Vektorszámítás tételek / vektoranalízis

1. Differenciálszámítás

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\cancel{f(x)} + f(x+\Delta x) - f(x) = f'(x) \cdot \Delta x + \varepsilon(x, \Delta x)$$

$$\lim_{\Delta x \to 0} \frac{\varepsilon(x, \Delta x)}{\Delta x} = 0$$

Tulajdonságok:

1. $f(x) = f_1(x) + f_2(x)$ $\quad f'(x) = f_1'(x) + f_2'(x)$

2. $f(x) = \lambda g(x)$ $\quad f'(x) = \lambda g'(x)$

$\frac{d}{dx} f(x)$ deriválás lineáris operátor

3. $F(x) = f(x) \cdot g(x)$

$F'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

4. $F(x) = \frac{f(x)}{g(x)}$ $\quad F'(x) = \frac{f'(x) g(x) - f(x) g'(x)}{g^2(x)}$

5. $F(x) = f(g(x))$ $\quad F'(x) = f'(g(x)) \cdot g'(x)$

$(e^x)' = e^x$

$(\ln x)' = \frac{1}{e^{\ln x}} = \frac{1}{x}$

$a^x = e^{x \ln a} = a^x \cdot \ln a$

$\operatorname{arsh} x = \operatorname{sh}^{-1}(x) = \ln(x + \sqrt{1+x^2})$

$y = \operatorname{sh} y = \frac{e^y - e^{-y}}{2}$

$x = \frac{e^y - e^{-y}}{2}$

$2x = e^y - \frac{1}{e^y}$

$2xe^y = e^{2y} - 1 \quad / \ln($

$\to$ ~~$\ln 2x + \ln x = \ln 2x - \ln 1$~~

~~$\ln(2x)\ln = y$~~

~~$\ln(2x) = y$~~?

~~$\ln(2x) + y = \ln$~~$(e^{2y} - 1)$

nem így

$y = \arc\sin x$

$$(\arc\sin x)' = \frac{1}{\cos(\arc\sin x)}$$

$\cos x = \sqrt{1 - \sin x \cdot \sin x}$

$\cos(\arc\sin x) = \sqrt{1 - \sin \arc\sin x \cdot \sin \arc\sin x}$

$$= \frac{1}{\sqrt{1-x^2}}$$

magasabb rendű differenciálhányadosok:

$y = f(x)$ $\quad y^{(n)} = f^{(n)}(x) = \underset{1}{\frac{d}{dx}}\,\underset{2}{\frac{d}{dx}} \cdots\cdots \underset{n}{\frac{d}{dx}} f(x) = \hat{D}^n f(x)$

$y' = f'(x)$

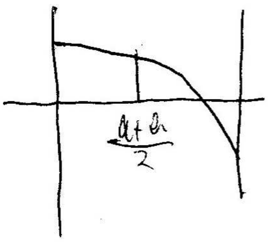
$$\hat{D}\big(u(x)\cdot v(x)\big) = \frac{d}{dx}\big(u'(x)\cdot v(x) + u(x)\,v'(x)\big)$$

$$\hat{D}^n\big(u(x)\,v(x)\big) = \sum_{i=0}^{n} \binom{n}{i} u^{(i)} v^{(n-i)}$$

Középérték tételek:

① Bolzano-tétel: f legyen folytonos az $[a,b]$ ~~intervallum~~-on

$f(a)\cdot f(b) < 0 \Rightarrow \exists\, \xi \quad f(\xi) = 0, \quad \xi \in [a,b]$



számítógépes felezés

$[a,b]$-n értelmezett, negatív és pozitív értéket is felvevő függvénynek van zérushelye.

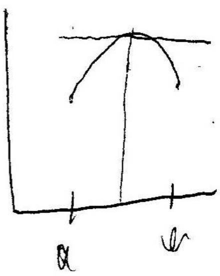
② Rolle-tétel:

$f(a) = f(b)$

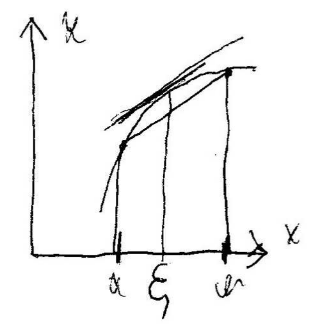
f folytonos

$\Rightarrow \exists\, \xi$

$f'(\xi) = 0$



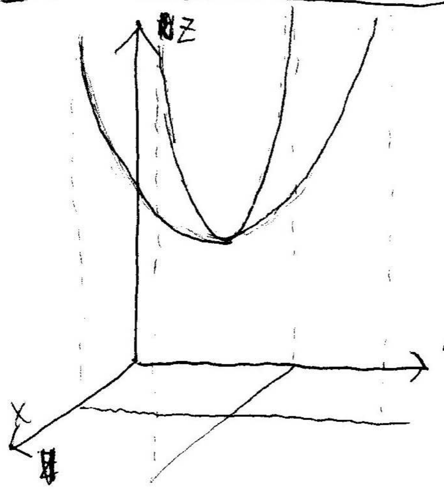
③ Lagrange-tétel:



f folytonos

$$\exists\, \xi \quad f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

Többváltozós függvények deriválása:



$$z = f(x,y)$$

$$\left.\frac{dz}{dx}\right|_{y=\text{áll}} = \left.\frac{df(x,y)}{dx}\right|_{y=\text{áll}} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \equiv \frac{\partial f(x,y)}{\partial x}$$

$$\left.\frac{dz}{dy}\right|_{x=\text{áll}} = \left.\frac{df(x,y)}{dy}\right|_{x=\text{áll}} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y} \equiv \frac{\partial f(x,y)}{\partial x}$$

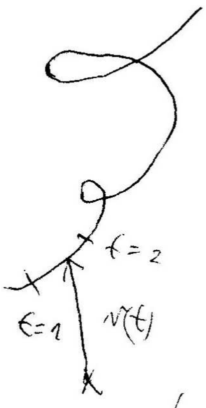
$\frac{\partial}{\partial x}$: x szerinti ~~parciális~~ parciális deriválás operátora

$$\frac{\partial}{\partial_x} = \partial_x = f_x'(x,y)$$

Young-tétel:

$f''_{yx} = f''_{xy}$ feltétel: f_x', f_y', f''_{xy}, f''_{yx} folytonosak

2. Vektorok, skalárok deriválása



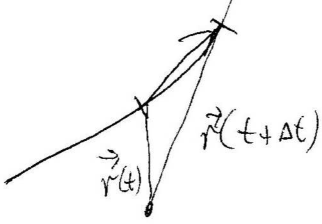
$\vec r(t)$: görbe

$\vec r(t)$

$$\dot{\vec r}(t) = \lim_{\Delta t\to 0} \frac{\vec r(t+\Delta t)-\vec r(t)}{\Delta t}$$

a görbe érintőjének irányába mutat

A definíció nem függ a paraméterezéstől



Pl: egyenletes körmozgás

$$\underline{r}(t) = R\cdot\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$

$$\dot{\underline{r}}(t) = R\begin{pmatrix}-\sin\omega t\cdot\omega\\ \cos\omega t\cdot\omega\end{pmatrix}$$

$$\ddot{\underline{r}}(t) = -R\omega^2\begin{pmatrix}\cos\omega t\\ \sin\omega t\end{pmatrix}$$

