sigma} + 4 e_\sigma e_\kappa e_\kappa\right) = \frac{1}{r^3}\, a_\varepsilon b_\rho \left(-2 a_\sigma \delta_{\sigma\varepsilon} e_\rho - a_\sigma \delta_{\rho\sigma} e_\varepsilon - \delta_{\rho\varepsilon} a_\sigma e_\sigma + 4 a_\sigma e_\sigma e_\varepsilon e_\rho\right) - \frac{1}{r^3} a_\rho b_\rho a_\sigma (-2e_\sigma) =$$

$$= \frac{1}{r^3}\, a_\varepsilon b_\rho \left(-2 a_\rho e_\rho - a_\rho e_\varepsilon - (\underline{a}\,\underline{e})\,\delta_{\rho\varepsilon} + 4(\underline{a}\,\underline{e})\, e_\rho e_\rho\right) - \frac{2}{r^3}\, a_\rho b_\rho (\underline{a}\,\underline{e}) =$$

$$= \frac{1}{r^3}\left(-2a^2(\underline{b}\,\underline{e}) - (\underline{a}\,\underline{b})(\underline{a}\,\underline{e}) - (\underline{a}\,\underline{e})(\underline{a}\,\underline{b}) + 4(\underline{a}\,\underline{e})^2(\underline{b}\,\underline{e}) + 2(\underline{a}\,\underline{b})(\underline{a}\,\underline{e})\right) =$$

$$= \frac{1}{r^3}(\underline{b}\,\underline{e})\left(-a^2 + 2(\underline{a}\,\underline{e})^2\right)$$

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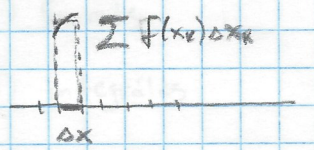
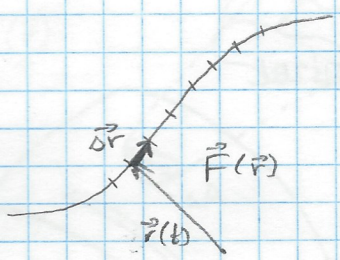
$$\nabla\left(\frac{(ac)}{(bc)}\right) = \partial_\kappa \frac{(ac)}{(bc)} = \frac{(\partial_\kappa (ac))(bc) - (ac)\partial_\kappa (bc)}{(bc)^2} = \frac{\partial_\kappa (a_\rho c_\rho)(bc) - (ac)\partial_\kappa (b_\varepsilon c_\varepsilon)}{(bc)^2} =$$

$$= \frac{ac\,\frac{\delta_{\kappa\varepsilon} - e_\kappa e_\varepsilon}{r}(bc) - (ac)\,bc\,\frac{\delta_{\kappa\varepsilon} - e_\kappa e_\varepsilon}{r}}{(bc)^2} = \frac{1}{r(bc)^2}\cdot\left((a_\kappa - (ac)e_\rho)(bc) - ac(\dots)\right)$$



measure
gauge



Δs
$|\Delta\vec{r}| \to 0$

$$\sum \vec{F}(\vec{r})\,\Delta\vec{r} \longrightarrow \int_G \vec{F}(\vec{r})\,d\vec{r}$$ VONALMENTI INTEGRÁL

$$\Delta\vec{r} = \frac{\Delta\vec{r}}{\Delta t}\Delta t = \dot{\vec{r}}(t)\,\Delta t \qquad t: \text{paraméterezés}$$

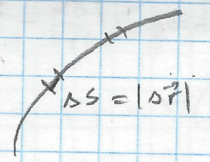
$$\Delta\underline{r} = \underline{\dot{r}}\,\Delta t$$

$$\underbrace{\underline{F}\,(\underline{r}(t))\,\underline{\dot{r}}(t)}_{f(t)}\,\Delta t$$

$$\sum f(t)\,\Delta t \longrightarrow \int_{t_1}^{t_2} f(t)\,dt$$

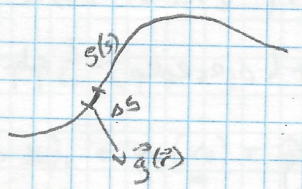
$$\sum \vec{F}(\vec{r}) \times \Delta\vec{r} \qquad \vec{B} = \int \vec{f}(r) \times d\vec{r}$$

$$\sum \vec{F}(\vec{r}) \circ \Delta\vec{r}$$



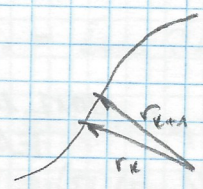
$\sum \Delta s \to s$

$\sum g(s)\,\Delta s$ ÍVHOSSZ MENTI INTEGRÁL



$\sum \vec{g}(\vec{r})\, s(s)\, \Delta s$

$\int \vec{g}(\vec{r})\, s(s)\, ds = \vec{G}$



$\phi(\vec{r}+\vec{\delta}) - \phi(\vec{r}) \approx \vec{g}(\vec{r})\,\Delta\vec{r}$

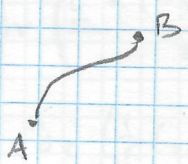
$\vec{g}(\vec{r})\,\Delta\vec{r} = \phi(\vec{r}_{k+1}) - \phi(\vec{r}_k)$

$\downarrow \sum$

$\phi_{\text{utolsó}} - \phi_{\text{első}}$

$$\int_A^B \underline{\nabla}\phi \, d\underline{r} = \phi(B) - \phi(A) \quad \Rightarrow \text{NEM FÜGG A GÖRBÉTŐL}$$

$$\int_G^{A} \vec{F}(\vec{r})\, d\vec{r} = -\int_G^{B} \vec{F}(\vec{r})\, d\vec{r}$$



$\phi(B) - \phi(A)$

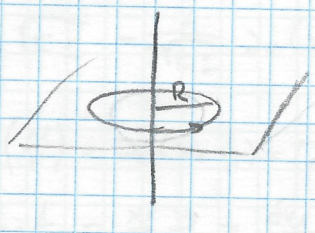
$\phi(A) - \phi(B)$

$$\boxed{\oint \underline{\nabla}\phi \, d\underline{r} = 0}$$ MÁSODIK GRADIENSTÉTEL

$v_x = \dfrac{-y}{x^2+y^2}$

$v_y = \dfrac{x}{x^2+y^2}$

$v_z = 0$



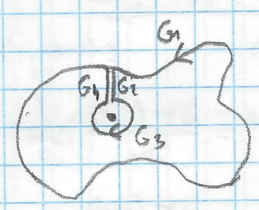
<u>paraméterezés</u>

$$r(t) = \begin{cases} x = R\cdot\cos t \\ y = R\cdot\sin t \\ z = 0 \end{cases} \qquad \underline{\dot{r}}(t) = \begin{pmatrix} -R\cdot\sin t \\ R\cos t \\ 0 \end{pmatrix}$$

$$v_x = \frac{-R\cdot\sin t}{R^2\cos^2 t + R^2\sin^2 t} = \frac{-R\cdot\sin t}{R^2} = -\frac{\sin t}{R}$$

$$v_y = \frac{R\cdot\cos t}{R^2\sin^2 t + R^2\cos^2 t} = \frac{\cos t}{R}$$

$$\oint \underline{v}\, d\underline{r} = \int_0^{2\pi} \underline{v}(r(t))\, \dot{r}(t)\, dt = \int_0^{2\pi} \frac{1}{R}\begin{pmatrix}-\sin t\\ \cos t\\ 0\end{pmatrix} \cdot \begin{pmatrix}R\sin t\\ R\cos t\\ 0\end{pmatrix} dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t)\, dt = 2\pi$$



$$\int_{G_1} + \int_{G_2} + \int_{G_3} + \int_{G_4} = 0$$

$$\int_{G_1} + \int_{G_3} = 0$$

$$\int_{G_1} = 2\pi$$

0) Hány dimenziós

1*) Paraméterek megválasztása

2*) Határok megválasztása

3) Szimbólumok

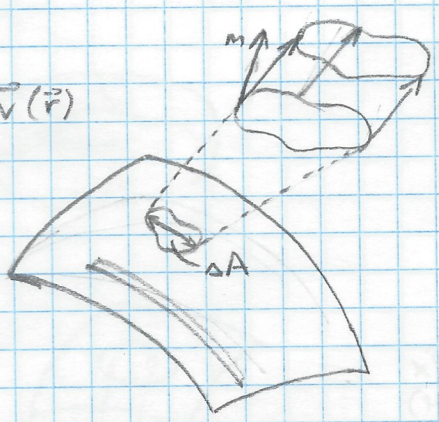
4) Deriválás

5) Vektorműveletek

6) Behelyettesítés

7) Szorzás

8) Integrálás



FELOSZTÁS: „kis elemekbe", átmérője tart 0-hoz
- síkkal közelíthetőek
- az integrálandó fv konstans

$$\Delta V = \Delta A \cdot n$$

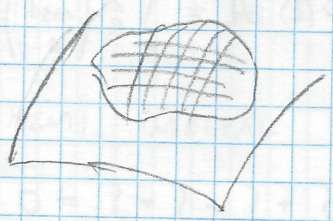
$$n = \vec{n}\,\Delta\vec{l}$$

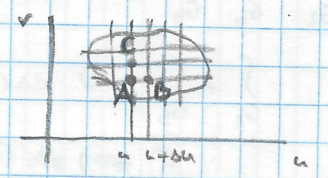
$$\Delta V = \Delta A \cdot n = \Delta A \cdot \vec{n} \cdot \Delta\vec{l} = \Delta A \cdot \vec{n} \cdot \vec{v}\,\Delta t$$

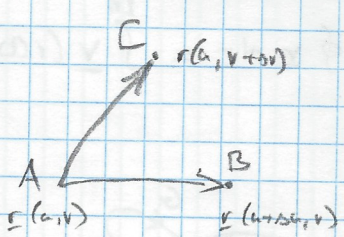
$$\frac{\Delta V}{\Delta t} = \underbrace{\Delta A \cdot \vec{n}}_{\Delta\vec{F}} \cdot \vec{v} = \vec{v}(\vec{r}) \cdot \Delta\vec{F}$$

felületelem - vektor

$$\frac{\Delta V}{\Delta t} = \sum \vec{v}(r)\,\Delta\vec{F} \longrightarrow \int \vec{v}(\vec{r})\, d\vec{F}$$ FELÜLETI INTEGRÁL







$$\Delta \underline{r}^{AB} = \frac{\partial \underline{r}(u,v)}{\partial u}\Delta u$$

$$\Delta \underline{r}^{AC} = \frac{\partial \underline{r}(u,v)}{\partial v}\Delta v$$

$$\Delta \underline{F} = \Delta \underline{r}^{AB} \times \Delta \underline{r}^{AC} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\Delta u \Delta v$$

$$|\Delta F| = \Delta A$$

$\Delta F \perp$ felület

$$\boxed{d\underline{F} = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\, du\, dv}$$

$$\boxed{dA = \left|\frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\right| du\, dv}$$

