ZH: március 22. (Hétfő), 8:15 - 9:45, -1.85 Hsz. tanterem

Elmarad márc. 25-én a gyak

11. gázmolekulák forgási, belső szab. fokai

$$\hat{H} = \sum_{n=1}^{N} \left(\frac{\hat{p}_n^2}{2m} + \hat{H}_{i,n} \right)$$

$\hat{H}_{i,n}$: n-edik molekula Ham.-op.

$\hookrightarrow \varepsilon_i$

$$Z = \frac{1}{N!} \cdot Z_1^N \qquad Z_1 = V \cdot \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3}{2}} \cdot \underbrace{\sum_i e^{-\beta \varepsilon_i}}_{Z_i}$$

$\frac{\hat{p}^2}{2m}$

$$F = -k_B T \ln Z = -N k_B T \ln Z_1 + k_B T (N \ln N - N)$$

$$dF = -S\,dT - p\,dV + \mu\,dN$$

$$p = -\left. \frac{\partial F}{\partial V} \right|_{T,N} = N k_B T \cdot \frac{1}{V}$$

id. gáz áll. egyenlet

A belső szabadsági fok nem befolyásolja (csak) az állapotegyenletet.

$$S = -\left. \frac{\partial F}{\partial T} \right|_{V,N} = \dots = N k_B \left[\frac{5}{2} + \ln \frac{V}{N \lambda^3} + \ln Z_i + T \frac{\partial \ln Z_i}{\partial T} \right]$$

$$\lambda = \frac{h}{\sqrt{2\pi m k_B T}} \qquad \frac{V}{N} \ll \lambda^3$$

$$E = -\frac{\partial \ln Z}{\partial \beta} = \dots = N k_B T \left[\frac{3}{2} + T \frac{\partial \ln Z_i}{\partial T} \right]$$

$$\bar{C}_V = \left. \frac{\partial E}{\partial T} \right|_{V,N} = \dots = N k_B \left[\frac{3}{2} + \frac{\partial}{\partial T} \left(T^2 \frac{\partial (\ln Z_i)}{\partial T} \right) \right]$$

↑ translații

$$\uparrow V = \frac{m}{M} R T = N k_B T$$

$$\frac{m}{M} R = N k_B$$

$$\bar{C}_V = \alpha \cdot N k_B = \alpha \cdot \frac{m}{M} R \quad \Rightarrow \quad c_V = \frac{\bar{C}_V}{m} = \alpha \cdot \frac{R}{M}$$

$$\varepsilon_i = \varepsilon_{rot} + \varepsilon_{vib} + \varepsilon_{el}$$

a) $\varepsilon_{rot} = l(l+1) \frac{\hbar^2}{2G}$ $\quad l = 0, 1, 2, \dots$

b) $\varepsilon_{vib} = \hbar \omega \left(n + \frac{1}{2} \right)$ $\quad n = 0, 1, 2, \dots$

$\varepsilon_{el} \lesssim \varepsilon_{disociație} \Rightarrow$

H_2 : 52000°

~~O_2 : 59000°~~

Cl_2 : 29000°

NO : 61000°

CO : 98000°

Iuăm O_2, NO

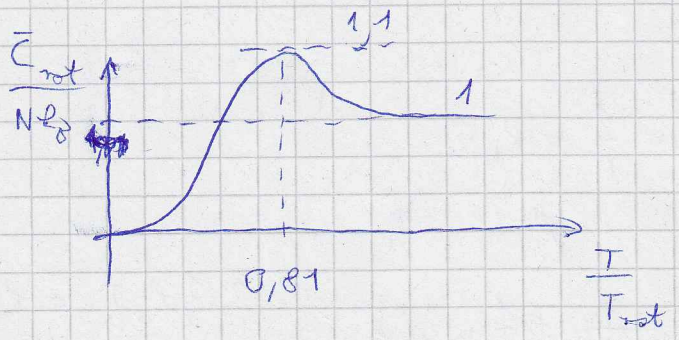
Ha $k_B T \ll \varepsilon_{dissoc} \Rightarrow \varepsilon_{el}$: alapállapotban van, ε_0