Vektor határértéke:

$$\lim_{t\to t_0}\vec r(t) = \vec r_0 \qquad \left(\vec r(t)-\vec r_0\right)\cdot\vec{\underline{x}} = \lambda_x(t)$$

$$\forall \underline{x}\text{-re}: \lim_{t\to t_0}\lambda_x(t)=0 \rightarrow \lim_{t\to t_0}\vec r(t) = \vec r_0$$

$$\lim_{r\to r_0}\hat A(r) = \hat A_0 \qquad \lim_{r\to r_0}\left[\left(\hat A(r)-\hat A_0\right)\underline{x}\right]\cdot\underline{y} = 0 \quad \forall\, \underline{x},\underline{y}\text{-ra}$$

$$\Downarrow$$

$$\lim_{r\to r_0}\hat A(r) = A_0$$

Tenzor deriváltja: (egy paraméteres)

$$\lim_{\Delta r\to 0}\frac{\hat A(r+\Delta r)-\hat A(r)}{\Delta r} = \hat A'(r)$$

$$\left[\underline{A}'(t)\right]_{ik} = A_{ik}'(t)$$ minden egyes elemet egyenként kell deriválni

$$\underline{\underline{O}}_z(\varphi) = \begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}$$

$$\underline{\underline{O}}_z'(\varphi) = \begin{pmatrix}-\sin\varphi & \cos\varphi\\ -\cos\varphi & -\sin\varphi\end{pmatrix}$$

$$\vec{c}(t) = \alpha\,\vec{a}(t) + \beta\,\vec{b}(t)$$
$$\dot{\vec{c}}(t) = \alpha\,\dot{\vec{a}}(t) + \beta\,\dot{\vec{b}}(t)$$
$$\dot{c}_k(t) = \alpha\,\dot{a}_k(t) + \beta\,\dot{b}_k(t)$$

Skalárszorzat:

$$c(t) = \vec{a}(t)\cdot\vec{b}(t) \qquad \frac{d}{dt}\,c(t) = \left(\sum_{k=1}^{n} a_k(t)\cdot b_k(t)\right)' = \sum_{k=1}^{n} \dot{a}_k(t)\cdot b_k(t) +$$
$$+ a_k(t)\cdot\dot{b}_k(t) = \dot{\vec{a}}(t)\,\vec{b}(t) + \vec{a}(t)\,\dot{\vec{b}}(t)$$

Keresztszorzat:

$$\vec{c}(t) = \vec{a}(t)\times\vec{b}(t)$$
$$\frac{d}{dt}\,\vec{c}(t) = \left(\sum_{i,j,k} \vec{e}^{(i)}\,\varepsilon_{ijk}\,a_j(t)\,b_k(t)\right)' = \sum_{i,j,k} \vec{e}^{(i)}\,\varepsilon_{ijk}\left(\dot{a}_j(t)\,b_k(t) + a_j(t)\,\dot{b}_k(t)\right) =$$
$$= \dot{\vec{a}}(t)\times\vec{b}(t) + \vec{a}(t)\times\dot{\vec{b}}(t)$$

Tenzor · vektor:

$$\vec{a}(t) = \hat{B}(t)\,\vec{b}(t)$$
$$a_k(t) = \sum_l B_{kl}(t)\cdot b_l(t)$$
$$\dot{a}_k(t) = \sum_l\left(\dot{B}_{kl}(t)\,b_l(t) + B_{kl}(t)\,\dot{b}_l(t)\right) \Rightarrow \dot{\vec{a}}(t) = \dot{\hat{B}}(t)\,\vec{b}(t) + \hat{B}(t)\,\dot{\vec{b}}(t)$$

Mátrixok deriválása:

$$\underline{\underline{C}}(t) = \underline{\underline{A}}(t)\cdot\underline{\underline{B}}(t)$$
$$C(t)_{ij} = \sum_k A_{ik}(t)\cdot B_{kj}(t)$$
$$\dot{C}(t)_{ij} = \sum_k\left(\dot{A}_{ik}(t)\,B_{kj}(t) + A_{ik}(t)\,\dot{B}_{kj}(t)\right)$$

$$\boxed{\dot{\underline{\underline{C}}} = \dot{\underline{\underline{A}}}\cdot\underline{\underline{B}} + \underline{\underline{A}}\cdot\dot{\underline{\underline{B}}}}$$ nem kommutatív!

Spec: $\underline{\underline{W}} = \underline{\underline{A}}\cdot\underline{\underline{B}}\cdot\underline{\underline{C}}\cdot\underline{\underline{D}}$

$$\underline{\underline{W}}' = \underline{\underline{A}}'\cdot\underline{\underline{B}}\cdot\underline{\underline{C}}\cdot\underline{\underline{D}} + \underline{\underline{A}}\cdot\underline{\underline{B}}'\cdot\underline{\underline{C}}\cdot\underline{\underline{D}} + \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}'\,\underline{\underline{D}} + \underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}\,\underline{\underline{D}}'$$

Spec: $\underline{\underline{W}}(t) = \underline{\underline{A}}^n(t)$

$$\dot{\underline{\underline{W}}}(t) = \dot{\underline{\underline{A}}}\,\underline{\underline{A}}\cdots\underline{\underline{A}} + \underline{\underline{A}}\,\dot{\underline{\underline{A}}}\cdots\underline{\underline{A}} + \cdots\cdots + \underline{\underline{A}}\,\underline{\underline{A}}\cdots\dot{\underline{\underline{A}}}$$

$$W = Tr(\hat{A}^n(t)) \Rightarrow \dot{W} = n \cdot Tr(\hat{\dot{A}}(t) \cdot \hat{A}^{n-1})$$

~~Skop~~ $Tr(\underline{\underline{A}}\,\underline{\underline{B}}\cdots\underline{\underline{J}}\,\underline{\underline{K}}) = Tr(\underline{\underline{K}}\,\underline{\underline{A}}\,\underline{\underline{B}}\cdots\underline{\underline{J}})$

$$Tr(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}) = Tr(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{B}})$$

$$Tr(\underline{\underline{A}}\,\underline{\underline{B}}\,\underline{\underline{C}}) = \sum_{k,l,m} A_{km} B_{ml} C_{lk} = \sum_{k,l,m} C_{lk} A_{km} B_{ml} = Tr(\underline{\underline{C}}\,\underline{\underline{A}}\,\underline{\underline{B}})$$

Tr vatt a mte - az ciklikusan generálható.