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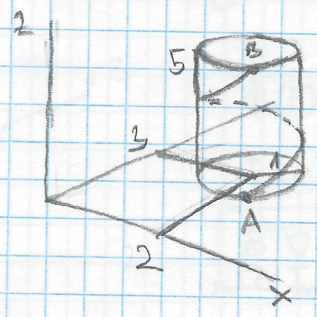
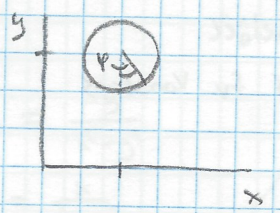


$$\underline{v}(\underline{r}) = \begin{pmatrix} 2x-3y \\ x+y+z \\ z-5x \end{pmatrix}$$

$$I = \int \underline{v}\, d\underline{r}$$



$\varphi \in [0, 2\pi]$

$$\begin{cases} x = 2 + \sin\varphi \\ y = 3 - \cos\varphi \\ z = \frac{5}{2\pi}\varphi \end{cases} \qquad \underline{r}(\varphi)$$

$$d\underline{r} = \dot{\underline{r}}(\varphi)\, d\varphi$$

$$\dot{\underline{r}} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ \frac{5}{2\pi} \end{pmatrix}$$

$$\underline{v}\left(\underline{r}(\varphi)\right) = \begin{pmatrix} 2(2+\sin\varphi) - 3(3-\cos\varphi) \\ (2+\sin\varphi) + (3-\cos\varphi) + \frac{5}{2\pi}\varphi \\ \frac{5}{2\pi}\varphi - 5(2+\sin\varphi) \end{pmatrix} = \begin{pmatrix} -5 + 2\sin\varphi + 3\cos\varphi \\ 5 + \sin\varphi - \cos\varphi + \frac{5}{2\pi}\varphi \\ -10 - 5\sin\varphi + \frac{5}{2\pi}\varphi \end{pmatrix}$$

$$\underline{v}\,\dot{\underline{r}} = (-5+2\sin\varphi+3\cos\varphi)\cdot\cos\varphi + (5+\sin\varphi-\cos\varphi+\tfrac{5}{2\pi}\varphi)\cdot\sin\varphi + (-10-5\sin\varphi+\tfrac{5}{2\pi}\varphi)\cdot\tfrac{5}{2\pi} =$$

$$= 3\cos^2\varphi + \sin^2\varphi + \sin\varphi\cdot\cos\varphi - 5\cos\varphi + 5\sin\varphi - \tfrac{25}{2\pi}\cdot\sin\varphi + \tfrac{5}{2\pi}\varphi\cdot\sin\varphi + \tfrac{25}{4\pi^2}\varphi - \tfrac{50}{2\pi}$$

$$\int_0^{2\pi}\cos^2\varphi\,d\varphi = \int_0^{2\pi}\frac{1-\cos 2\varphi}{2}\,d\varphi = \pi$$

$$\underline{I} = \int_0^{2\pi} d\varphi\,(\underline{v}\,\dot{\underline{r}}) = 3\pi + \pi + \frac{5}{2\pi}\int_0^{2\pi}\varphi\sin\varphi\,d\varphi + \frac{25}{4\pi^2}\cdot 2\pi^2 - \frac{50}{2\pi}2\pi$$

$$\nabla\times\underline{v} = \begin{pmatrix}\partial_y v_z - \partial_z v_y\\ \partial_z v_x - \partial_x v_z\\ \partial_x v_y - \partial_y v_x\end{pmatrix} = \begin{pmatrix}-1\\5\\4\end{pmatrix}$$

$$\underline{r}(z) = \begin{pmatrix}2\\2\\z\end{pmatrix}$$

$$\dot{\underline{r}}(z) = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad \underline{v}(\underline{r}(z)) = \begin{pmatrix}-2\\4+z\\z-10\end{pmatrix}$$

$z\in[0,5]$

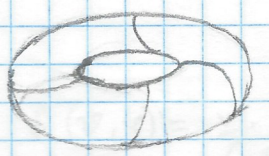
$x=2$

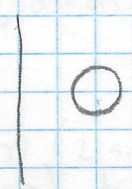
$y=2$

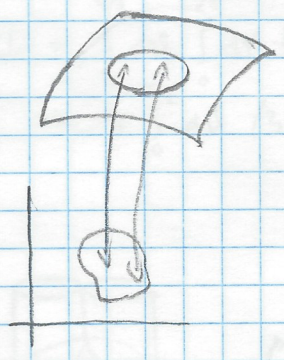
$z=z$

$$\underline{v}\,\dot{\underline{r}} = z-10$$

$$\int_{G_z}\underline{v}\,d\underline{r} = \int_0^5 (z-10)\,dz = \left[\frac{z^2}{2}-10z\right]_0^5 = \frac{75}{2}-50 = -\frac{75}{2}$$



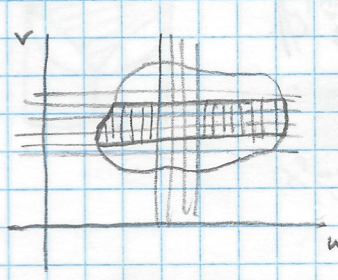




$$x^2+y^2 = R^2$$

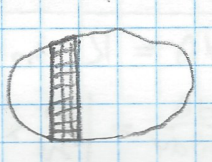
$$y = \pm\sqrt{R^2-x^2}$$

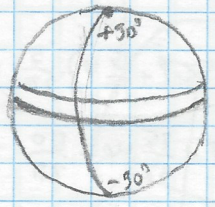
$$2\int_{-R}^{R}\sqrt{R^2-x^2}\,dx$$



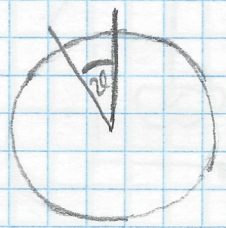
$$\int_{v_{alsó}}^{v_{felső}} dv \int_{\alpha v_1}^{\beta(v_1)} du\, f(u,v) \quad , \text{ fordítva is lehet: } \int_{u_{alsó}}^{u_{felső}} du \int_{\gamma(u)}^{\delta(u)} dv\, f(u,v)$$

Határok: $u=\alpha(v)$, $v=\beta(u)$

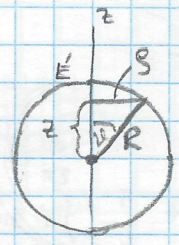




$\lambda \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



$\vartheta \in [0, \pi]$

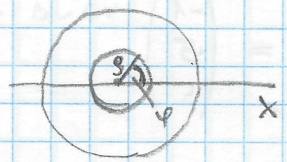


$z = R \cdot \cos\vartheta$

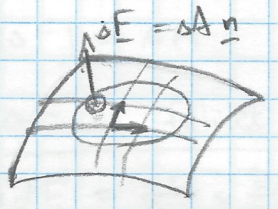
$s = R \cdot \sin\vartheta$

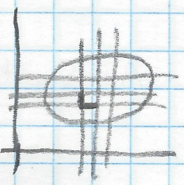
$x = s \cos\varphi$

$y = s \cdot \sin\varphi$



$$\underline{r}(\vartheta,\varphi) = \begin{pmatrix} R \cdot \sin\vartheta \cdot \cos\varphi \\ R \cdot \sin\vartheta \cdot \sin\varphi \\ R \cdot \cos\vartheta \end{pmatrix}$$





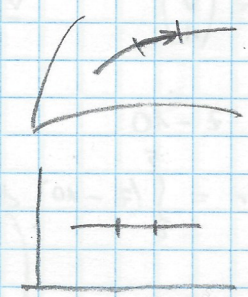
$$A = \lim_{\Delta A \to 0} \sum \Delta A$$

$d\underline{r} = \dot{\underline{r}}\, dt$

$\underline{r}(u,v)$

$$\Delta \underline{r}_u = \frac{\partial \underline{r}(u,v)}{\partial u} \Delta u$$

$$\Delta \underline{r}_v = \frac{\partial \underline{r}(u,v)}{\partial v} \Delta v$$



$$d\underline{F} = \Delta \underline{r}_u \times \Delta \underline{r}_v = \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v}\, du\, dv$$

$$dA = \left| d\underline{F} \right| = \left| \frac{\partial \underline{r}}{\partial u} \times \frac{\partial \underline{r}}{\partial v} \right| du\, dv$$

Élek sarkon: szingularitás → probléma lehet ($\sin 0$-val osztás)

$\vartheta \in [0, \pi]$

$\varphi \in [0, 2\pi[$

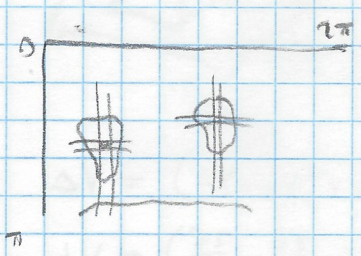
$$\frac{\partial \underline{r}}{\partial \vartheta} = \begin{pmatrix} R \cdot \cos\vartheta \cdot \cos\varphi \\ R \cdot \cos\vartheta \cdot \sin\varphi \\ -R \cdot \sin\vartheta \end{pmatrix} \qquad \frac{\partial \underline{r}}{\partial \varphi} = \begin{pmatrix} -R \cdot \sin\vartheta \cdot \sin\varphi \\ R \cdot \sin\vartheta \cdot \cos\varphi \\ 0 \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \vartheta} \times \frac{\partial \underline{r}}{\partial \varphi} = R^2 \cdot \sin\vartheta \cdot \begin{pmatrix} \cos\vartheta \cos\varphi \\ \cos\vartheta \cdot \sin\varphi \\ -\sin\vartheta \end{pmatrix} \times \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = R^2 \cdot \sin\vartheta \cdot \begin{pmatrix} \sin\vartheta \cdot \cos\varphi \\ \sin\vartheta \cdot \sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$\boxed{dA = R^2 \cdot \sin\vartheta \, d\vartheta \, d\varphi}$$

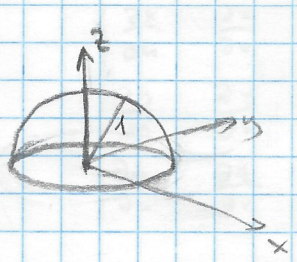
$$\boxed{\frac{dA}{d\vartheta d\varphi} = \left| \frac{\partial \underline{r}}{\partial \vartheta} \times \frac{\partial \underline{r}}{\partial \varphi} \right| = R^2 \cdot \sin\vartheta}$$

$$A = \int dA = \int_{\varphi=0}^{2\pi} d\varphi \int_{\vartheta=0}^{\pi} d\vartheta\, R^2 \cdot \sin\vartheta = R^2 \int_0^{2\pi} d\varphi \left[-\cos\vartheta\right]_0^{\pi} = 2R^2 \int_0^{2\pi} d\varphi = 2R^2 2\pi = 4\pi R^2$$



$R^2 \cdot \sin\vartheta \rightarrow$ torzítási tényező

JACOBI - FÜGGVÉNY



$$\underline{v} = \begin{pmatrix} 2x - 3y \\ x + y + z \\ z - 5x \end{pmatrix}$$

$$I = \int \underline{v}\, d\underline{F}$$

$$\underline{r} = \begin{pmatrix} \sin\vartheta \cdot \cos\varphi \\ \sin\vartheta \cdot \sin\varphi \\ \cos\vartheta \end{pmatrix} \qquad \vartheta \in [0, \pi] \qquad \varphi \in [\pi, 2\pi[$$

$$d\underline{F} = \sin\vartheta \cdot \begin{pmatrix} \sin\vartheta \cdot \cos\varphi \\ \sin\vartheta \cdot \sin\varphi \\ \cos\vartheta \end{pmatrix} d\vartheta\, d\varphi$$

$$\underline{v}(\underline{r}(\vartheta,\varphi)) = \begin{pmatrix} 2\sin\vartheta\cos\varphi - 3\sin\vartheta\cdot\sin\varphi \\ \sin\vartheta\cdot\cos\varphi + \sin\vartheta\sin\varphi + \cos\vartheta \\ \cos\vartheta - 5\sin\vartheta\cdot\cos\varphi \end{pmatrix}$$

$$I = \int \underline{v}\, d\underline{F} = \int \underline{v} \frac{d\underline{F}}{d\vartheta\, d\varphi} d\vartheta\, d\varphi$$

$$f(\vartheta,\varphi) = \underline{v} \cdot \frac{d\underline{F}}{d\vartheta\, d\varphi} = \sin\vartheta \left(2\sin^2\vartheta\cdot\cos^2\varphi - 3\sin^2\vartheta\cdot\sin\varphi\cdot\cos\varphi + \sin^2\vartheta\cos\varphi\sin\varphi + \sin^2\vartheta\sin^2\varphi + \right.$$
$$\left. + \cos\vartheta\cdot\sin\vartheta\cdot\sin\varphi + \cos^2\vartheta - 5\sin\vartheta\cdot\cos\vartheta\cdot\cos\varphi\right)$$

$$I = \int_0^{\pi/2} d\vartheta \int_0^{2\pi} d\varphi\, f(\vartheta,\varphi) = \int_0^{\pi/2} d\vartheta \left(2\sin^3\vartheta\,\pi + \sin^3\vartheta\,\pi + \sin\vartheta\cdot\cos^2\vartheta\cdot 2\pi\right) =$$
$$= \pi \int_0^{\pi/2} d\vartheta \left(3\sin^3\vartheta + 2\sin\vartheta\cdot\cos^2\vartheta\right) = \pi \left[-3\cos\vartheta + \frac{\cos^3\vartheta}{3}\right]_0^{\pi/2} = \pi\left(3 - \frac{1}{3}\right) = \frac{8}{3}\pi$$