$$Z_i \overset{(\ddagger)}{=} g_i e^{-\beta \varepsilon_0} \cdot Z_{rot} \cdot Z_{vib}$$

$$Z_{rot} = \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{-\beta \ell(\ell+1)\frac{\hbar^2}{2\Theta}} = \sum_{\ell=0}^{\infty} (2\ell+1)\, e^{-\frac{T_{rot}}{T}\ell(\ell+1)}$$

$$T_{rot} = \frac{\hbar^2}{2\Theta k_B}$$

$$Z_{vib} = \sum_{n=0}^{\infty} e^{-\beta \hbar\omega\left(n+\frac{1}{2}\right)} = \dots = \frac{1}{2 \operatorname{sh} \beta \frac{\hbar\omega}{2}}$$



klassikusan $T \gg T_{rot}$

$$Z_{rot} = \frac{2\Theta}{\hbar^2 \beta} \longleftarrow H = \frac{1}{2\Theta}\left(p_\vartheta^2 + \frac{p_\varphi^2}{\sin^2\vartheta}\right)$$

$$E = -\frac{\partial(-\ln \beta)}{\partial \beta} = \frac{1}{\beta} = k_B T$$

$$Z_{vibr} = \frac{1}{2\sinh\left(\frac{\beta\hbar\omega}{2}\right)} = \frac{1}{2\sinh\frac{T_{vibr}}{2T}}$$

$$\downarrow$$

$$k_B T_{vibr} = \hbar\omega$$



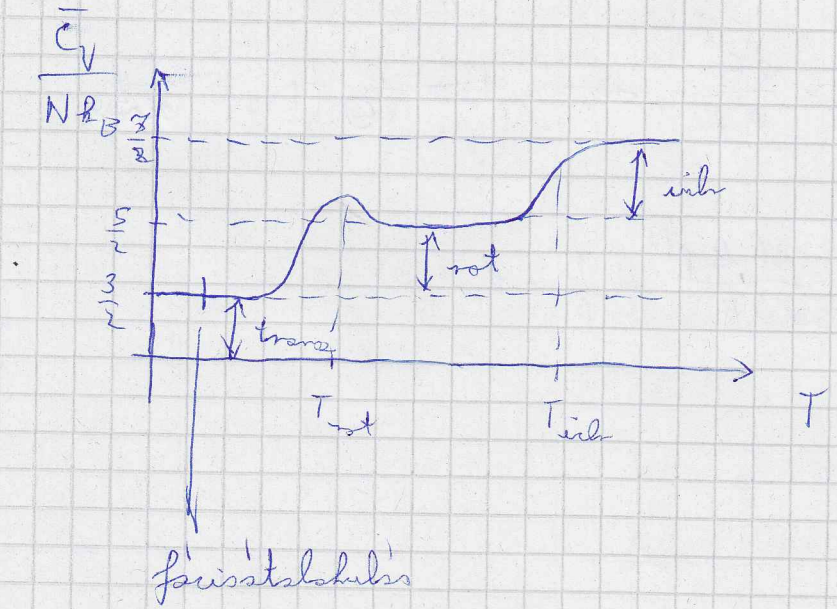
$$T \to \infty \;,\; \beta \to 0 \Rightarrow Z_{vibr} = \frac{1}{2\cdot\frac{\beta\hbar\omega}{2}} = \frac{1}{\beta\hbar\omega}$$

$$E = -\frac{\partial \ln Z}{\partial \beta} = \frac{1}{\beta} = k_B T$$

$$\uparrow$$

$$\frac{p^2}{2m} + \frac{1}{2} m\omega^2 x^2 \qquad 2\cdot\frac{1}{2} k_B T = k_B T$$

$$C_V = N k_B$$



Többatomos mol.: $\varepsilon_{el} = \varepsilon_0$ deg.: g_e $(=1)$

magspin degeneráció

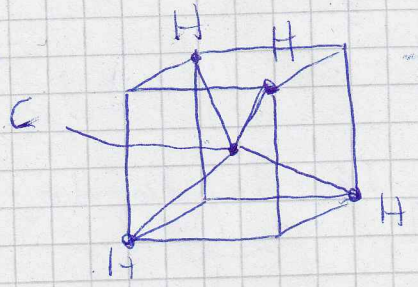
$$Z_{(mol)} = g_e \cdot \frac{g_{mag} \cdot Z_{rot}^{klasszikus}}{\sigma} \cdot Z_{vib}$$