$$\dot{W} = \frac{d}{dt} Tr(\hat{A}^n(t)) = Tr((\underline{\underline{A}}\cdots\underline{\underline{A}})') = Tr(\dot{\underline{\underline{A}}}\,\underline{\underline{A}}\cdots\underline{\underline{A}} + \underline{\underline{A}}\,\dot{\underline{\underline{A}}}\cdots\underline{\underline{A}} + \cdots + \underline{\underline{A}}\,\underline{\underline{A}}\cdots\dot{\underline{\underline{A}}})$$

<u>Inverz ~~d~~ mátrix deriváltja:</u>

$$A^{-1}(t) = B(t)$$

$$\underline{\underline{A}}(t)\,B(t) = \underline{\underline{E}}$$

$\underline{\underline{B}}\cdot|$ $\quad \dot{A}(t)\cdot B(t) + \underline{\underline{A}}(t)\cdot\dot{\underline{\underline{B}}}(t) = 0$

$$\underline{\underline{B}}\,\dot{\underline{\underline{A}}}\,\underline{\underline{B}} + \underbrace{\underline{\underline{B}}\,\underline{\underline{A}}}_{\underline{\underline{E}}}\,\dot{\underline{\underline{B}}} = 0 \Rightarrow \dot{\underline{\underline{B}}} = -\underline{\underline{B}}\,\dot{\underline{\underline{A}}}\,\underline{\underline{B}}$$

$$\frac{d}{dt} A^{-1}(t) = -\underline{\underline{A}}^{-1}(t)\cdot\dot{\underline{\underline{A}}}(t)\cdot\underline{\underline{A}}^{-1}(t)$$

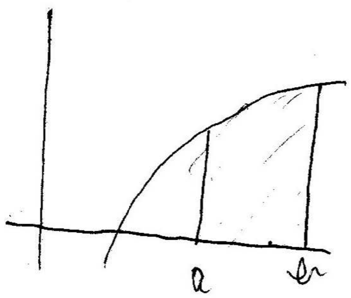
<u>Determináns deriváltja:</u>

$$\frac{d}{dt}(\det A(t)) = \frac{d}{dt}\left(\sum_{i_1}^{n}\sum_{i_2}^{n}\cdots\sum_{i_n}^{n} \varepsilon_{i_1 i_2\ldots i_n} A_{1i_1} A_{2i_2}\cdots A_{ni_n}\right) =$$

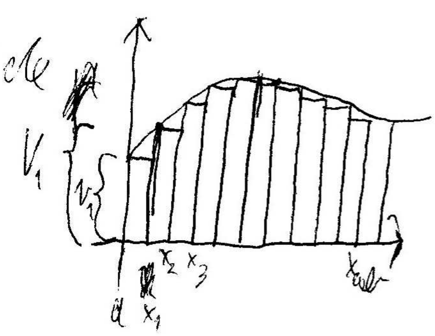
$$= \sum_{i_1}\sum_{i_2}\cdots\sum_{i_n} \varepsilon_{i_1 i_2\ldots i_n} (\dot{A}_{1i_1}(t) A_{2i_2}(t)\cdots A_{ni_n}(t)) =$$

$$= \det\begin{vmatrix} \dot{A}_{11} & \dot{A}_{12} & \cdots & \dot{A}_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \end{vmatrix} + \det\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \dot{A}_{21} & \dot{A}_{22} & \cdots & \dot{A}_{2n} \\ \vdots & & & \end{vmatrix} + \cdots + \det\begin{vmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ \dot{A}_{n1} & \dot{A}_{n2} & \cdots & \dot{A}_{nn} \end{vmatrix}$$

3. Integrálszámítás egy változóban

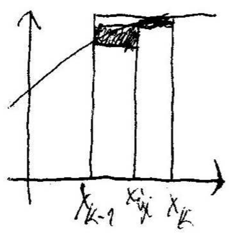


$$\int_a^b f(x)\,dx$$



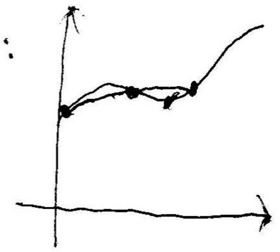
Riemann-integrál: $\sum_{i=1}^{N} v_i \Delta x_i \le \int_a^b f(x)\,dx \le \sum_{k=1}^{N} V_k \Delta x_k$

$\uparrow s$ $\qquad\qquad$ $\uparrow S$



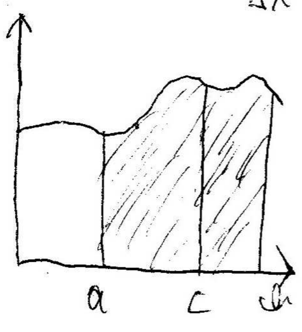
Numerikus integrálás: trapézokkal közelítünk

Simpson-összeg:



3 egymás utáni pontra parabolát illesztek

$$\sum_{i=1}^{N} f(x_i) \underset{\substack{\uparrow \\ \Delta x}}{W_i} \simeq \int_a^b f(x)\,dx$$



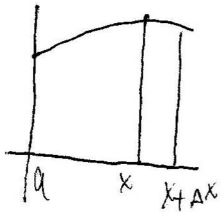
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

a = b esetén

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\frac{d}{dx}\int_a^x f(x')\,dx = \lim_{\Delta x\to 0}\frac{\int_a^{x+\Delta x} f(x)\,dx - \int_a^x f(x)\,dx}{\Delta x} = f(x)$$



$$\approx \frac{f(x)\cdot\Delta x}{\Delta x} = f(x)$$

$$F(a,x) = \int_a^x f(x)\,dx \qquad \frac{d}{dx}F(x,a) = \int_x^a f(x')\,dx' = -f(x')$$

$$\frac{d}{dx}F(a,x) = f(x)$$

Határozatlan integrál:

$\int f(x)\,dx = F(x)$, ha $F'(x) = f(x)$ de $F(x)+c$ is az

Newton-Leibniz formula:

$$\int_a^b f(x)\,dx = \left[F(x)\right]_a^b \equiv F(b) - F(a)$$

Spec: $\int_a^b f'(x)\,dx = \left[f(x)\right]_a^b = f(b) - f(a)$

~~Elnevezett integrálok~~:

$$li\,x = \int\frac{1}{\ln x}$$

$$Si\,x = \int_0^x \frac{\sin x}{x}\,dx$$

Lineáris művelet: $\int_a^b \alpha f(x) + \beta g(x) = \alpha\int f(x)\,dx + \beta\int g(x)\,dx$

Parciális határozott:

$$\int_a^b\left(\frac{du}{dx}\right)v(x)\,dx = \left[u(x)\cdot v(x)\right]_a^b - \int_a^b u(x)\left(\frac{dv(x)}{dx}\right)dx$$

pl.: $\int_a^b 1\cdot\ln x\,dx = \left[x\cdot\ln x\right]_a^b - \int_a^b x\cdot\frac{1}{x}\,dx = \left[x\cdot\ln x\right]_a^b - \left[x\right]_a^b = \left[x\cdot\ln x - x\right]_a^b$