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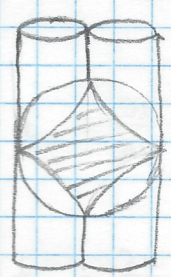
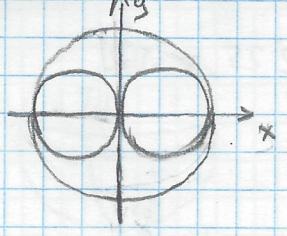


$$x = R\cdot\sin\vartheta\cos\varphi$$

$$y = R\cdot\sin\vartheta\cdot\sin\varphi$$

$$z = R\cdot\cos\vartheta$$

$$x^2 + y^2 + z^2 = R^2$$



$$\left(x-\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$

$$x^2-xR+\left(\frac{R}{2}\right)^2+y^2=\left(\frac{R}{2}\right)^2$$

$$\boxed{x^2+y^2=xR}$$

$$(R\sin\vartheta\cdot\cos\varphi)^2+(R\cdot\sin\vartheta\cdot\sin\varphi)^2=R(R\sin\vartheta\cdot\cos\varphi)$$

$$R^2\cdot\sin^2\vartheta=R^2\cdot\sin\vartheta\cdot\sin\varphi$$

$$\sin^2\vartheta=\sin\vartheta\cdot\sin\varphi$$

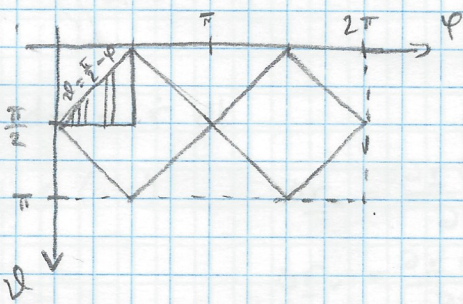
$\sin\vartheta=0$ | $\sin\vartheta=\cos\varphi$

$$\sin\vartheta=\sin\left(\frac{\pi}{2}-\varphi\right)$$

$$\boxed{\vartheta=\frac{\pi}{2}-\varphi+2k\pi,\quad k\in\mathbb{Z}}$$

$$\pi-\vartheta=\frac{\pi}{2}-\varphi+2k\pi$$

$$\boxed{\vartheta=\frac{\pi}{2}+\varphi-2k\pi\qquad k\in\mathbb{Z}}$$

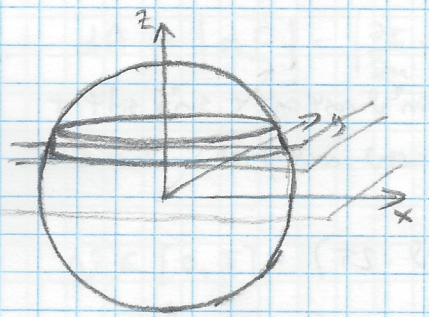


$$A=\int dA=\iint R^2\sin\vartheta\,d\vartheta\,d\varphi=$$

$$=\int_{\varphi=0}^{\pi/2}d\varphi\left(\int_{\vartheta=\frac{\pi}{2}-\varphi}^{\pi}d\vartheta\,R^2\cdot\sin\vartheta\right)=R^2\int_0^{\pi/2}d\varphi\left[-\cos\vartheta\right]_{\frac{\pi}{2}-\varphi}^{\pi}=$$

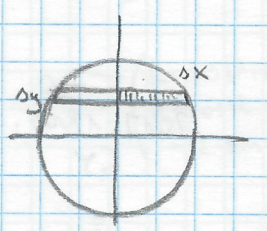
$$=R^2\int_0^{\pi/2}d\varphi\cdot\cos\left(\frac{\pi}{2}-\varphi\right)=R^2\int_0^{\pi/2}d\varphi\sin\varphi=R^2\cdot\left[-\cos\varphi\right]_0^{\pi/2}=$$

$$=R^2\left(-\cos\pi/2+\cos 0\right)=R^2$$



$$s^2+z^2=R^2$$

$$s=\sqrt{R^2-z^2}$$



$$\sum_{z=-R}^{R}\left(\sum_{x=-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}}\left(\sum_{x=-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}}\Delta x\right)\Delta y\right)\Delta z$$

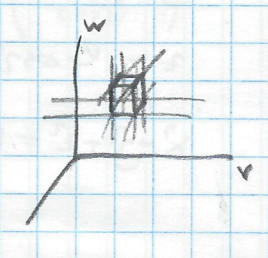
$$V=\int_{-R}^{R}dz\left(\int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}}dy\left(\int_{-\sqrt{R^2-z^2-y^2}}^{\sqrt{R^2-z^2-y^2}}1\,dx\right)\right)$$

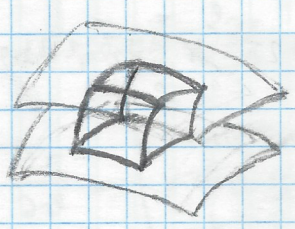
$$\underline{r}(r,\vartheta,\varphi)=\begin{pmatrix}r\cdot\sin\vartheta\cdot\cos\varphi\\ r\cdot\sin\vartheta\cdot\sin\varphi\\ r\cdot\cos\vartheta\end{pmatrix}$$

$$r\in[0,R]$$

$$\vartheta\in[0,\pi]$$

$$\varphi\in[0,2\pi[$$





$$\Delta \underline{r}_u = \frac{\partial \underline{r}(u,v,w)}{\partial u}\Delta u$$

$$\Delta \underline{r}_v = \frac{\partial \underline{r}}{\partial v}\Delta v$$

$$\Delta \underline{r}_w = \frac{\partial \underline{r}}{\partial w}\Delta w$$

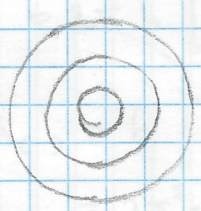
$$\Delta V = (\Delta \underline{r}_u, \Delta \underline{r}_v, \Delta \underline{r}_w) = \left(\frac{\partial \underline{r}}{\partial u}, \frac{\partial \underline{r}}{\partial v}, \frac{\partial \underline{r}}{\partial w}\right)\Delta u \Delta v \Delta w$$

$$dV = \left(\frac{\partial \underline{r}}{\partial u}, \frac{\partial \underline{r}}{\partial v}, \frac{\partial \underline{r}}{\partial w}\right) du\,dv\,dw$$

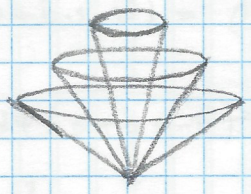
TÉRFOGATI INTEGRÁL

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|$$

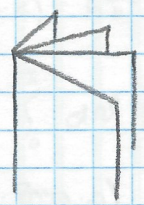
JACOBI - DETERMINÁNS



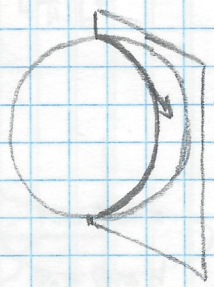
r Konstans

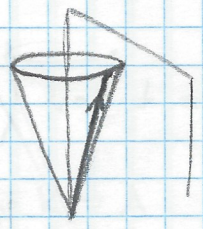


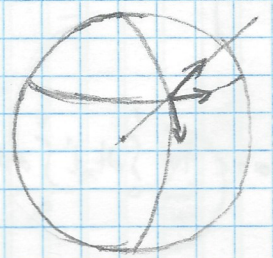
ϑ Konstans



φ Konstans







Gömbi Koordinátarendszer → ORTONORMÁLT

$$\underline{t}_r = \frac{\partial \underline{r}}{\partial r} = \begin{pmatrix} \sin\vartheta\cdot\cos\varphi \\ \sin\vartheta\cdot\sin\varphi \\ \cos\vartheta \end{pmatrix} = \underline{e}_r$$

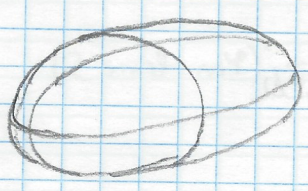
$$\underline{t}_\vartheta = \frac{\partial \underline{r}}{\partial \vartheta} = \begin{pmatrix} r\cdot\cos\vartheta\cdot\cos\varphi \\ r\cdot\cos\vartheta\cdot\sin\varphi \\ -\sin\vartheta \end{pmatrix} = r\cdot\begin{pmatrix} \cos\vartheta\cdot\cos\varphi \\ \cos\vartheta\cdot\sin\varphi \\ -\sin\vartheta \end{pmatrix} = r\cdot\underline{e}_\vartheta$$

$$\underline{t}_\varphi = \frac{\partial \underline{r}}{\partial \varphi} = \begin{pmatrix} -r\cdot\sin\vartheta\cdot\sin\varphi \\ r\cdot\sin\vartheta\cdot\cos\varphi \\ 0 \end{pmatrix} = r\cdot\sin\vartheta\cdot\underline{e}_\varphi$$

$$J = \frac{dV}{dr\,d\vartheta\,d\varphi} = (\underline{t}_r, \underline{t}_\vartheta, \underline{t}_\varphi) = r^2\cdot\sin\vartheta$$

$$V = \int dV = \int \frac{dV}{dr\,d\vartheta\,d\varphi}\, dr\,d\vartheta\,d\varphi = \int_{r=0}^{R}\int_{\vartheta=0}^{\pi}\int_{\varphi=0}^{2\pi} r^2 \sin\vartheta\, dr\,d\vartheta\,d\varphi = 2\pi \int_0^R dr \int_0^\pi d\vartheta\, r^2 \sin\vartheta =$$

$$= 2\pi \int_0^R dr\, r^2 \int_0^\pi d\vartheta \sin\vartheta = 2\pi \left[\frac{r^3}{3}\right]_0^R \cdot \left[-\cos\vartheta\right]_0^\pi = \frac{4}{3}\pi R^3$$



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$x = a \cdot \sin\vartheta \cdot \cos\varphi$$

$$y = b \cdot \sin\vartheta \cdot \sin\varphi$$

$$z = c \cdot \cos\vartheta$$

$$\underline{r}(t,\vartheta,\varphi) = \begin{pmatrix} a\,t \sin\vartheta \cos\varphi \\ b\,t \cdot \sin\vartheta \cdot \sin\varphi \\ c \cdot t \cos\vartheta \end{pmatrix} \qquad \begin{aligned} t &\in [0,1] \\ \vartheta &\in [0,\pi] \\ \varphi &\in [0,2\pi[\end{aligned}$$

$$\frac{\partial \underline{r}}{\partial t} = \begin{pmatrix} a \cdot \sin\vartheta \cdot \cos\varphi \\ b \cdot \sin\vartheta \cdot \sin\varphi \\ c \cdot \cos\vartheta \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \vartheta} = \begin{pmatrix} a\,t \cos\vartheta \cdot \cos\varphi \\ b\,t \cdot \cos\vartheta \cdot \sin\varphi \\ -c\,t \sin\vartheta \end{pmatrix} = t \begin{pmatrix} a \cdot \cos\vartheta \cdot \cos\varphi \\ b \cdot \cos\vartheta \cdot \sin\varphi \\ -c \cdot \sin\vartheta \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \varphi} = \begin{pmatrix} -a\,t \sin\vartheta \cdot \sin\varphi \\ b\,t \cdot \sin\vartheta \cdot \cos\varphi \\ 0 \end{pmatrix} = t \cdot \sin\vartheta \begin{pmatrix} -a \cdot \sin\varphi \\ b \cdot \cos\varphi \\ 0 \end{pmatrix}$$

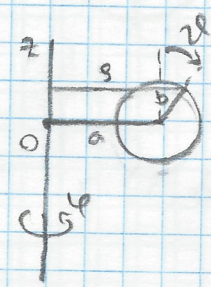
$$J = \left(\underline{t}_r, \underline{t}_\vartheta, \underline{t}_\varphi\right) = t^2 \cdot \sin\vartheta \begin{vmatrix} a \cdot \sin\vartheta \cdot \cos\varphi & a \cdot \cos\vartheta \cos\varphi & -a \cdot \sin\varphi \\ b \cdot \sin\vartheta \cdot \sin\varphi & b \cos\vartheta \sin\varphi & b \cdot \cos\varphi \\ c \cdot \cos\vartheta & -c \cdot \sin\vartheta & 0 \end{vmatrix} = abc\, t^2 \cdot \sin\vartheta$$