szimm. paraméter

$$Z_{rot}^{klasszikus} = \frac{2\Theta}{\hbar^2 \beta} \qquad T \gg T_{rot}$$

$$\Theta_1 = \Theta_2 = \Theta_3 \Leftarrow \underline{\underline{\Theta}}$$

$$H = \frac{1}{2} \underline{J}\, \underline{\underline{\Theta}}^{-1} \cdot \underline{J}$$

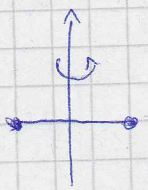


CH_4

4 db 3 forgási szim.

tengelye

$\sigma = 3 \cdot 4 = 12$



H_2: $\sigma = 2$

Ha n atom van a molekulában: $3n$ = molekula fokok száma

$3n-3-3$ vagy $3n-3-2$

↑ ↑ ↑ ↑

transz forgási transz forg.

(nemlineáris molekula) (lineáris molekula)

$$Z_{vibr} = \begin{cases} \prod_{i=1}^{3n-5} \frac{1}{2\sinh\frac{\beta\hbar\omega_i}{2}} & \text{lineáris mol} \\ \prod_{i=1}^{3n-6} \frac{1}{2\sinh\frac{\beta\hbar\omega_i}{2}} & \text{nemlineáris mol.} \end{cases}$$

klassz. $\underline{\underline{\Theta}} \rightarrow \underline{\underline{\Theta}} = \begin{pmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_3 \end{pmatrix}$

ξ, η, φ

$$H = \frac{J_\xi^2}{2\Theta_1} + \frac{J_\eta^2}{2\Theta_2} + \frac{J_\varphi^2}{2\Theta_3}$$

$$Z_{klasz} = \frac{1}{\sigma}\cdot\frac{1}{h^3}\int dJ_\xi\, dJ_\eta\, dJ_\varphi\, d\varphi_\xi\, d\varphi_\eta\, d\varphi_\varphi \cdot e^{-\beta H} =$$

$$\int d\Omega_\varphi = 4\pi$$

$$\int d\varphi_\eta = 2\pi$$

$$= \frac{4\pi \cdot 2\pi}{\sigma} \sqrt{\frac{\pi}{\beta \cdot \frac{1}{2}\Theta_1}} \sqrt{\frac{\pi}{\beta \cdot \frac{1}{2}\Theta_2}} \cdot \sqrt{\frac{\pi}{\beta \frac{1}{2}\Theta_3}} = \frac{\sqrt{\pi}}{\sigma} \left(\frac{2}{\hbar^2 \beta}\right)^{\frac{3}{2}} \cdot \underbrace{\sqrt{\Theta_1 \Theta_2 \Theta_3}}_{\det \underline{\underline{\Theta}}}$$

Kub.: 212 old. (3/4) képlet

London V. 172. old.

kvantumosan:

- gömbi pörgettyű

$$\Theta_1 = \Theta_2 = \Theta_3 = \Theta \Rightarrow \varepsilon_{rot} = \ell(\ell+1)\frac{\hbar^2}{2\Theta}$$

- szimmetrikus pörgettyű

$\Theta_1 = \Theta_2 \neq \Theta_3$ ~~(kardán)~~

London III. 484. old

(1036) képlet

Példák:



H $3 \cdot \frac{1}{2} k_B T$

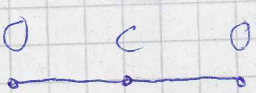
$n = 2$

$3n - 5 = 1$

forgás: $2 \cdot \frac{1}{2} k_B T \leftarrow \frac{1}{2\Theta}\left(\uparrow_\Theta^2 + \frac{\uparrow_\varphi^2}{2\pi^2 \Theta}\right)$