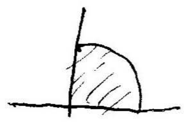
Helyettesítéses integrál:

$$\int_a^b f(x)\,dx = \int_{\varphi^{-1}(a)}^{\varphi^{-1}(b)} f(\varphi(t))\,dt$$

$x = \varphi(t)$

$t = \varphi^{-1}(x)$

$$dx = \frac{dx}{dt} = \varphi'(t)\,dt$$



Pl:

$$\int_0^R \sqrt{R^2-x^2}\,dx = \int_0^{\pi/2} \sqrt{R^2-R^2\sin^2 t}\,\cos t\,dt =$$

$x = R\sin t \qquad t = \arcsin\left(\frac{x}{R}\right)$

$dx = R\cos t$

$$= \int_0^{\pi/2} R^2\cdot\sqrt{1-\sin^2 t}\cdot\cos t\,dt =$$

$$= R^2\int_0^{\pi/2}\cos^2 t\,dt = R^2\int_0^{\pi/2}\frac{1+\cos 2t}{2}\,dt =$$

$$= \frac{R^2}{2}\left[t+\frac{\sin 2t}{2}\right]_0^{\pi/2} = \frac{R^2}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-0\right] = \frac{R^2\pi}{4}$$

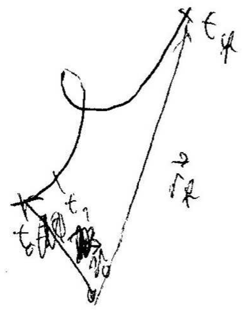
$$\int\frac{f'(x)}{f(x)}\,dx = \ln|f(x)|+c$$

$$\int a^x\,dx = \frac{a^x}{\ln a}$$

$\int a^x dx = \frac{a^x}{\ln a} \qquad (a^x)' = a^x\cdot\ln a$

Integrálás görbe mentén:

$$L = \sum_{i=1}^{N}\Delta r_i = \int_{\text{útvonal}} |d\vec r| = \int_{t_0}^{t_k}\frac{|d\vec r(t)|}{|dt|}\,dt \qquad t\in[t_0, t_k]$$



$\vec r_i = \vec r(t_i)$

$$\vec r(t) = \begin{pmatrix} x(t)\\ y(t)\\ z(t)\end{pmatrix} \qquad \left|\frac{d\vec r}{dt}\right| = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$

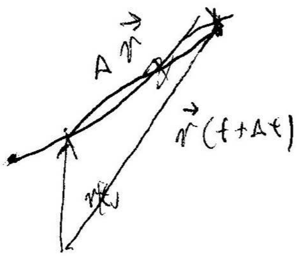
$$\frac{d\vec r}{dt} = \begin{pmatrix} x'(t)\\ y'(t)\\ z'(t)\end{pmatrix}$$

Példa: $\underline{r}(t) = \begin{pmatrix} R\cos t \\ R\sin t \\ \alpha t \end{pmatrix}$ $\qquad \left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{R^2+\alpha^2}$

$$\frac{d\vec{r}}{dt} = \begin{pmatrix} -R\sin t \\ R\cos t \\ \alpha \end{pmatrix} \qquad \int_0^{4\pi} \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^{4\pi} \sqrt{R^2+\alpha^2} = \sqrt{R^2+\alpha^2}\cdot 4\pi$$

$t := [0, 4\pi]$

Vonalintegrálok:



$$\Delta\vec{r} = \vec{r}(t+\Delta t) - \vec{r}(t)$$

$$\vec{u}\cdot d\vec{r} \approx \sum_i \vec{u}(\vec{r}_i)\cdot\Delta\vec{r}_i$$

1. típus: $\displaystyle\int_{\text{útvonal}} \vec{u}\, d\vec{r} = \int_{t_0}^{t_1} \left(\vec{u}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\right) dt$

paraméterezés

Példa: $\underline{u}(\underline{r}) = \begin{pmatrix} 5z \\ x \\ 2x-y \end{pmatrix}$ $\qquad \underline{r}(t) = \begin{matrix} 2\cos t \\ 2\sin t \\ 5t \end{matrix}$ $\qquad t\in[0,5\pi]$

$$\int_{\text{útvonal}} \vec{u}\, d\vec{r} = \int_0^{5\pi} \begin{pmatrix} 25t \\ 2\sin t \\ 4\cos t - 2\sin t \end{pmatrix} \cdot \begin{pmatrix} -2\sin t \\ 2\cos t \\ 5 \end{pmatrix} dt =$$

$$= \int_0^{5\pi} -50t\sin(t) + 4\cos t\sin t \cancel{20} + 20\cos t - 10\sin t \, dt =$$

$$= \left[50\cos t - 50\sin t - 2\cos t + 20\sin t + 10\cos t\right]_0^{5\pi} =$$

$$= -250\pi - 10 - (-10) = -250\pi - 20$$

A lényeg, hogy vonalintegrálnál mindig beszorzunk dt-vel

(1) $$\int_{t_0}^{t^*} \vec{u}(\vec{r}(t))\left(\frac{d\vec{r}}{dt}\right) dt$$

(2) $$\int_{t_0}^{t^*} u(\vec{r}(t))\left(\frac{d\vec{r}}{dt}\right) dt$$

(3) $$\int_{t_0}^{t^*} \vec{u}(\vec{r}(t) \cdot \left|\frac{d\vec{r}}{dt}\right| dt$$

(4.) $$\int_{t_0}^{t^*} u(\vec{r}(t)) \left|\frac{d\vec{r}}{dt}\right| dt$$

(5.) $$\int_{t_0}^{t^*} \left[\vec{u}(\vec{r}(t)) \times \left(\frac{d\vec{r}}{dt}\right)\right] dt$$

4. Skalár- és vektormezők deriválása

Gradiens:

$$f(x+\Delta x) - f(x) = f'(x)\cdot\Delta x + \mathcal{E}(x,\Delta x) \leftarrow \lim_{\Delta x\to 0}\frac{\mathcal{E}(x,\Delta x)}{\Delta x} = 0$$

$$\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \lim_{\Delta x\to 0}\frac{\mathcal{E}(x,\Delta x)}{\Delta x}$$

$\cancel{f(\vec{x}+\Delta\vec{x})=}$

gradiens: grad $f(x)$ → most jelöljük $\vec{A}$-val

$f(\vec{x})$ $\quad f(\vec{x}+\Delta\vec{x}) - f(\vec{x}) = \vec{A}\cdot\Delta\vec{x} + \mathcal{E}(\vec{x},\Delta\vec{x})$

skalár

$$\lim_{\Delta x\to 0}\frac{\mathcal{E}(\vec{x},\Delta\vec{x})}{|\Delta\vec{x}|} = 0$$