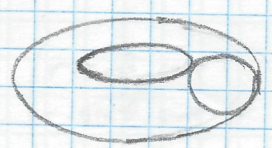
$$V = \int dV = \int \frac{dV}{dt\,d\vartheta\,d\varphi}\, dt\,d\vartheta\,d\varphi = \int_0^1 dt \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi\; abc\, t^2 \cdot \sin\vartheta = abc\, 2\pi \int_0^1 dt\, t^2 \int_0^\pi d\vartheta \sin\vartheta =$$

$$= \frac{4\pi}{3} abc$$

Kúp → gömb, paraméterezés, ϑ konstans

Tórusz





$$R = a + b \cdot \sin\vartheta$$

$$z = b \cdot \cos\vartheta$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (a + b\sin\vartheta) \cdot \cos\varphi \\ (a + b\sin\vartheta) \cdot \sin\varphi \\ b \cdot \cos\vartheta \end{pmatrix} \qquad \begin{aligned} \vartheta &\in [0, 2\pi[\\ \varphi &\in [0, 2\pi[\end{aligned}$$

$$d\underline{F} = \frac{\partial \underline{r}}{\partial \vartheta} \times \frac{\partial \underline{r}}{\partial \varphi}\, d\vartheta\, d\varphi$$

$$\frac{\partial \underline{r}}{\partial \vartheta} = \begin{pmatrix} b\cdot\cos 2\vartheta\cdot\cos\varphi \\ b\cdot\cos 2\vartheta\cdot\sin\varphi \\ -b\cdot\sin 2\vartheta \end{pmatrix} = b\cdot\begin{pmatrix} \cos 2\vartheta\cdot\cos\varphi \\ \cos 2\vartheta\,\sin\varphi \\ -\sin 2\vartheta \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \varphi} = \begin{pmatrix} -(a+b\cdot\sin 2\vartheta)\cdot\sin\varphi \\ (a+b\cdot\sin 2\vartheta)\cdot\cos\varphi \\ 0 \end{pmatrix} = (a+b\cdot\sin 2\vartheta)\cdot\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \vartheta} \times \frac{\partial \underline{r}}{\partial \varphi} = b\,(a+b\cdot\sin 2\vartheta)\begin{pmatrix} \sin 2\vartheta\cdot\cos\varphi \\ \sin 2\vartheta\cdot\sin\varphi \\ \cos 2\vartheta \end{pmatrix}$$

$$dA = b\,(a+b\cdot\sin 2\vartheta)\cdot d\vartheta\, d\varphi$$

$$A = \int dA = \int \frac{dA}{d\vartheta\, d\varphi}\cdot d\vartheta\, d\varphi = \int b(a+b\cdot\sin 2\vartheta)\, d\vartheta\, d\varphi = b\cdot\int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi\,(a+b\cdot\sin 2\vartheta) =$$

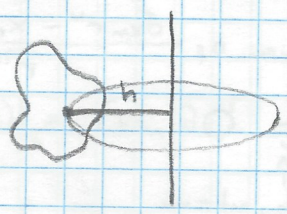
$$= 2\pi b\int_0^{2\pi} d\vartheta\,(a+b\cdot\sin 2\vartheta) = (2\pi b)\cdot(2\pi\cdot a) = 4\pi^2 ab$$

$$A_{cs} = b\int_0^{\pi} d\vartheta \int_0^{2\pi} d\varphi\,(a+b\cdot\sin 2\vartheta) = 2\pi b\int_0^{\pi} d\vartheta\,(a+b\cdot\sin 2\vartheta) = 2\pi b\left[a\vartheta - b\cdot\cos 2\vartheta\right]_0^{\pi} =$$

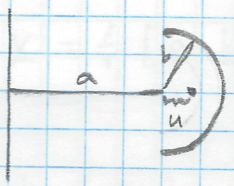
$$= 2\pi^2 ab + 4\pi b^2$$

$$\lim_{a\to 0} A_{cs} = A_{gömb} = 4\pi b^2$$

Guldin I.



$$2\pi h\cdot K = A$$



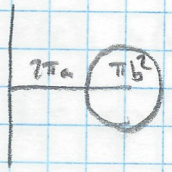
$$2\pi(a+u)\cdot\pi b = 2\pi^2 ab + 4\pi b^2$$

$$2\pi^2 ub = 4\pi b^2$$

$$\pi u = 2b$$

$$u = \frac{2}{\pi} b$$

Guldin II.



$$V = 2\pi^2 a b^2$$

2017.12.07.

GY

$\operatorname{div} \underline{v} = \phi$ $\qquad T_{ke}(r)$

$\partial_k v_k$ $\qquad \partial_l T_{kl} = v_k$

$\operatorname{Div} \underline{\underline{T}} = \underline{v}$

$\underline{v} = \operatorname{rot} (\underline{a} \times \underline{e})$

$v_k = \varepsilon_{kem}\, \partial_l (\underline{a} \times \underline{e})_m = \varepsilon_{kem}\, \partial_l (\varepsilon_{mpq}\, a_p e_q) = \varepsilon_{kem}\varepsilon_{mpq}\, a_p\, \partial_l e_q$

* $\partial_l x_k = \delta_{lk}$

$\partial_l r = e_l$

$\partial_l e_k = \dfrac{\delta_{lk} - e_l e_k}{r}$

$v_k = \varepsilon_{kem}\varepsilon_{mpq}\, a_p \dfrac{\delta_{lq} - e_l e_q}{r} = \dfrac{1}{r}(\delta_{kp}\delta_{lq} - \delta_{kq}\delta_{lp})\, a_p (\delta_{lq} - e_l e_q) =$

$= \dfrac{1}{r}\left[a_k (\underbrace{\delta_{ll}}_{3} - \underbrace{e_l e_l}_{1}) - a_l (\delta_{lk} - e_l e_k)\right] = \dfrac{1}{r}\left[2a_k - a_k + a_l e_l e_k\right] =$

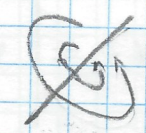
$= \dfrac{1}{r}\left[a_k + (\underline{a}\cdot\underline{e}) e_k\right] = \dfrac{1}{r}\left(\underline{a} + (\underline{a}\underline{e})\underline{e}\right)$

$\operatorname{div}\left(\dfrac{\underline{a}\times\underline{r}}{r^N}\right) = \partial_k \left[\dfrac{1}{r^N}(\underline{a}\times\underline{r})_k\right] = \partial_k \left[\dfrac{1}{r^N}\varepsilon_{kem} a_e x_m\right] = \varepsilon_{kem} a_e\, \partial_k \left(\dfrac{x_m}{r^N}\right) =$

$= \varepsilon_{kem} a_e \dfrac{(\partial_k x_m) r^N - x_m \partial_k r^N}{r^{2N}} = \varepsilon_{kem} a_e \dfrac{\delta_{km} r^N - x_m N r^{N-1} \partial_k r}{r^{2N}} =$

$= \varepsilon_{kem} a_e \dfrac{\delta_{km} r^N - N \cdot r^{N-2} x_m x_k}{r^{2N}} = 0$

↑ szimm., antiszimm.



$\Delta \left[(\underline{a}\times\underline{e})(\underline{b}\underline{e})\right] = \underline{v}$

* $\Delta\phi = \operatorname{div}\operatorname{grad}\phi$

$\partial_k (\partial_k \phi)$

$(\partial_k \partial_k)\phi$

∇^2

$(\Delta \underline{v})_k = \partial_l \partial_l v_k = \Delta(v_k)$

$\operatorname{Grad} \underline{v} = \underline{\nabla} \circ \underline{v}$

$(\operatorname{Grad} \underline{v})_{ke} = \partial_k v_e$

$[\operatorname{Div}(\operatorname{Grad} \underline{v})]_e = \partial_k (\partial_k v_e)$

$\partial_l T_{kl}$ → transzportálási problémák

$$v_k = \Delta\left[(\underline{a}\times\underline{e})(\underline{b}\underline{e})\right]_k = \Delta\left[(\ )_k(\ )\right] = \Delta\left[(\varepsilon_{kem}a_e e_m)(b_p e_p)\right] = \partial_s\partial_s\left[\varepsilon_{kem}a_e e_m b_p e_p\right] =$$

$$= \varepsilon_{kem}a_e b_p\left(\partial_s\partial_s e_m e_p\right) = \varepsilon_{kem}a_e b_p\,\partial_s\left((\partial_s e_m)e_p + e_m(\partial_s e_p)\right) =$$

$$= \varepsilon_{kem}a_e b_p\,\partial_s\left(\frac{\delta_{sm}-e_s e_m}{r}\cdot e_p + e_m\cdot\frac{\delta_{sp}-e_s e_p}{r}\right) = \varepsilon_{kem}a_e b_p\,\partial_s\left(\frac{1}{r}(\delta_{sm}e_p+\delta_{sp}e_m-2e_s e_m e_p)\right) =$$

$$= \varepsilon_{kem}a_e b_p\left\{-\frac{1}{r^2}(\partial_s r)(\delta_{sm}e_p+\delta_{sp}e_m-2e_s e_m e_p)+\frac{1}{r}\left[\delta_{sm}\frac{\delta_{sp}-e_s e_p}{r}+\delta_{sp}\frac{\delta_{sm}-e_s e_m}{r}-\right.\right.$$

$$\left.\left.-2\left(\frac{2}{r}\cdot e_m e_p + e_s\frac{\delta_{sm}-e_s e_m}{r}e_p + e_s e_m\frac{\delta_{sp}-e_s e_p}{r}\right)\right]\right\} = \frac{1}{r^2}\varepsilon_{kem}a_e b_p\left\{\underbrace{(-e_m e_p - e_p e_m + 2e_m e_p)}_{0}+\right.$$

$$\left.+\delta_{mp}-e_m e_p+\delta_{pm}-e_p e_m-2\left(2e_m e_p+\underbrace{e_m e_p - e_m e_p}_{0}+\underbrace{e_m e_p - e_m e_p}_{0}\right)\right\} =$$

$$= \frac{1}{r^2}\varepsilon_{kem}a_e b_p\left(2\delta_{pm}-6e_m e_p\right) = \frac{2}{r^2}\left(\varepsilon_{kem}a_e b_m - 3\varepsilon_{kem}a_e e_m b_p e_p\right) =$$

$$= \frac{2}{r^2}\left[(\underline{a}\times\underline{b})_k - 3(\underline{b}\underline{e})(\underline{a}\times\underline{e})_k\right]$$

$$\underline{v} = \frac{2}{r^2}\left[(\underline{a}\times\underline{b}) - 3(\underline{b}\underline{e})(\underline{a}\times\underline{e})\right]$$

$$\Delta(\underline{e}\,\underline{\underline{S}}\,\underline{e}) = \partial_p\partial_p\left(e_k S_{ke} e_e\right) = S_{ke}\partial_p\partial_p(e_k e_e) = S_{ke}\,\partial_p\left(\frac{\delta_{pk}-e_k e_p}{r}+\frac{\delta_{pe}-e_e e_k}{r}\right) =$$

$$= S_{ke}\,\partial_p\frac{\delta_{pk}e_e+\delta_{pe}e_k-2e_k e_p e_e}{r} = S_{ke}\left[-\frac{1}{r^2}e_p(\delta_{pk}e_e+\delta_{pe}e_k-2e_k e_e e_p)+\right.$$

$$\left.+\frac{1}{r}\left(\delta_{pk}\frac{\delta_{pe}-e_p e_e}{r}+\delta_{pe}\frac{\delta_{pk}-e_p e_k}{r}-2\left(\frac{2}{r}e_k e_e+0\right)\right)\right] = S_{ke}\frac{1}{r^2}\left(\delta_{ke}-e_k e_e+\right.$$

$$\left.+\delta_{ek}-e_e e_k-4e_k e_e\right) = \frac{2}{r^2}S_{ke}\left(\delta_{ke}-3e_k e_e\right) = \frac{2}{r^2}\left(S_{kk}-3e_k S_{ke}e_e\right) =$$

$$= \frac{2}{r^2}\left[S_p\underline{\underline{S}}-3\,\underline{e}\,\underline{\underline{S}}\,\underline{e}\right]$$

$$v_k = \mathrm{rot}\left[\underbrace{\underline{a}(\underline{e}\circ\underline{b})}_{(\underline{a}\underline{e})\underline{b}}\right]_k = \varepsilon_{kem}\partial_e[\ ]_m = \varepsilon_{kem}\partial_e\left[(ae)b_m\right] = \varepsilon_{kem}b_m\partial_e(e_p e_p) =$$

$$= \varepsilon_{kem}b_m a_p\frac{\delta_{ep}-e_e e_p}{r} = \frac{1}{r}\left[\varepsilon_{kem}a_e b_m - (a_p e_p)\varepsilon_{kem}a_e b_m\right]$$

$$\underline{v} = \frac{1}{r}\left[\underline{a}\times\underline{b} - (\underline{a}\,\underline{e})(\underline{e}\times\underline{b})\right]$$

$$\mathrm{div}\left(\frac{\underline{e}}{\underline{a}\,\underline{\underline{S}}\,\underline{e}}\right) = \partial_k(\ )_k = \frac{(\partial_k e_k)(\underline{a}\underline{e})-e_k\partial_k(\underline{a}\underline{e})}{(\underline{a}\underline{e})^2} = \frac{\frac{2}{r}(\underline{e}\underline{e})-e_k\partial_k(a_e e_e)}{(ae)^2} =$$

$$= \frac{2}{r(\underline{a}\underline{e})} = \frac{2}{(\underline{a}\underline{r})}$$