1 r. rádiusz: $2 \cdot \frac{1}{2} k_B T \leftarrow \frac{\uparrow^2}{2m} + \frac{1}{2} m \omega^2 x^2$

$\bar{C}_V = \frac{7}{2} N k_B$



tr. $3 \cdot \frac{1}{2} k_B T$

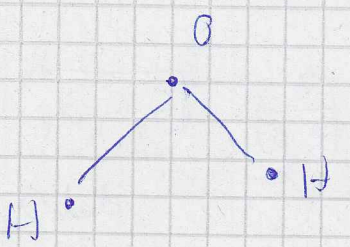
$n = 3$

$3n - 5 = 4$

forgás $2 \cdot \frac{1}{2} k_B T$

$\bar{C}_V = \frac{13}{2} k_B N$

4 (~~forgás~~) rezgési módus $4 \cdot 2 \cdot \frac{1}{2} k_B T$



$n = 3$

$3n - 6 = 3$

tr. $3 \cdot \frac{1}{2} k_B T$

forgási: $3 \cdot \frac{1}{2} k_B T$

rezgési módus $3 \cdot 2 \cdot \frac{1}{2} k_B T$

$$\bar{C}_V = \frac{12}{2} k_B N = 6 k_B N$$

nagyon nehéz: AB ← a) kétatomos lineáris

S_A S_B

$2S_A + 1$ $2S_B + 1$

$g_{rog} = (2S_A + 1)(2S_B + 1)$ degenerált

b) azonos rog $A\Delta$

$(2S_A + 1)^2$

↙ ↘

rsz (db) antisz

$\uparrow^2 = \frac{\uparrow(\uparrow - 1)}{2} = \frac{\uparrow(\uparrow + 1)}{2}$

dönt

$f_1, \ldots, f_p$

$\frac{f_i(1) f_j(2) + f_i(2) f_j(1)}{\sqrt{2}}$ rsz

$\frac{f_i(1) f_j(2) + f_i(2) f_j(1)}{\sqrt{2}}$ antisz

$\frac{\uparrow(\uparrow - 1)}{2}$ db

$\uparrow = 2S_A + 1$

→ szim $(S_A+1)(2S_A+1)$ ℓ = páratlan — ortho-H_2

→ antiszim $S_A(2S_A+1)$ ℓ = páros — para-H_2

H_2: $S_A = \frac{1}{2}$ $\left(2\cdot\frac{1}{2}+1\right)^2 = 4$ db

1 antiszim (szinglett) — 3 szim (triplett)

$$Z = S_A(2S_A+1)\, Z_{\text{páros}} + (S_A+1)(2S_A+1)\, Z_{\text{páratlan}}$$

para-H_2 ← $Z_{\text{páros}}$; ortho-H_2 ← $Z_{\text{páratlan}}$

$$Z_{\text{páros}} = \sum_{\ell=0,2,4,\ldots} \ell(\ell+1)\, e^{-\beta \frac{\hbar^2}{2\Theta}\ell(\ell+1)}$$

$$Z_{\text{páratlan}} = \sum_{\ell=1,3,5,\ldots} \ell(\ell+1)\, e^{-\beta \frac{\hbar^2}{2\Theta}\ell(\ell+1)}$$

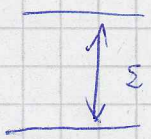
Elektronjelek a fajhőben

Ha T olyan, hogy $\varepsilon_{el} = \varepsilon_0$ alapáll., akkor nem számít.

Kivéve O_2, NO

g_1 / $\varepsilon_1 = \varepsilon_0 + \varepsilon$

g_0 / ε_0



$$Z_{el} = g_0 \cdot e^{-\beta \varepsilon_0} + g_1 \cdot e^{-\beta \varepsilon_1} =$$