$\vec{A}(x) = \text{grad}\, f(x)$ (vektor)

$$\Delta \underline{x}_i = \begin{pmatrix}0\\0\\\Delta x\\0\\0\end{pmatrix} \leftarrow i$$

Δx speciális megválasztásával $f(x_1,x_2\ldots,x_i+\Delta x,\ldots) - f(x_1,\ldots,x_i,\ldots x_n)$
$= A_i\,\Delta x + \mathcal{E}(\vec{x},\Delta\vec{x})$

$$\lim_{\Delta x\to 0}\frac{f(x_1,\ldots x_i+\Delta x\ldots) - f(x_1\ldots\ldots x_n)}{|\Delta\vec{x}|} = A_i(\vec{x}) + \lim_{\Delta x\to 0}\frac{\mathcal{E}(\vec{x},\Delta\vec{x})}{|\Delta x|}$$

a konstant

$$\frac{\partial f}{\partial x_i} = A_i(\vec{x}) = (\text{grad} f)_i$$

$$\text{grad} f = \begin{pmatrix}\frac{\partial f}{\partial x_1}\\ \frac{\partial f}{\partial x_2}\\ \vdots\\ \frac{\partial f}{\partial x_n}\end{pmatrix} = \begin{pmatrix}\partial_1\\ \partial_2\\ \partial_3\\ \vdots\\ \partial_n\end{pmatrix} f = \vec{\nabla}(f)$$

Pl.: $\left(\text{grad}\,(\vec{x}\cdot\vec{x})^{100}\right)_i = \partial_{x_i}\,(x_1^2 + x_2^2 + \dots + x_n^2)^{100} = 100\cdot(x_1^2+\dots+x_n^2)^{99} \cdot 2x_i$

$$\Rightarrow \text{grad}(\vec{x}\cdot\vec{x})^{100} = 100(\vec{x}\cdot\vec{x})^{99} \sum_i x_i\, \vec{e}^{(i)} = 200(\vec{x}\cdot\vec{x})^{99}\cdot\vec{x}$$

Irány menti derivált:

$$f(\vec{x}+\varepsilon\vec{n}) - f(\vec{x}) = (\text{grad}\,f)\cdot\varepsilon\vec{n} + \varepsilon(\vec{x},\Delta\vec{x})$$

$\Delta\vec{x} = \varepsilon\vec{n} \leftarrow \vec{n}$ rögzített,

~~$\vec{x}$ rögzített~~ ebbe az irányba vizsgálod a deriváltat

ε-konstans

$$\lim_{\varepsilon\to 0}\frac{f(\vec{x}+\varepsilon\vec{n}) - f(\vec{x})}{\varepsilon} = \frac{\partial f}{\partial\vec{x}} = (\text{grad}\,f)\cdot\vec{n} + \lim_{\varepsilon\to 0}\frac{\varepsilon(\vec{x},\Delta\vec{x})}{\varepsilon}$$

$$\left(\frac{\partial f}{\partial\vec{x}}\right)_{\vec{n}} = (\text{grad}\,f)\cdot\vec{n}$$

$$\Delta f = f(\vec{x}+\Delta\vec{x}) - f(\vec{x})$$

$$\Delta f = (\text{grad}\,f)\cdot\Delta\vec{x} + \varepsilon(\vec{x},\Delta\vec{x})$$

$\Downarrow$

Δx infinitezimálisan kicsi

$$df = (\text{grad}\,f)\cdot d\vec{x} = \left(\frac{\partial f}{\partial x_1}\right)dx_1 + \left(\frac{\partial f}{\partial x_2}\right)dx_2 \dots + \left(\frac{\partial f}{\partial x_n}\right)dx_n$$

Kapcsolat a láncszabállyal:

$$\frac{df(\vec{x}(t))}{dt} = (\text{grad}\,f)\cdot\vec{v}(t)$$

$$(\text{grad}\,|\vec{r}|)_i = \frac{\partial}{\partial r_i}\sqrt{r_1^2+r_2^2+\dots r_n^2} = \frac{1}{2\sqrt{r_1^2+r_2^2+\dots+r_n^2}}\,\frac{\partial}{\partial r_i}(r_1^2+r_2^2+\dots+r_n^2) =$$

$$= \frac{r_i}{|\vec{r}|} \quad \text{grad}\,|\vec{r}| = \sum_i \frac{r_i\,\vec{e}^{(i)}}{|\vec{r}|} = \frac{\vec{r}}{|\vec{r}|}$$

Radiális függvény esetén:

$$\text{grad}(|\vec{r}|) = \frac{df}{d|\vec{r}|} \qquad \text{grad}\,|\vec{r}| = f'(\vec{r})\cdot\frac{\vec{r}}{|\vec{r}|}$$

Példák: $\phi(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\cdot\frac{Q}{|\vec{r}|}$

$$\vec{E}(\vec{r}) = -\,\text{grad}\,\phi(\vec{r}) = -\frac{Q}{4\pi\varepsilon_0}\left(-\frac{1}{r^2}\right)\cdot\frac{\vec{r}}{|\vec{r}|}$$

$$|\vec{E}(\vec{r})| = \frac{Q}{4\pi\varepsilon_0}\,\frac{\vec{r}}{|\vec{r}|^3} = \frac{Q}{4\pi\varepsilon_0}\cdot\frac{1}{r^2}$$

lex:

$$\Delta f = \text{grad}(f)\cdot\Delta\vec{x} + \varepsilon$$

megvalósítandó

Az $f(\vec{x})$ skalármező gradiense az a vektormező, amely minden pontban a skalármező legnagyobb növekedésének irányába mutat (a skalárszorzat miatt), abszolút értéke pedig az ebbe az irányba mutató iránymenti derivált értéke.

Deriválttenzor:

$$\vec{f}(\vec{x}+\Delta\vec{x}) - \vec{f}(\vec{x}) = \underline{\underline{A}}(\vec{x})\cdot\Delta\vec{x} + \vec{\varepsilon}(\vec{x},\Delta\vec{x})$$

$$\lim_{\Delta x\to 0}\frac{\vec{\varepsilon}(\vec{x},\Delta\vec{x})}{\Delta\vec{x}} = 0$$

$$f_i(\vec{x}+\Delta\vec{x}) - f_i(\vec{x}) = \sum_j A_{ij}(\vec{x})\cdot\Delta x_j + \varepsilon_i(\vec{x},\Delta\vec{x})$$

$$\Delta\underline{x} = \begin{pmatrix} 0\\ 0\\ \vdots\\ \Delta x\\ 0\\ 0\end{pmatrix} \quad f_i(x_1,\dots x_j+\Delta x,\dots x_m) - f_i(x_1,\dots,x_j\dots x_m) =$$

$$= A_{ij}(\vec{x})\cdot\Delta x + \varepsilon_i(\vec{x},\Delta\vec{x}) = A_{ij}(\vec{x}) + \frac{\varepsilon_i(\vec{x},\Delta\vec{x})}{\Delta x}$$

$$\lim_{\Delta x\to 0}\frac{f_i(x_1,\dots,x_j+\Delta x,\dots x_m) - f_i(x_1,\dots,x_j,\dots x_m)}{\Delta x} =$$

$$= A_{ij}(\vec{x}) + \lim_{\Delta x\to 0}\frac{\varepsilon_i(\vec{x},\Delta\vec{x})}{|\Delta\vec{x}|}$$