$\underline{v}(\underline{r})$, $\underline{u}(\underline{r})$

$\underline{z} = \underline{v}\times\mathrm{rot}\,\underline{u}$

$z_k = (\underline{v}\times\mathrm{rot}\,\underline{u})_k = \varepsilon_{kem}v_e(\mathrm{rot}\,\underline{u})_m = \varepsilon_{kem}v_e\left(\varepsilon_{mpq}\partial_e u_p\right) = \left(\delta_{ep}\delta_{eq}-\delta_{kq}\delta_{ep}\right)v_e\partial_e u_p =$

$= v_e\partial_k u_e - v_e\partial_e u_k$

$\underline{s}(\underline{r}) = \underline{u} \times \mathrm{rot}\,\underline{v}$

$s_k = u_l \partial_k v_l - u_l \partial_l v_k$

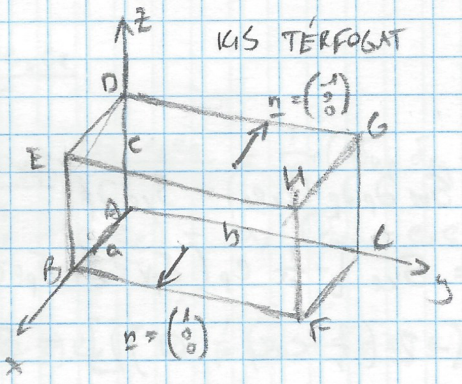
$\underline{w} = \underline{z} + \underline{s} = \underline{v} \times \mathrm{rot}\,\underline{u} + \underline{u} \times \mathrm{rot}\,\underline{v}$

$$w_k = \underbrace{v_l \partial_k u_l + u_l \partial_k v_l}_{\substack{\partial_k (u_l v_l) \\ [\nabla(\underline{u}\underline{v})]_k}} - v_l \partial_l u_k - u_l \partial_l v_k$$

$$\boxed{\underline{\nabla}(\underline{u}\underline{v}) = \underline{v} \times \mathrm{rot}\,\underline{u} + \underline{u} \times \mathrm{rot}\,\underline{v} + (\underline{u}\underline{\nabla})\underline{v} + (\underline{v}\underline{\nabla})\underline{u}}$$

2017. 17.07.

EA



$I = \oint \underline{v}(\underline{r})\, d\underline{F} =$

$= \oint v(\underline{r})\underline{n}(\underline{r})\, dA =$

$= \left(\int_{ACGD} + \int_{BFHE}\right) +$

$+ \left(\int_{ABED} + \int_{CGHF}\right) +$

$+ \left(\int_{ABFC} + \int_{DEHG}\right)$

$\int_{ACGD} \underline{v}\, d\underline{F} = \underline{v}(ACGD) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} bc =$

$= -v_x(ACGD) \cdot bc$

$\int_{BFHE} \underline{v}\, d\underline{F} = \underline{v}(BFHE) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} bc =$

$= v_x(BFHE) \cdot bc$

$v_x(ACGD) = v_x(x=0, y, z)$

$v_x(BFHE) = v_x(x = 0+a, y, z) \approx v_x(x=0, y, z) + \frac{\partial v_x}{\partial x} a$

$\rightarrow \int_{ACGD} + \int_{BFHE} = \left(\frac{\partial v_x}{\partial x} a\right) bc$

$\int_{ABED} \underline{v}\, d\underline{F} = \underline{v}(ABED) \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} ac = -v_y(ABED) \cdot ac = v_y(x, y=0, z)$

$\int_{CGHF} \underline{v}\, d\underline{F} = \underline{v}(CGHF) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} ac = v_y(CGHF) \cdot ac \approx v_y(x, y=0, z) + \frac{\partial v_y}{\partial y} b$

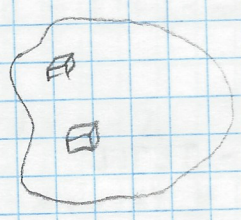
$\rightarrow \int_{ABED} + \int_{CGHF} = \left(\frac{\partial v_y}{\partial y} b\right) ac$

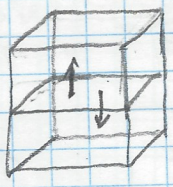
$\rightarrow \int_{ABFC} + \int_{DEHG} = \left(\frac{\partial v_z}{\partial z} c\right) ab$

$\Rightarrow \oint \underline{v}\, d\underline{F} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) abc = (\mathrm{div}\,\underline{v})\, \Delta V$

$\Downarrow$

· FORRÁSERŐSSÉG





$$\oint \underline{v}\, d\underline{F} = \sum \operatorname{div} \underline{v}\, \Delta V \to \int \operatorname{div} \underline{v}\, dV$$

$$\boxed{\oint_{\partial V} \underline{v}\, d\underline{F} = \int_V \operatorname{div} \underline{v}\, dV}$$

GAUSS - TÉTEL

↓

Azaz igaz, ha a térfogaton belül a függvény mindenhol értelmezhető és kétszer div.

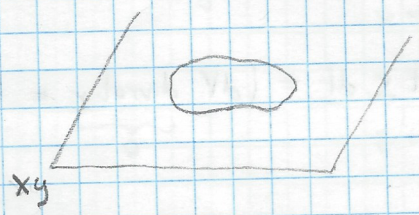
$$\int_A^B f'(x)\, dx = f(B) - f(A)$$

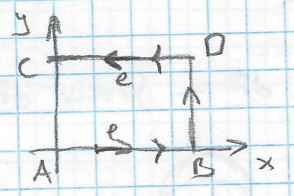
$$\int_T \partial( ) = \int_{\partial t} ( )$$

$$\underline{v} = \begin{pmatrix} 2x - 3y \\ x + y + z \\ x - y \end{pmatrix}$$

$$\operatorname{div} \underline{v} = 3$$

$$I = \oint \underline{v}\, d\underline{r}$$





$$\oint \underline{v}\, d\underline{r} = \boxplus \underline{v}\, d\underline{r} = \left(\int_{AB} + \int_{DC}\right) + \left(\int_{BC} + \int_{CA}\right)$$

$$d\underline{r} = \underline{e}\, ds$$

AB $\quad e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\qquad$ CD $\quad e = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$

$$\int_{AB} \underline{v}\, d\underline{r} = \underline{v}(AB) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} a = v_x(AB)\, a = v_x(x, y=0) \cdot a$$

$$\int_{DC} \underline{v}\, d\underline{r} = \underline{v}(DC) \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} a = -v_x(DC) \cdot a = v_x(x, y=b) \cdot a \approx \left(v_x(y=0) + \frac{\partial v_x}{\partial y}\, b\right) \cdot a$$

$$\to \int_{AB} + \int_{DC} = -\frac{\partial v_x}{\partial y} \cdot b \cdot a$$

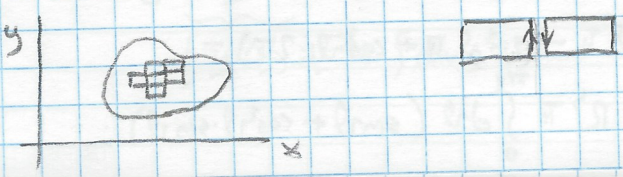
$$-\int_{CA} \underline{v}\, d\underline{r} = \underline{v}(CA) \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} b = -v_y(CA) \cdot b = -v_y(0, y) \cdot b$$

$$\int_{BD} \underline{v}\, d\underline{r} = \underline{v}(BD) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot b = v_y(BD) \cdot b = v_y(x=a, y) \cdot b \approx \left(v_y(0, y) + \frac{\partial v_y}{\partial x} \cdot a\right) \cdot b$$

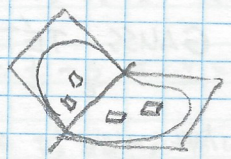
$$\to \int_{CA} + \int_{BD} = \frac{\partial v_y}{\partial x} \cdot ab$$

$$\Rightarrow \oint \underline{v}\, d\underline{r} = \boxplus \underline{v}\, d\underline{r} = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) \cdot ab = (\operatorname{rot} \underline{v})_z \cdot ab = (\operatorname{rot} \underline{v}) \cdot \underline{n} \cdot \Delta A$$

$$\oint \underline{v}\, d\underline{r} = (\operatorname{rot} \underline{v}) \cdot \Delta \underline{F}$$

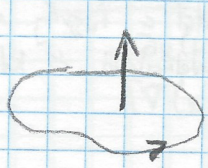


$$\oint_G \underline{v}\, d\underline{r} = \sum \operatorname{rot} \underline{v}\, \underline{n}\, dA \rightarrow \int \operatorname{rot} \underline{v}\, d\underline{F}$$

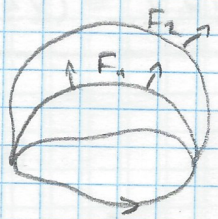


$$\boxed{\oint_{\partial F} \underline{v}\, d\underline{r} = \int_F \operatorname{rot} d\underline{F}}$$

STOKES-TÉTEL



Jobbkéz szabály



$$\int_{F_2} \operatorname{rot} \underline{v}\, dF - \int_{F_1} \operatorname{rot} \underline{v}\, d\underline{F} = \int_{F_2} \operatorname{rot} \underline{v}\, d\underline{F} - \int \operatorname{rot} \underline{v}\, (-d\underline{F}) =$$

$$= \int \operatorname{rot} \underline{v}\, d\underline{F} + \int \operatorname{rot} \underline{v}\, d\underline{F} = \oint \operatorname{rot} \underline{v}\, d\underline{F} = \int dV\, \underbrace{\operatorname{div} \operatorname{rot} \underline{v}}_{0} = 0$$

$$\oint_{\partial V} \underline{v}\, d\underline{F} = \int_V \operatorname{div} \underline{v}\, dV$$

$$\oint_{\partial F} \underline{v}\, d\underline{r} = \int_F \operatorname{rot} \underline{v}\, d\underline{F}$$

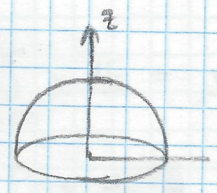
$$\oint v\, dr = \int \operatorname{rot} v\, dF \neq \int \operatorname{div} \operatorname{rot} \underline{v}\, dV = 0$$

NEM ZÁRT FELÜLET

$\partial\partial V = 0$: Zárt felület határa $\emptyset$

rot grad $= 0$

div rot $= 0$



$$\underline{v} = \begin{pmatrix} 2-3y \\ x-y+z \\ z-x \end{pmatrix} = R \begin{pmatrix} 2\sin\vartheta\cdot\cos\varphi - 3\sin\vartheta\sin\varphi \\ \sin\vartheta\cos\varphi - \sin\vartheta\sin\varphi + \cos\vartheta \\ \cos\vartheta - \sin\vartheta\cos\varphi \end{pmatrix}$$

$$\underline{r} = \begin{pmatrix} R\cdot\sin\vartheta\cdot\cos\varphi \\ R\cdot\sin\vartheta\cdot\sin\varphi \\ R\cos\vartheta \end{pmatrix}$$

$$I = \int_{\frown} v\, dF$$

$$d\underline{F} = R^2\cdot\sin\vartheta \begin{pmatrix} \sin\vartheta\cdot\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} d\vartheta\, d\varphi$$