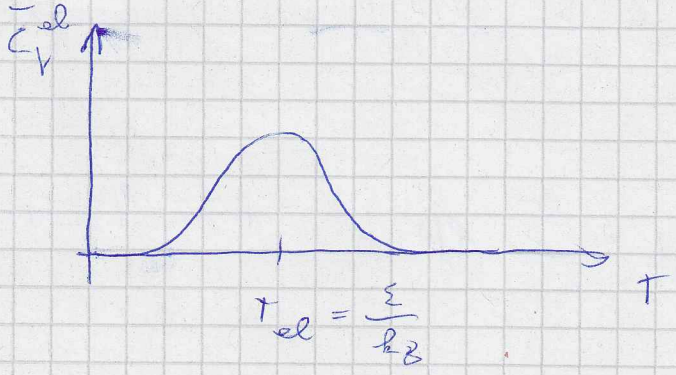
$$= e^{-\beta \varepsilon_0} \left(g_0 + g_1 \cdot e^{-\beta \varepsilon} \right)$$

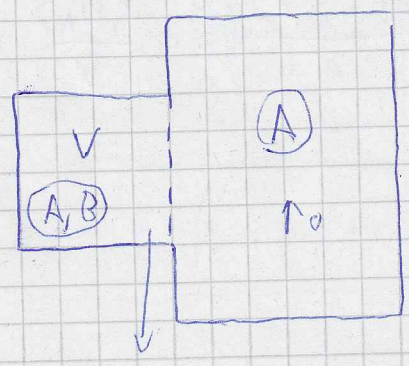
$$\Downarrow$$

$$\bar{C}_V^{el} = \dots$$

$$E = -\frac{\partial \ln Z_{el}}{\partial \beta} = \varepsilon_0 + \frac{\partial \ln \left(g_0 + g_1 e^{-\beta \varepsilon} \right)}{\partial \beta}$$

$$\bar{C}_V^{el} = \frac{\partial E}{\partial T} = \dots = N k_B \frac{g_0 g_1 (\beta \varepsilon)^2 e^{+\beta \varepsilon}}{\left(g_0 + g_1 \cdot e^{\beta \varepsilon} \right)^2}$$





félig áteresztő fal:

A-t áteresztí → egyensúlyban } vegyes rendszer

B-t nem → kizárólag

$$Z = \left(\sum_{N_A=0}^{\infty} \frac{1}{N_A!} Z_{1A}^{N_A} \cdot e^{\beta \mu_A N_A} \right) \frac{1}{N_B!} Z_{1B}^{N_B} =$$

$$= \exp\left(Z_{1A} \cdot e^{\beta \mu_A} \right) \cdot \frac{1}{N_B!} \cdot Z_{1B}^{N_B}$$

$$Z_{1A} = V \cdot \left(\frac{2\pi m_A k T}{h^2} \right)^{\frac{3}{2}} \qquad Z_{1B} = V \cdot \left(\frac{2\pi m_B k T}{h^2} \right)^{\frac{3}{2}}$$

$$F(T, V, N_B, \mu_A) = -k_B T \ln Z$$

$$dF = -S\,dT - p\,dV + \mu_B\,dN_B - N_A\,d\mu_A$$

$$\langle N_A \rangle = -\left. \frac{\partial F}{\partial \mu_A} \right|_{T,V,N_B} = e^{\beta \mu_A} \cdot V \cdot \left(\frac{2\pi m_A k T}{h^2} \right)^{\frac{3}{2}}$$

$$p = -\left. \frac{\partial F}{\partial V} \right|_{T,N_B,\mu_A} = k T \left(N_B \cdot \frac{1}{V} \cdot e^{\beta \mu_A} \left(\frac{2\pi m_A k T}{h^2} \right)^{\frac{3}{2}} \right) = \frac{\left(\langle N_A \rangle + N_B \right) k T}{V}$$

Kint.: $\frac{N_{kint}}{V_{kint}} = \frac{\uparrow_0}{kT} = e^{\beta \mu_A} \cdot \left(\frac{2\pi m_A kT}{h^2} \right)^{\frac{3}{2}} = \frac{\langle N_A \rangle}{V}$

$$\rightarrow \langle N_A \rangle = \frac{\uparrow_0 \cdot V}{kT}$$

$$\uparrow = \uparrow_0 + \frac{N_B kT}{V}$$

II. megoldás: $\mu_A = -kT \left(\ln \frac{V}{N_A} + \frac{3}{2} \ln \frac{2\pi m_A kT}{h^2} \right) =$

$$= -kT \left(\ln \frac{V_{kint}}{N_{kint}} + \frac{3}{2} \ln \frac{2\pi m_A kT}{h^2} \right)$$

$$\frac{V}{N_A} = \frac{V_{kint}}{N_{kint}} = \frac{kT}{\uparrow_0} \Rightarrow N_A = \frac{\uparrow_0 V}{kT}$$