0-hoz tart

$$\left(\frac{\partial f_i}{\partial x_j}\right) = A_{ij}$$

$$\underline{\underline{T}} = \begin{pmatrix} \partial_1 t_1 & \partial_2 t_1 & \dots\dots & \partial_n t_1 \\ \vdots & & & \\ \partial_1 t_n & \dots\dots & & \partial_n t_n \end{pmatrix}$$ (Jacobi)

Spec: $\vec{x}$: 3 komponensű
$\vec{t}$: 3 komponensű

$$\begin{pmatrix} \partial_1 t_1 & \partial_2 t_1 & \partial_3 t_1 \\ \partial_1 t_2 & \partial_2 t_2 & \partial_3 t_2 \\ \partial_1 t_3 & \partial_2 t_3 & \partial_3 t_3 \end{pmatrix}$$

<u>Divergencia:</u>

$$Sp\left(\frac{\partial \vec{t}}{\partial \vec{x}}\right)_{ij} = \sum_{i=1}^{n} \frac{\partial \vec{t}}{\partial x_i} = \vec{\nabla}\cdot\vec{t} = div(\vec{t})$$

<u>Rotáció:</u>

$$(rot\,\vec{t})_i = \sum_{j,k} \varepsilon_{ijk}\, \partial_j t_k$$

<u>Néhány egyszerű példa indexes deriválásra:</u>
/gyakorlás/

$$\frac{\partial r_i}{\partial r_j} = \delta_{ij} \qquad \underline{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$div(\vec{r}) = 3$$

Pl.: $\vec{v}(\vec{r}) = \vec{\omega}\times\vec{r}$

$$v_i = \sum_{jk} \varepsilon_{ijk}\,\omega_j r_k$$

$$rot(\vec{v}) = \sum_{j,k} \varepsilon_{ijk}\,\partial_j v_k = \sum_{jk} \varepsilon_{ijk}\,\partial_j\Big(\overset{\sum_{lm}}{\varepsilon_{klm}\,\omega_l r_m}\Big) =$$

$$= \sum_{j,k}\sum_{l,m} \overset{\varepsilon_{kij}}{\varepsilon_{ijk}}\,\varepsilon_{klm}\,\partial_j(\omega_l r_m) = \sum_{jklm}(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\,\partial_j(\omega_l r_m) =$$

$$= \delta_{il}\,\partial_j(\omega_l r_j) - \delta_{jl}\,\partial_j(\omega_l r_i) = \partial_j(\omega_i r_j) - \underbrace{\partial_l(\omega_l r_i)}_{\omega_l\delta_{li}} =$$

$$= 3\vec{\omega} - \vec{\omega} = 2\vec{\omega}$$

$$\mathrm{div}\,\mathrm{rot}\,\vec{x} = \sum_{i=1}^{3} \partial_i (\mathrm{rot}\,\vec{x})_i = \sum_{i=1}^{3} \partial_i \left[\varepsilon_{ijk}\, \partial_j x_k\right] =$$

$$= \sum_{ijk}^{3} \partial_i \left[\cancel{\varepsilon_{jk}}\, \varepsilon_{kij}\, \partial_j x_k\right] = \sum_{ijk}^{3} \varepsilon_{ijk}\, \partial_i \partial_j x_k = \underline{0}$$

szimmetrikus összegzése ~~egy~~ ~~an~~ antiszimmetrikussal mindig nullát ad

$$\boxed{\mathrm{div}\,\mathrm{rot}\,\vec{x} = \underline{0}} \quad \forall \vec{x}\text{-re}$$

$$(\mathrm{rot}\,\mathrm{grad}\,x)_i = \sum_{jk} \varepsilon_{ijk}\, \partial_j \partial_k x = 0$$

megint

$$\boxed{\mathrm{rot}(\mathrm{grad}\,x) = \underline{0}} \quad \forall x\text{-re}$$

$$\vec{\nabla}(\vec{\nabla} a) = \mathrm{div}(\mathrm{grad}\,a) = \Delta a$$

Laplace operátor

$$\Delta a = \sum_i \partial_i \partial_i a$$

5) Kettős és hármas integrálok

Kettős:

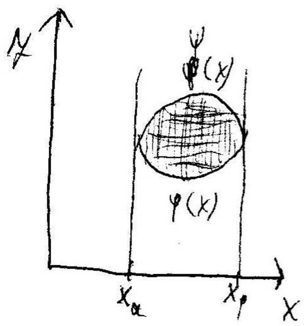
$$\int f(x,y)\,dx\,dy \approx \sum_{i=1}^{N} f(x_i, y_i)\,\Delta x_i\,\Delta y_i$$

pl. ha $\rho(x,y)$ felületi sűrűség

$$\frac{\int \rho(x,y)\,dx\,dy}{\int \rho(x,y)\,dx\,dy}$$

$$\Delta A_i = \Delta x_i \cdot \Delta y_i$$

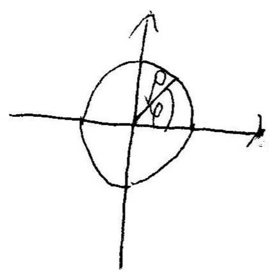
$$\int_{\text{felületen}} f(x,y)\,dy = \int_{x_a}^{x_f} \int_{\varphi(x)}^{\psi(x)} f(x,y)\,dy\,dx$$



1 kör területe

Példa:

$$\int_0^R \int_0^{2\pi} f(\underbrace{\rho\cos\varphi}_{x}, \underbrace{\rho\sin\varphi}_{y})\,\rho\,d\rho\,d\varphi$$



$x = \rho\cos\varphi$

$y = \rho\sin\varphi$

$\rho = [0, R]$

$\varphi = [0, 2\pi)$

Általánosabban:

$x = \varphi(u,v)$

$y = \psi(u,v)$

$$dx\,dy = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$\Leftarrow$

Jacobi determináns: $\begin{vmatrix} \partial_u x & \partial_v x \\ \partial_u y & \partial_v y \end{vmatrix} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

A példában:

$\psi_1(\rho,\varphi) = \rho\cos\varphi$

$\psi(\rho,\varphi) = \rho\sin\varphi$

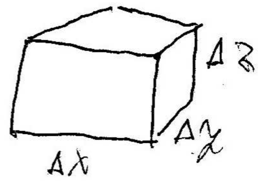
$$\begin{vmatrix} \partial_\rho \psi_1 & \partial_\varphi \psi_1 \\ \partial_\rho \psi_2 & \partial_\varphi \psi_2 \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix} = \rho(\cos^2\varphi + \sin^2\varphi) = \rho$$

$$dx\,dy = \rho\,d\rho\,d\varphi$$

$$\int_0^{2\pi}\int_0^{R} \rho\,d\rho\,d\varphi = \int_0^{2\pi} \frac{R^2}{2}\,d\varphi = \pi R^2$$