$\vartheta \in [0, \frac{\pi}{2}]$

$\varphi \in [0, 2\pi[$

$$\frac{\underline{v}\, d\underline{F}}{d\vartheta\, d\varphi} = R^3\cdot\sin\vartheta \begin{pmatrix} 2\sin^2\vartheta\cos^2\varphi - 3\sin^2\vartheta\sin\varphi\cos\varphi \\ \sin^2\vartheta\cos\varphi\sin\varphi - \sin^2\vartheta\sin^2\varphi + \cos\vartheta\sin\vartheta\sin\varphi \\ \cos^2\vartheta - \sin\vartheta\cos\vartheta\cos\varphi \end{pmatrix}$$

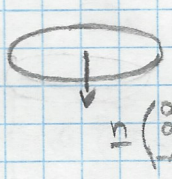
$$I = \int_0^{\pi/2} d\vartheta \int_0^{2\pi} d\varphi = \int_0^{\pi/2} d\vartheta\, R^3 \sin\vartheta\, (2\sin^2\vartheta\,\pi - \sin^2\vartheta\,\pi + \cos^2\vartheta\cdot 2\pi) =$$

$$= R^3\pi \int_0^{\pi/2} d\vartheta\cdot\sin\vartheta\, (\sin^2\vartheta + 2\cos^2\vartheta) = R^3\pi \int_0^{\pi/2} d\vartheta\, (\sin\vartheta + \cos^2\vartheta\cdot\sin\vartheta)$$

$$I = R^3\pi\left[-\cos\varphi - \frac{\cos^3\varphi}{3}\right]_0^{\pi/2} = \frac{4}{3}R^3\pi$$

$$\operatorname{div}\underline{v} = 2$$

$$\int \operatorname{div}\underline{v}\, dV = 2\cdot\frac{2\pi}{3}R^3$$

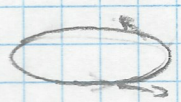


$$\underline{n}\begin{pmatrix}0\\0\\-1\end{pmatrix}$$

Alapkörre $\rightarrow$ $\int \underline{v}\, d\underline{F} = \int \underline{v}\begin{pmatrix}0\\0\\-1\end{pmatrix} dA = -\int v_z\, dA =$

$$= \iint (x-2)\, dA = \int x\, dA = 0$$

↓ szimmetrikus



$$J = \oint \underline{v}\, d\underline{r}$$

$x = R\cdot\cos\varphi$

$y = R\cdot\sin\varphi$ $\rightarrow \underline{r}(\varphi) \rightarrow \dot{\underline{r}}(\varphi) = \begin{pmatrix}-R\sin\varphi\\R\cdot\cos\varphi\\0\end{pmatrix}$

$z = 0$

$$\underline{v}\,\dot{\underline{r}} = -R\cdot\sin\varphi\,(2R\cdot\cos\varphi - 3R\cdot\sin\varphi) + R\cdot\cos\varphi\,(R\cdot\cos\varphi - R\cdot\sin\varphi) =$$

$$= R^2\left[-2\cos\varphi\cdot\sin\varphi + 3\sin^2\varphi + \cos^2\varphi - \cos\varphi\cdot\sin\varphi\right]$$

$$J = \int_0^{2\pi} d\varphi\, \underline{v}\,\dot{\underline{r}} = 4\pi R^2$$

$$\operatorname{rot}\underline{v} = \begin{pmatrix}\partial_y v_z - \partial_z v_y\\ \partial_z v_x - \partial_x v_z\\ \partial_x v_y - \partial_y v_x\end{pmatrix} = \begin{pmatrix}0-1\\0+1\\1+3\end{pmatrix} = \begin{pmatrix}-1\\1\\4\end{pmatrix}$$

$$\int_O \operatorname{rot}\underline{v}\; d\underline{F} = \int_O \operatorname{rot}\underline{v}\;\underline{n}\, dA = \int (\operatorname{rot}\underline{v})_z\, dA = 4\int dA = 4R^2\pi$$

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EA

$$\oint \underline{v}\, d\underline{F} = \int \operatorname{div}\underline{v}\, dV$$

$\underline{\underline{T}}(\underline{r})$

$$\underline{V} = \oint \underline{\underline{T}}(\underline{r})\, d\underline{F}$$

$$\underline{a}\,\underline{V} = \oint \underline{a}\,\underline{\underline{T}}(\underline{r})\, d\underline{F}$$

$\oint \underline{u}(\underline{r})\, d\underline{F} = \int \operatorname{div}\underline{u}\, dV$

$$\oint v_k\, dF_k = \int \partial_k v_k\, dV$$

$$V_k = \oint T_{ke}(\underline{r})\, dF_e$$

$$a_k V_k = a_k \oint T_{ke}(\underline{r})\, dF_e =$$

$$= \oint (a_k T_{ke}(\underline{r}))\, dF_e$$

$\oint u_e(\underline{r})\, dF_e = \int (\partial_e u_e)\, dV$

$$u_k = a_k T_{ke}(r)$$

$$\partial_e u_e = \partial_e (a_k T_{ke}) = a_k (\partial_e T_{ke})$$

$$\operatorname{div} u = \operatorname{div}(\underline{a}\,\underline{\underline{I}}) = a\,(\operatorname{Div} T) = \underline{a}\,\underline{w}$$

$$\oint u_e(\underline{r})\, dF_e = \int (\partial_e u_e)\, dV = \int a_k w_k\, dV = a_k \int w_k\, dV$$

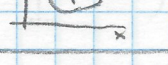
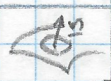
$$\Rightarrow \quad a_k \left(V_k - \int w_k\, dV \right) = 0$$

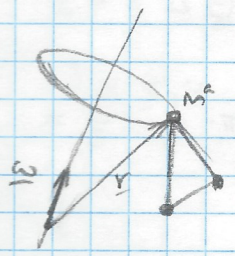
$$\underline{V} = \int \underline{w}\, dV \quad \Rightarrow \quad \boxed{\oint \underline{\underline{T}}(\underline{r})\, d\underline{F} = \int (\operatorname{Div} T)\, dV}$$

TENZORIÁLIS GAUSS-TÉTEL

$$\oint T_{ke}\, dF_e = \int (\partial_e T_{ke})\, dV$$

$$\boxed{\oint_{\partial V} (\;)_e\, dF_e = \int_V \partial_e (\;)_e\, dV} \rightarrow \text{Általános Gauss-tétel (akárhány dimenzióra)}$$

dim = 1 görbe		dim = 2 felület		dim = 3 térfogat
$\int_a^b f(x)\, dx$			$\int f(x,y)\, dx\, dy$	$\int f(x,y,z)\, dx\, dy\, dz$
ds	$d\underline{r}$	dA	$d\underline{F}$ 	dV
$\int ds$	~~$\int dx$~~	$A = \int dA$	$\int dF$	$V = \int dV$
$\int f(s)\, ds$	$\int f(s)\, d\underline{r}$	$\int f(A)\, dA$	$\int f(A)\, d\underline{F}$	$\int \phi(\underline{r})\, dV$
$\int \underline{v}\, ds$	$\int \underline{v}(s)\, d\underline{r}$	$\int \underline{v}(A)\, dA$	$\int \underline{v}(A)\, d\underline{F}$	$\int \underline{v}(\underline{r})\, dV$
$\int \phi(\underline{r})\, ds$	$\int \phi(\underline{r})\, d\underline{r}$	$\int \phi(\underline{r})\, dA$	$\int \phi(\underline{r})\, d\underline{F}$	
$\int \underline{v}(\underline{r})\, ds$	* $\int \underline{v}(\underline{r})\, d\underline{r}$	$\int \underline{v}(\underline{r})\, dA$	* $\int \underline{v}(\underline{r})\, d\underline{F}$	
	$\int \underline{v}(\underline{r}) \times d\underline{r}$		$\int \underline{v}(\underline{r}) \times d\underline{F}$	
	$\int \underline{v}(\underline{r}) \circ d\underline{r}$		$\int \underline{v}(\underline{r}) \circ d\underline{F}$	
	$\int \underline{\underline{T}}\, d\underline{r}$		$\int \underline{\underline{T}}(\underline{r})\, d\underline{F}$	



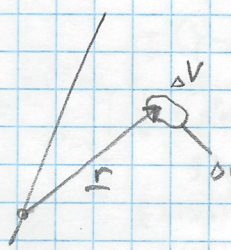
$$\underline{v}^a = \underline{\omega} \times \underline{r}^a$$

$$\underline{N}^a = \underline{r}^a \times \underline{p}^a = \underline{r}^a \times m_a \underline{v}^a = m_a \underline{r}^a \times (\underline{\omega} \times \underline{r}^a)$$

$$\underline{N} = \sum \underline{N}^a = \sum_a m_a \underline{r}^a \times (\underline{\omega} \times \underline{r}^a) = \sum_a m_a \left[\underline{\omega}(\underline{r}^a \underline{r}^a) - \underline{r}^a (\underline{\omega} \underline{r}^a)\right] =$$

$$= \underbrace{\left(\sum_a m_a \left[(r^a)^2 \underline{\underline{I}} - \underline{r}^a \circ \underline{r}^a\right]\right)}_{\underline{\underline{\Theta}}} \underline{\omega}$$

$$N_k = \sum_a m_a \left(\omega_k (r^a)^2 - r^a_k \sum_\ell \omega_\ell r^a_\ell\right) = \sum_\ell \underbrace{\left(\sum_a m_a \left((r^a)^2 \delta_{k\ell} - r^a_k r^a_\ell\right)\right)}_{\Theta_{k\ell}} \omega_\ell$$

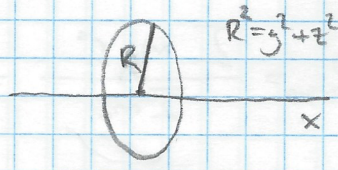


$$\Delta \underline{N} = \Delta m \, \underline{r} \times (\omega \times \underline{r}) = \Delta V \, \rho(\underline{r}) \, \underline{r} \times (\underline{\omega} \times \underline{r})$$

$$\underline{N} = \int dV \, \rho(\underline{r}) \, \underline{r} \times (\underline{\omega} \times \underline{r}) =$$

$$= \underbrace{\left[\int dV \, \rho(\underline{r}) \left(r^2 \underline{\underline{I}} - \underline{r} \circ \underline{r}\right)\right]}_{\underline{\underline{\Theta}}} \underline{\omega}$$

$$\Theta_{k\ell} = \int dV \, \rho(\underline{r}) \cdot \left(r^2 \delta_{k\ell} - x_k x_\ell\right)$$



$$\Theta_{11} = \Theta_{xx} = \int dV \, \rho(r) \left(r^2 - x^2\right) = \int dV \, \rho(r) \left(y^2 + z^2\right)$$

$$\Theta_{22} = \Theta_{yy} = \int dV \, \rho(r) \left(x^2 + z^2\right)$$

$$\Theta_{33} = \Theta_{zz} = \int dV \, \rho(r) \left(x^2 + y^2\right)$$

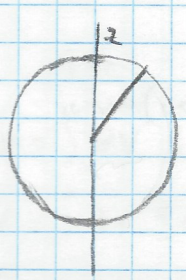
$$\Theta_{12} = \Theta_{21} = \Theta_{xy} = \Theta_{yx} = -\int dV \, \rho(r) \, xy$$

$$\Theta_{13} \qquad \Theta_{xz} = -\int dV \, \rho(r) \, xz$$

$$\Theta_{23} \qquad \Theta_{yz} = -\int dV \, \rho(r) \, yz$$

Sajátértékprobléma → főtengelytranszformáció (diagonálelemek pozitívak)

$$\Theta_{22} + \Theta_{33} > \Theta_{11}$$



$$\underline{r}(r, \vartheta, \varphi) = \begin{pmatrix} r \cdot \sin\vartheta \cos\varphi \\ r \cdot \sin\vartheta \cdot \sin\varphi \\ r \cdot \cos\vartheta \end{pmatrix}$$

$$M = \frac{4}{3} R^3 \pi \rho$$

$$\rho = \frac{3M}{4\pi} \cdot \frac{1}{R^3}$$

$$dV = \left(\frac{\partial \underline{r}}{\partial r}, \frac{\partial \underline{r}}{\partial \vartheta}, \frac{\partial \underline{r}}{\partial \varphi}\right) dr \, d\vartheta \, d\varphi = r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi$$