Perturbációszámítás

Klasszikus rendszer

$$H = H_0 + \lambda K$$

$$Z = \int d\Gamma \, e^{-\beta H} = \int e^{-\beta H_0} \underbrace{e^{-\beta \lambda K}}_{\parallel \; 1 - \beta \lambda K} d\Gamma =$$

$$= Z_0 - \lambda \beta Z_0 \cdot \frac{1}{Z_0} \int K \cdot e^{-\beta H_0} d\Gamma =$$

$$= Z_0 - \lambda \beta Z_0 \langle K \rangle_0$$

, ahol $\boxed{\langle K \rangle_0 = \int K \cdot \frac{e^{-\beta H_0}}{Z_0} \, d\Gamma}$

$$\boxed{Z = Z_0 \left(1 - \lambda \beta \langle K \rangle_0\right)}$$

$$F = -kT \ln Z = -kT \left(\ln Z_0 + \underbrace{\ln\left(1 - \lambda \beta \langle K \rangle_0\right)}_{-\lambda \beta \langle K \rangle_0} \right) =$$

$$= \underbrace{-kT \ln Z_0}_{F_0} + \lambda \langle K \rangle_0$$

$$\boxed{F = F_0 + \lambda \langle K \rangle_0}$$

Kvantummechanika

$$E_n = E_n^{(0)} + \lambda \langle n | \hat{K} | n \rangle$$

$$Z = \sum_n e^{-\beta E_n} = \sum_n e^{-\beta E_n^{(0)}} \cdot e^{-\beta \lambda \langle n | \hat{K} | n \rangle}$$

$$= \sum_n e^{-\beta E_n^{(0)}} \left(1 - \lambda \beta \langle n | \hat{K} | n \rangle\right)$$

$$= Z_0 \left(1 - \frac{1}{Z_0} \lambda \beta \cdot \sum_n \langle n | \hat{K} | n \rangle \, e^{-\beta E_n^{(0)}}\right) =$$

$$= \boxed{Z_0 \left(1 - \lambda \beta \langle \hat{K} \rangle_0\right) = Z}$$

, ahol: $\langle \hat{K} \rangle_0 = \frac{1}{Z_0} \sum_n \langle n | \hat{K} | n \rangle \, e^{-\beta E_n^{(0)}}$

Fields: $H = \underbrace{\frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2}_{H_0} + \lambda x^4$

1 dof

$$Z_0 = \frac{1}{h} \int e^{-\beta H_0} \, dp \, dx = \frac{1}{\beta h \omega}$$

$$\langle x^4 \rangle_0 = \frac{\frac{1}{h} \int x^4 e^{-\beta \frac{1}{2} m \omega^2 x^2} \, dx}{\frac{1}{h} \int e^{-\beta \frac{1}{2} m \omega^2 x^2} \, dx} = \frac{\int x^4 e^{-u x^2} \, dx}{\int e^{-u x^2} \, dx} =$$

$$u = \frac{1}{2} m \omega^2 \beta$$

$$= \frac{\frac{\partial^2}{\partial u^2} \int_{-\infty}^{+\infty} e^{-u x^2} \, dx}{\int_{-\infty}^{+\infty} e^{-u x^2} \, dx} = \frac{\frac{\partial^2}{\partial u^2} \sqrt{\frac{\pi}{u}}}{\sqrt{\frac{\pi}{u}}} = \frac{3}{(\beta m \omega^2)^2}$$

$$Z = Z_0 \left(1 - \lambda \beta \langle x^4 \rangle_0\right) = \frac{1}{\beta h \omega} \left(1 - \frac{3 \lambda \beta}{(\beta m \omega^2)^2}\right)$$

$$E = -\frac{\partial (\ln Z)}{\partial \beta} = kT - \frac{3\lambda}{m^2 \omega^4} (kT)^2$$

$$C = k - \frac{6\lambda}{m^2 \omega^4} k^2 T$$

Kv. gázok (bozon, fermion)

$$\Phi(T,V,\mu) = -pV = -k_B T \ln Q$$

$$\uparrow$$

G.-D.