Hármas integrál:

$$\int_{V_0} f(x,y,z)\,dx\,dy\,dz \cong \sum_i f(x_i,y_i,z_i)$$



$$x = \Psi_1(u,v,w)$$
$$y = \Psi_2(u,v,w)$$
$$z = \Psi_3(u,v,w)$$

$$dx\,dy\,dz = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Jacobi determináns

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \partial_u \Psi_1 & \partial_v \Psi_1 & \partial_w \Psi_1 \\ \partial_u \Psi_2 & \partial_v \Psi_2 & \partial_w \Psi_2 \\ \partial\ \Psi_3 & \partial_v \Psi_3 & \partial_w \Psi_3 \end{vmatrix}$$

Példák:

Hengerkoordinátáknál:

$$x = \rho\cos\varphi \qquad (x,y,z)\to(\rho,\varphi,z)$$
$$y = \rho\sin\varphi$$
$$z = z$$

$$dx\,dy\,dz \Rightarrow \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} \cdot d\rho\,d\varphi\,dz = \rho\,d\rho\,d\varphi\,dz$$

$\rho \in [0,R]$

$\varphi \in [0,2\pi]$

$z \in [0,m]$

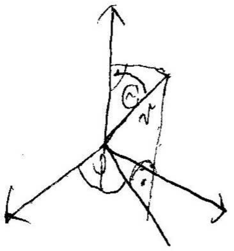
$$\int_0^m\int_0^{2\pi}\int_0^R \rho\,d\rho\,d\varphi\,dz = \int_0^m\int_0^{2\pi} \frac{R^2}{2}\,d\varphi\,dz = \int_0^m \frac{R^2 2\pi}{2}\,dz =$$
$$= R^2\pi\cdot m$$

gömbkoordinátákkal:

$x = r \sin\theta \cos\varphi$

$y = r \sin\theta \sin\varphi$

$z = r \cos\theta$



$$dx\,dy\,dz = J\,dr\,d\theta\,d\varphi$$

$$J = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix} =$$

$$= r^2\cos^2\theta\sin\theta\,\overset{\cos^2}{\cancel{\sin^2}}\varphi + r^2\sin^3\theta\sin^2\varphi +$$

$$+ r^2\sin\theta\cos^2\theta\sin^2\varphi + r^2\sin^3\theta\cos^2\varphi =$$

$$= r^2\sin\theta\left(\cos^2\theta\,\overset{\cos^2}{\cancel{\sin^2}}\varphi + \sin^2\theta\,\cancel{\cos}\sin^2\varphi + \cos^2\theta\sin^2\varphi +\right.$$

$$\left. + \sin^2\theta\cos^2\varphi\right) = r^2\sin\theta\left(\cos^2\theta + \sin^2\theta\right)\cdot\left(\cos^2\varphi + \overset{\sin}{\cancel{\cos}}{}^2\varphi\right) =$$

$$= r^2\sin\theta$$

$r \in [0, R]$

$\theta \in [0, \pi]$

$\varphi \in [0, 2\pi]$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin\theta\,dr\,d\theta\,d\varphi = \int_0^{2\pi}\int_0^{\pi} \frac{R^3}{3}\sin\theta\,d\theta\,d\varphi =$$

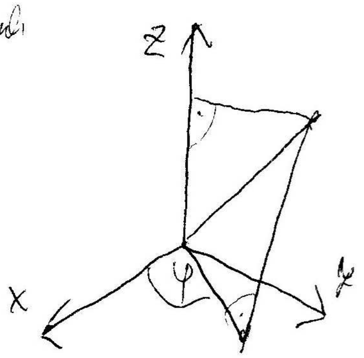
$$= \int_0^{2\pi} \cancel{r^2}\,\frac{R^3}{3}\,d\varphi = \frac{2R^3\pi}{3}$$

6. Felületi és térfogati integrálok

Felületi integrál:

Felület

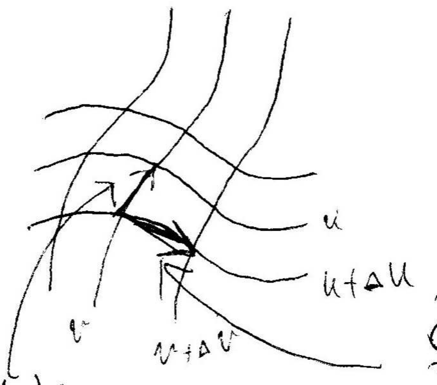
gömb: $\vec{r}(\theta, \varphi)$ ($\theta \leftarrow u$, $\varphi \leftarrow v$)



$\vec{r}(u, v)$

↓ ↓
paraméterek

$$\vec{r}(\theta, \varphi) = \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R \cos\theta \end{pmatrix} \quad R = \text{állandó}$$



$\frac{\partial \vec{r}}{\partial v} \cdot dv$

$\frac{\partial \vec{r}}{\partial u} du$

$$\vec{r}(u, v+\Delta v) - \vec{r}(u,v) = \frac{\partial \vec{r}}{\partial v} \cdot dv$$

$$d\vec{f} = \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right) du\, dv$$

fajták:
1, $\int A(\vec{r})\, df$
2, $\int \vec{A}(\vec{r})\, df$
3, $\int \vec{A}(\vec{r}) \cdot d\vec{f}$
4, $\int A(\vec{r})\, d\vec{f}$
5, $\int \vec{A}(\vec{r}) \times d\vec{f}$

adott valami felület:
$\vec{r}(u, v)$, $u \in [u_a, u_f]$
$v \in [v_a, v_f]$

③ $$\int_{u_a}^{u_f}\int_{v_a}^{v_f} \vec{A}(\vec{r}(u,v)) \cdot \left(\frac{\partial \vec{r}(u,v)}{\partial u} \times \frac{\partial \vec{r}(u,v)}{\partial v} \right) du\, dv$$

① $$\int_{u_a}^{u_f}\int_{v_a}^{v_f} A(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dv\, du$$

Példa: gömb felszíne

$$\vec{r}(\underset{\downarrow\, u}{\Theta}, \underset{\downarrow\, v}{\varphi}) = \begin{pmatrix} R \cdot \sin\Theta \cos\varphi \\ R \cdot \sin\Theta \sin\varphi \\ R \cdot \cos\Theta \end{pmatrix} = R\vec{n}$$