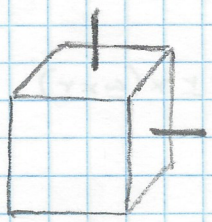
$$\Theta_{33} = \int dV\, \rho\,(x^2+y^2) = \int_{r=0}^{R} dr \int_{\vartheta=0}^{\pi} d\vartheta \int_{\varphi=0}^{2\pi} d\varphi\; r^2 \sin\vartheta \cdot \rho \cdot \underbrace{r^2 \sin^2\vartheta}_{x^2+y^2} =$$

$$= \rho \cdot 2\pi \cdot \int_0^R dr\, r^4 \int_0^{\pi} d\vartheta \sin^3\vartheta = \rho\, 2\pi \cdot \frac{R^5}{5} \cdot \frac{4}{3} = 3 \cdot \frac{M}{4\pi} \frac{1}{R^3} \cdot 2\pi \cdot \frac{R^5}{5} \cdot \frac{4}{3} =$$

$$= \frac{2}{5} M R^2$$

→ Ebben a síkban minden vektor sajátvektor

↳ MINDEN SZAB. N-SÖG ALAPÚ HASÁBRA



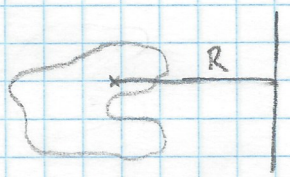
$$\Theta_{zz} = \int dV\, \rho\,(x^2+y^2) = \rho \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz\,(x^2+y^2) =$$

$$= \rho\, a \int_{-a/2}^{a/2} dx \int_{-a/2}^{a/2} dy\,(x^2+y^2) = \rho\, a \int_{-a/2}^{a/2} dx \left[x^2 y + \frac{y^3}{3}\right]_{y=-a/2}^{y=a/2} =$$

$$= \rho\, a \int_{-a/2}^{a/2} dx \left(x^2 a + \frac{a^3}{12}\right) = \rho a \left[\frac{x^3}{3} a + \frac{a^3}{12} x\right]_{x=-\frac{a}{2}}^{\frac{a}{2}} =$$

$$= \rho\, a \left[\frac{a^3}{8} \cdot \frac{a}{3} + \frac{a^3}{12} a\right] = \rho \frac{a^5}{6} = \frac{1}{6} M a^2$$

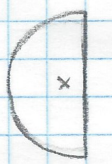
$$\underline{R}_{TKP} = \frac{\sum m_a \underline{r}^a}{\sum m_a} = \frac{\int dV\, \rho(\underline{r})\, \underline{r}}{\int dV\, \rho(\underline{r})} \qquad \rightarrow \text{M.K az integrálási határok?}$$



$$A = 2\pi R\, K$$

$$V = 2\pi R\, T$$

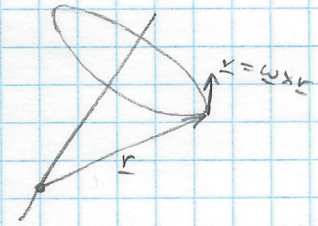
HF.



TKP kiszámítani Guldin

2017. 11. 12.

GY



$$K = \frac{mv^2}{2} = \frac{m}{2}(\underline{\omega} \times \underline{r})^2 = \frac{m}{2}(\omega^2 r^2 - (\underline{\omega}\,\underline{r})^2) =$$

$$= \frac{m}{2}(\omega_\kappa \omega_\kappa r^2 - \omega_\kappa x_\kappa \omega_\kappa x_\kappa) = \frac{m}{2}(\delta_{\kappa\kappa} r^2 - x_\kappa x_\kappa)\omega_\kappa \omega_\kappa =$$

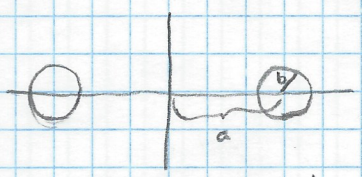
$$K = \frac{1}{2}\left[\sum_a m_a (\delta_{\kappa\kappa} r^{a2} - x_\kappa^a x_\kappa^a)\right] \omega_\kappa \omega_\kappa$$

$$K = \frac{1}{2} \omega_\kappa \Theta \omega_\kappa \qquad \underline{\omega} = \omega \underline{n}$$

$$K = \frac{1}{2} \omega^2 (\underline{n} \Theta \underline{n})$$

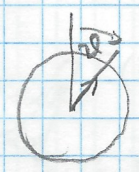
$$K = \frac{1}{2}\,\underline{\omega}\,\underline{\underline{\Theta}}\,\underline{\omega} = \frac{1}{2}\,\underline{\omega}\,\underline{J}$$

$$\underline{J} = \underline{\underline{\Theta}}\,\underline{\omega}$$



$a > b$

$$\Theta_{11} = \int dV\, \rho\, (y^2+z^2)$$



$$\left.\begin{aligned} x &= (a + r\cdot\sin\vartheta)\cdot\cos\varphi \\ y &= (a + r\cdot\sin\vartheta)\cdot\sin\varphi \\ z &= r\cdot\cos\vartheta \end{aligned}\right\} \underline{r}\,(r,\vartheta,\varphi) \qquad \begin{aligned} r &\in [0,b] \\ \vartheta &\in [0,2\pi[\\ \varphi &\in [0,2\pi[\end{aligned}$$

$$dV = \left(\frac{\partial \underline{r}}{\partial r}, \frac{\partial \underline{r}}{\partial \vartheta}, \frac{\partial \underline{r}}{\partial \varphi}\right) dr\, d\vartheta\, d\varphi$$

$$\frac{\partial \underline{r}}{\partial r} = \begin{pmatrix} \sin\vartheta\cdot\cos\varphi \\ \sin\vartheta\cdot\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \vartheta} = \begin{pmatrix} r\cdot\cos\vartheta\cdot\cos\varphi \\ r\cdot\cos\vartheta\cdot\sin\varphi \\ -r\cdot\sin\vartheta \end{pmatrix} = r \begin{pmatrix} \cos\vartheta\cdot\cos\varphi \\ \cos\vartheta\,\sin\varphi \\ -\sin\vartheta \end{pmatrix}$$

$$\frac{\partial \underline{r}}{\partial \varphi} = \begin{pmatrix} -(a+r\sin\vartheta)\cdot\sin\varphi \\ (a+r\cdot\cos\vartheta)\cdot\cos\varphi \\ 0 \end{pmatrix} = (a + r\cdot\sin\vartheta)\cdot\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}$$

$$J = r\cdot(a + r\cdot\sin\vartheta)$$

$$\Theta_{11} = \int dV\, \rho\, (y^2+z^2) = \rho \int_0^b dr \int_0^{2\pi} d\vartheta \int_0^{2\pi} d\varphi\; r(a+r\cdot\sin\vartheta)\left[(a+r\cdot\sin\vartheta)^2\sin^2\varphi + r^2\cos^2\vartheta\right] =$$

$$= \rho\pi \int_0^b dr \int_0^{2\pi} d\vartheta\; r(a+r\cdot\sin\vartheta)\cdot\left[(a+r\sin\vartheta)^2 + 2r^2\cos^2\vartheta\right] =$$

$$= \rho\pi \int_0^b dr \int_0^{2\pi} d\vartheta\; r\,(a+r\cdot\sin\vartheta)\,\underbrace{(a^2 + 2ra\sin\vartheta + r^2\sin^2\vartheta + 2r^2\cos^2\vartheta)}_{a^2+2r^2+2ra\sin\vartheta - r^2\sin^2\vartheta} =$$

$$= \rho\pi \int_0^b dr \int_0^{2\pi} d\vartheta\, \left[a(a^2+2r^2) + r(a^2+2r^2)\sin\vartheta + 2a^2r\sin\vartheta + 2a^2r\sin\vartheta - ar^2\sin^2\vartheta - r^3\sin^3\vartheta\right]$$

$$= \rho\pi \int_0^b dr\; r\left[2a(a^2+2r^2) + 2ar^2 - ar^2\right] = \rho\pi^2 \int dr\; r\,(5ar^2 + 2a^3) =$$

$$= \rho\pi^2 \int_0^b dr\,(5ar^3 + 2a^3r) = \rho\pi^2\left[5a\frac{r^4}{4} + 2a^3\frac{r^2}{2}\right]_0^b = \rho\pi^2\left(\frac{5}{4}ab^4 + a^3b^2\right)$$

$\rho(\underline{r})$

$$q = \int dV\, \rho(\underline{r}) \qquad q = \int dV\, \rho(\underline{r})$$

$$\underline{d} = \int dV\, \rho(\underline{r})\,\underline{r} \qquad d_k = \int dV\, \rho(\underline{r})\, x_k$$

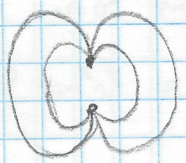
$$\underline{\underline{Q}} = \int dV\, \rho(\underline{r})\, \underline{r}\circ\underline{r} \qquad Q_{kl} = \int dV\, \rho(r)\, x_k x_l$$

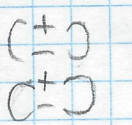
$\Rightarrow$ Összes ilyen (akárhány indexes) tenzort ismerve megkapható a töltéseloszlás

q $\frac{1}{r}$

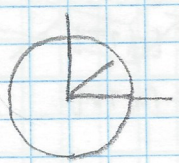
d $\frac{1}{r^2}$

Q $\frac{1}{r^3}$



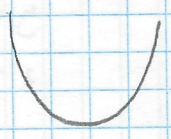


Paramétrages



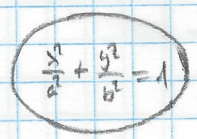
$$\begin{cases} x = R\cos\varphi \\ y = R\cdot\sin\varphi \\ z = 0 \end{cases}$$

$\underline{r}(\varphi) \quad \varphi \in [0, 2\pi[$



$x = x$

$y = ax^2$

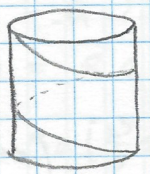


$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$x = a\cdot\cos t$

$y = b\cdot\sin t$

$z = 0$

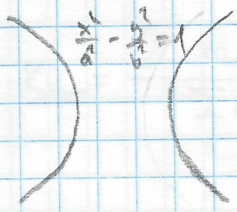


$t \in [0, 2k\pi]$

$x = R\cdot\cos t$

$y = R\cdot\sin t$

$z = kt$



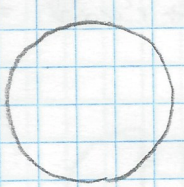
$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$x = a\cdot\mathrm{ch}\,t$

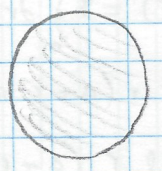
$y = b\cdot\mathrm{sh}\,t$

$z = 0$

$t \in]-\infty, \infty[$



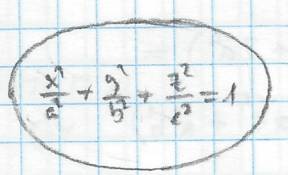
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = R\begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$



$$\underline{r}(r,\vartheta,\varphi) = \begin{pmatrix} r\cdot\sin\vartheta\cos\varphi \\ r\cdot\sin\vartheta\cdot\sin\varphi \\ r\cdot\cos\vartheta \end{pmatrix}$$

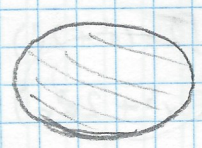
$r \in [0, R] \quad \varphi \in [0, 2\pi[$

$\vartheta \in [0, \pi]$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} a\cdot\sin\vartheta\cos\varphi \\ b\cdot\sin\vartheta\cdot\sin\varphi \\ c\cdot\cos\vartheta \end{pmatrix}$$



$$\underline{r}(t,\vartheta,\varphi) = \begin{pmatrix} t\cdot a\cdot\sin\vartheta\cdot\cos\varphi \\ t\cdot b\cdot\sin\vartheta\cdot\sin\varphi \\ t\cdot c\cdot\cos\vartheta \end{pmatrix}$$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$