$\vec{n}$: sugárirányú egységvektor

$$\left(\frac{\partial \vec{r}}{\partial \Theta} \times \frac{\partial \vec{r}}{\partial \varphi} \right) = \begin{pmatrix} R\cos\Theta\cos\varphi \\ R\, \cancel{\sin}\cos\Theta\sin\varphi \\ -R\sin\Theta \end{pmatrix} \times \begin{pmatrix} -R\sin\Theta\sin\varphi \\ R\sin\Theta\cos\varphi \\ 0 \end{pmatrix} =$$

$$= \begin{pmatrix} R^2 \sin^2\Theta \cos\varphi \\ R^2 \sin^2\Theta \sin\varphi \\ R^2 (\cos\Theta \sin\Theta \cos^2\varphi + \cos\Theta \sin\Theta \sin^2\varphi) \end{pmatrix} = R^2 \sin\Theta\, \vec{n}$$

$$\left| \frac{\partial \vec{r}}{\partial \Theta} \times \frac{\partial \vec{r}}{\partial \varphi} \right| = R^2 \sin\Theta\, |\vec{n}|$$

$$\int_0^{\pi}\int_0^{2\pi} \underset{A(\vec{r})}{1} \cdot \underbrace{\left| \frac{\partial \vec{r}}{\partial \Theta} \times \frac{\partial \vec{r}}{\partial \varphi} \right|}_{R^2 \sin\Theta} d\varphi\, d\Theta = R^2 \left[\int_0^{\pi} \sin\Theta\, d\Theta \right] \left[\int_0^{2\pi} 1\, d\varphi \right] = R^2 4\pi$$

$$\int_0^{\pi} \sin\Theta\, d\Theta \rightarrow \left[-\cos\Theta \right]_0^{\pi} \rightarrow 1+1=2$$

$$\int_0^{2\pi} 1\, d\varphi \rightarrow 2\pi$$

példa: R sugarú gömb felülete része adott térnyolcadban

$x > 0, \ y > 0, \ z > 0$

paraméterezés

$$\vec{r}(\theta, \varphi) = \begin{pmatrix} R \sin\theta \cos\varphi \\ R \sin\theta \sin\varphi \\ R \cos\theta \end{pmatrix} \qquad \begin{matrix} \theta \in [0, \pi/2] \\ \varphi \in [0, \pi/2] \end{matrix}$$

$$\vec{A}(\vec{r}) = \begin{pmatrix} 5z \\ x \\ 2x - y \end{pmatrix}$$

$$\int_{\text{felület}} \vec{A} \, d\vec{f} = \int_{\text{felület}} \vec{A}(\vec{r}(\theta, \varphi)) \left(\frac{\partial \vec{r}}{\partial \theta} \times \frac{\partial \vec{r}}{\partial \varphi} \right) d\theta \, d\varphi =$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \underbrace{\begin{pmatrix} 5R\cos\theta \\ R\sin\theta\sin\varphi \\ 2R\sin\theta\cos\varphi - R\sin\theta\sin\varphi \end{pmatrix}}_{\vec{A} \text{ értéke a felület pontjaiban}} \cdot R^2 \sin\theta \begin{pmatrix} \sin\theta\cos\varphi \\ \sin\theta\sin\varphi \\ \cos\theta \end{pmatrix} d\theta \, d\varphi =$$

$R := 3$

$$= \frac{9(12 + \pi)}{2}$$

Integrál átalakító tételek:

① Gradiens tétel:

$\vec{v} = \operatorname{grad} \varphi(\vec{r}) \qquad \vec{r}(t) \qquad t \in [t_A, t_B]$

$$\int \underbrace{\operatorname{grad} \varphi}_{\text{vektor}} \underbrace{\cdot}_{\text{skaláris szorzás}} \underbrace{d\vec{r}}_{\text{vektor}} = \varphi(\vec{r}(t_B)) - \varphi(\vec{r}(t_A))$$

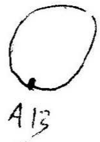
független az útvonaltól

Bizonyítás: $\int_A^B \operatorname{grad}\varphi \, d\vec r = \int_{t_A}^{t_B} \operatorname{grad}\varphi(\vec r(t)) \cdot \left(\frac{d\vec r}{dt}\right) dt =$

$$= \int_{t_A}^{t_B} \left(\frac{\partial \varphi}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial \varphi}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial \varphi}{\partial z} \cdot \frac{dz}{dt} \right) dt = \int_{t_A}^{t_B} \frac{d\varphi(\vec r(t))}{dt} dt =$$

$$= \left[\varphi(\vec r(t))\right]_{t_A}^{t_B} = \varphi(\vec r(t_B)) - \varphi(\vec r(t_A))$$

Következménye: ha körbe megyünk $\varphi(t_B) = \varphi(t_A)$



ha ~~ro~~ rot $\vec v = 0$, akkor $\exists \varphi$, melyre $\vec v = \operatorname{grad}\varphi$

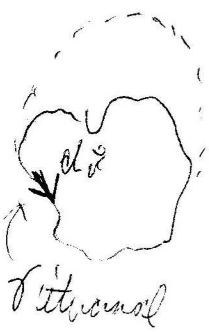
$$\boxed{\oint \vec v \, d\vec r = 0, \text{ ha rot } \vec v = 0}$$

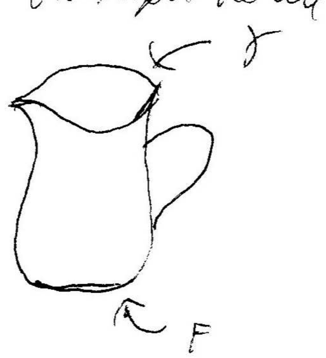
körintegrál zárt útra

2. Stokes tétel:

ha rot $\vec v \neq 0$

$$\oint_\gamma \vec v \, d\vec r = \int_F \operatorname{rot} \vec v \times d\vec F$$

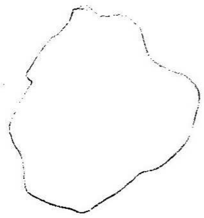




γ az F felület határa

másképp: Ha van egy zárt görbe, és a görbén integrálunk egy vektormezőt, akkor ez megegyezik a rotáció felületre vett integráljával.

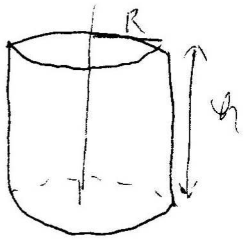
3. Gauss tétel:



zárt felület (F) V térfogatot határol

$$\oint\!\!\oint \vec{v}(\vec{r})\, d\vec{F} = \iiint_V \mathrm{div}\, \vec{v}(F)\, dV$$

Egyszerű példa:



$$\oint\!\!\oint \vec{v}\, d\vec{F} = 8 R^2 \pi \cdot h$$

$\vec{v} = 5x\,\vec{i} + 2y\,\vec{j} + z\cdot\vec{k}$ $\mathrm{div}\,\vec{v} = 5+2+1 = 8$

Megj.: Akkor igaz, ha a felület belsejében a térfogat értelmezhető, és ezek a mennyiségek léteznek