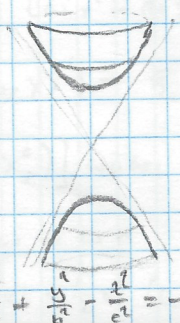
$x = a\cdot\mathrm{ch}\,t\cdot\cos\varphi$

$y = b\cdot\mathrm{ch}\,t\cdot\sin\varphi$

$z = c\cdot\mathrm{sh}\,t$

$t \in]-\infty, \infty[$

$\varphi \in [0, 2\pi[$



$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1$

$x = a\cdot\mathrm{sh}\,t\cdot\cos\varphi$

$y = b\cdot\mathrm{sh}\,t\cdot\sin\varphi$

$z = \pm c\cdot\mathrm{ch}\,t$

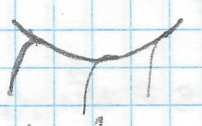
$t \in [0, \infty[$

$x = t \cdot a \cdot \cos\varphi$

$y = t \cdot b \cdot \sin\varphi$

$z = t^2$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = z$



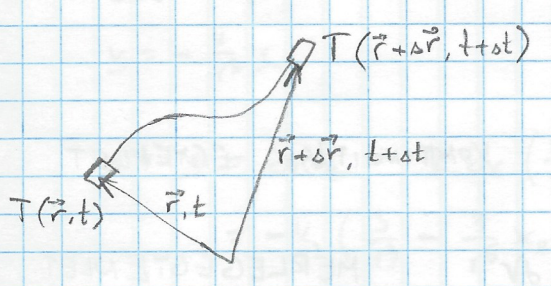
$x = t \cdot a \cdot \mathrm{ch}\,\varphi$

$y = t \cdot b \cdot \mathrm{sh}$

$z = t^2$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = z$

2017.12.14.
GY



$$\phi(\vec{r}+\Delta\vec{r}) \approx \phi(\vec{r}) + \vec{\nabla}\phi \cdot \Delta\vec{r}$$

$$T(\vec{r}+\Delta\vec{r}, t+\Delta t) \approx T(\vec{r},t) + \vec{\nabla}T \cdot \Delta\vec{r} + \frac{\partial T}{\partial t}\cdot\Delta t$$

$$T(x+\Delta x, y+\Delta y, z+\Delta z, t+\Delta t) \approx T(x,y,z,t) + \underbrace{\frac{\partial T}{\partial x}\cdot\Delta x + \frac{\partial T}{\partial y}\cdot\Delta y + \frac{\partial T}{\partial z}\cdot\Delta z}_{\mathrm{grad}\,T\,\cdot\,\Delta\vec{r}} + \frac{\partial T}{\partial t}\cdot\Delta t$$

$$\frac{\Delta T}{\Delta t} = \frac{T(\vec{r}+\Delta\vec{r}, t+\Delta t) - T(\vec{r},t)}{\Delta t} = \frac{\vec{\nabla}T\cdot\Delta\vec{r} + \frac{\partial T}{\partial t}\Delta t}{\Delta t} = \vec{\nabla}T\,\frac{\Delta\vec{r}}{\Delta t} + \frac{\partial T}{\partial t} \to \vec{v}\,\vec{\nabla}T + \frac{\partial T}{\partial t}$$

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{v}\,\vec{\nabla}T$$

$$\boxed{\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v}\,\vec{\nabla}}$$ SZUBSZTANCIÁLIS DERIVÁLT

Sebességmező: $\frac{d\vec{v}}{dt} = \frac{\partial\vec{v}}{\partial t} + (\vec{v}\,\vec{\nabla})\,\vec{v}$

$m\vec{a} = \vec{F}$

$\rho(\vec{r},t)\,\Delta V\,\frac{d\vec{v}}{dt} = -\Delta V\cdot\vec{\nabla}p + \rho\cdot\Delta V\cdot\vec{g}(\vec{r},t)$

$$\boxed{\rho\left(\frac{\partial\vec{v}}{\partial t} + (\vec{v}\vec{\nabla})\vec{v}\right) = -\vec{\nabla}p + \rho\cdot\vec{g}}$$

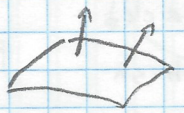
HIDRODINAMIKA EULER-EGYENLETE

$\rho(\vec{r},t)$

$\vec{v}(\vec{r},t)$

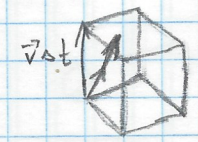
$$M(t) = \int \rho(\vec{r},t)\,dV$$

$$\frac{dM}{dt} = \frac{d}{dt}\int\rho(\vec{r},t)\,dV = \int\frac{\partial\rho}{\partial t}\,dV$$



$\int \vec{v}\, d\vec{F}$

$\frac{\Delta M}{\Delta t} = \rho \vec{v}\, \Delta \vec{F}$



$-\oint \rho \vec{v}\, d\vec{F}$

$\frac{dM}{dt} = \int \frac{\partial \rho}{\partial t} dV = -\oint \rho \vec{v}\, d\vec{F}$ Anyagmegmaradás → lokálisan is teljesül

$\rho \vec{j} \rightarrow Q = \int \rho\, dV$

$\frac{dQ}{dt} = \int \frac{\partial \rho}{\partial t} dV = -\oint \vec{j}\, d\vec{F}$ KONTINUITÁSI EGYENLET

$\frac{dQ}{dt} = \int \frac{\partial \rho}{\partial t} dV = -\oint \vec{j}\, d\vec{F} + \int q\, dV$ MÉRLEGEGYENLET

$q$uelle

Fényes Imre → Modern fizikai kisenciklopédia

→ legtöbb fizikai egyenlet: mérlegegyenlet alakban
(pl. Newton-törvények: impulzus mérlegegyenletei)

$\oint \underline{v}\, d\underline{F}$

$\int \left(\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} - q \right) dV = 0$

$$\boxed{\frac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = q}$$ MÉRLEGEGYENLET DIFFERENCIÁLIS ALAKJA

$F = -\frac{\gamma M m}{r^2}$

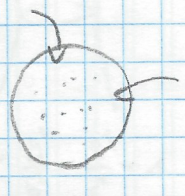
$\vec{F} = -\frac{\gamma M m}{r^3} \cdot \vec{r}$ $\quad m\vec{a} = m\vec{g}$ $\quad \vec{a} = \vec{g}$

M

$\vec{g}(\vec{r}, t)$

$\operatorname{rot} \vec{g} = 0$

$\vec{g} = -\vec{\nabla} \phi$



$-\oint \vec{g}\, d\vec{F} = 4\pi\gamma \int \rho(\vec{r}, t)\, dV$

$-\int \operatorname{div} \vec{g}\, dV$

$$-\operatorname{div}\vec{g} = 4\pi\gamma\varrho$$

$$\boxed{\Delta\phi = 4\pi\gamma\varrho}$$ POISSON-EGYENLET

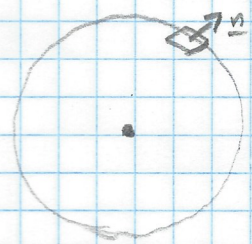
$\varrho \rightarrow [\ \Delta\ ] \rightarrow \phi$

$$\vec{g} = -\frac{\gamma M}{r^3}\vec{r}$$

$$\underline{v} = -\frac{k}{r^3}\underline{r}$$

$$\operatorname{div}\underline{v} = \partial_k v_k = -k\,\partial_k\left(\frac{x_k}{r^3}\right) = -k\left[(\partial_k x_k)\frac{1}{r^3} + x_k\left(-\frac{3}{r^4}\right)\partial_k r\right] =$$

$$= -k\left(\frac{3}{r^3} - \frac{3}{r^5}x_k x_k\right) = 0$$



$$\oint \underline{v}\,d\underline{F} = \int\left(-\frac{k}{r^2}\underline{e}\right)\underline{e}\,dA = -\int\frac{k}{r^2}dA = \int -\frac{k}{r^2}r^2\sin\vartheta\,d\vartheta\,d\varphi =$$

$$= -k\int_{\dots} = -4\pi k$$

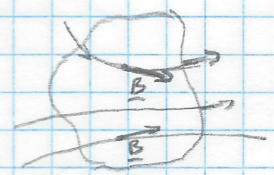
$\underline{v} = -\frac{k}{r^2}\underline{e}$

$dA = r^2\sin\vartheta\,d\vartheta\,d\varphi$

Divergencia 0, felületi integrál nem → szingularitás (Dirac-delta)

2017. 12. 14.

EA

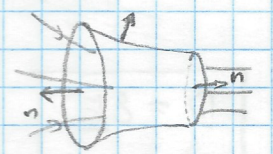


$$\boxed{\oint \underline{B}\,d\underline{F} = 0 \qquad \operatorname{div}\underline{B} = 0}$$

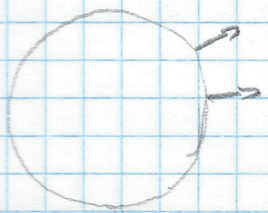
$$\frac{\partial\varrho}{\partial t} + \nabla(\varrho\underline{v}) = 0$$

Spec. $\varrho = \text{Konst.}$

$$\nabla\underline{v} = 0$$



<u>ERŐVONALAK</u>: akkor lehet, ha a divergencia 0 (mágneses mező, periodikus t. tere)



$\underline{E}(r)$

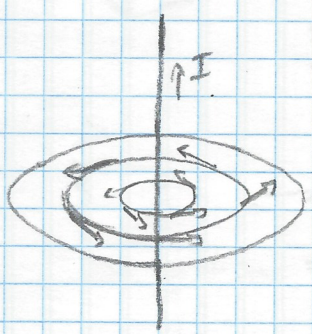
$$\oint \underline{E}\,d\underline{F} = \underbrace{\int \varrho\,dV}_{Q}$$

* megfelelő egységrendszerben

$$\operatorname{div}\underline{E} = \varrho$$

$$\oint \underline{D}\, d\underline{F} = \int \rho\, dV \qquad \underline{D} = \varepsilon \underline{E}$$

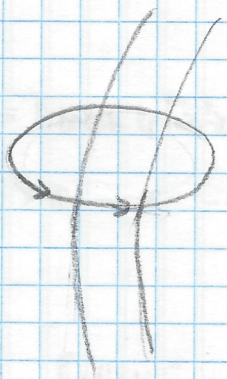
$$\operatorname{div} D = \rho$$



$$\oint \underline{H}\, d\underline{r} = \frac{1}{c} \int \underline{j}\, d\underline{F}$$

$$\int \operatorname{rot} \underline{H}\, d\underline{F}$$

$$\operatorname{rot} \underline{H} = \frac{1}{c} \underline{j} \qquad \underline{H} = \frac{1}{\mu} \underline{B}$$

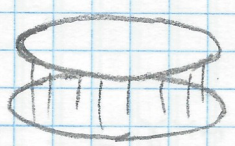


$$\oint \underline{E}\, d\underline{s} = -\frac{1}{c} \frac{d}{dt} \int \underline{B}\, d\underline{F}$$

$$\int \operatorname{rot} \underline{E}\, d\underline{F} = -\frac{1}{c} \int \frac{\partial B}{\partial t}\, dF$$

$$\boxed{\operatorname{rot} \underline{E} = -\frac{1}{c} \frac{\partial \underline{B}}{\partial t}}$$

Gibbs
Heaviside
} vektorok



$$\boxed{\operatorname{rot} \underline{H} = \frac{1}{c} \left(\underline{j} + \frac{\partial \underline{D}}{\partial t} \right)}$$

$$\boxed{\operatorname{div} \underline{B} = 0 \qquad \operatorname{rot} \underline{E} + \frac{1}{c} \frac{\partial \underline{B}}{\partial t} = 0}$$

$$\boxed{\operatorname{div} \underline{D} = \rho \qquad \operatorname{rot} \underline{H} - \frac{1}{c} \frac{\partial \underline{D}}{\partial t} = \frac{1}{c} \underline{j}}$$

$$\boxed{\underline{D} = \varepsilon \underline{E} \qquad \underline{H} = \frac{1}{\mu} \underline{B}}$$

$\varepsilon(\nu) \qquad n = \sqrt{\varepsilon}$

$D(t) = \int d\tau\, E(t)\, G(t-\tau)$

ZH. 3.67.-ben