$$E = TS - pV + \mu N$$

$$\Phi = E - TS - \mu N = -pV$$

F: $n_i = 0, 1$

B: $n_i = 0, 1, 2, \ldots$

$$Q = \sum_{\{n_i\}} e^{-\beta \sum_i n_i(\varepsilon_i - \mu)} = \prod_i \sum_{n_i} e^{-\beta n_i(\varepsilon_i - \mu)}$$

$$F: \quad Q = \prod_i \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)$$

$$B: \quad Q = \prod_i \sum_{n_i=0}^{\infty} \left(e^{-\beta(\varepsilon_i - \mu)}\right)^{n_i} = \prod_i \frac{1}{1 - e^{-\beta(\varepsilon_i - \mu)}}$$

$$F: \quad \Phi = -k_B T \ln Q = -k_B T \sum_i \ln\left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)$$

$$B: \quad \Phi = -k_B T \sum_i -\ln\left(1 - e^{-\beta(\varepsilon_i - \mu)}\right)$$

$$\boxed{\Phi = \mp k_B T \sum_i \ln\left(1 \pm e^{-\beta(\varepsilon_i - \mu)}\right)}$$

Allgemeines

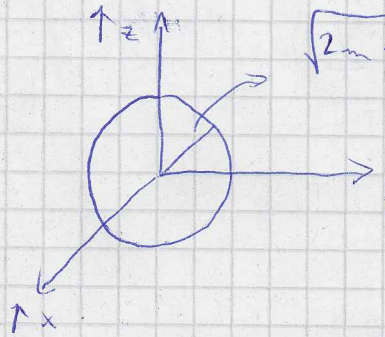


$$dN = g(E)\,dE$$

$$g(E) = \frac{d\Omega_0(E)}{dE}$$

$$H = \frac{\hbar^2}{2m} \rightarrow \varepsilon(\hbar) = \frac{\hbar^2}{2m} = \text{const} = \varepsilon$$

$$\hbar_x^2 + \hbar_y^2 + \hbar_z^2 = 2m\varepsilon$$



$g = 2s+1$ Degeneration

$$\Omega_0(\varepsilon) = \frac{g}{h^3}\int d\hbar\, d\varphi =$$

$$= \frac{g}{h^3}\cdot V \int d^3\hbar = \frac{g}{h^3} V \cdot \frac{4\pi}{3}\underbrace{\left(\sqrt{2m\varepsilon}\right)^3}_{(2m\varepsilon)^{\frac{3}{2}}}$$

$$g(\varepsilon) = \frac{d\Omega_0(\varepsilon)}{d\varepsilon} = \frac{g\cdot V}{h^3}\cdot\frac{4\pi}{3}\cdot\frac{3}{2}(2m\varepsilon)^{\frac{1}{2}}\cdot 2m = \tilde{\alpha}\cdot\sqrt{\varepsilon}$$

$$\Phi = \mp k_B T \int_0^\infty g(\varepsilon)\, d\varepsilon \, \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right)$$

Legyen $\varepsilon(\vec{p}) = \alpha \cdot |\vec{p}|^\nu$

d dim $\vec{p} = (p_1, p_2, \dots, p_d)$

Pl.: id. g.	$\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m} \Rightarrow \nu = 2,\ d = 3$	$g \sim \sqrt{\varepsilon}$
ultrarel. g.	$\varepsilon(\vec{p}) = c \cdot |\vec{p}| \Rightarrow \nu = 1,\ d = 3$	$g \sim \varepsilon^2$
grafén	$\varepsilon(\vec{p}) = c \cdot |\vec{p}| \Rightarrow \nu = 1,\ d = 2$	$g \sim \varepsilon$

$g(\varepsilon) = ?$

$$\Sigma_0(\varepsilon) = \frac{1}{h^d} \int d\vec{p}\, d\vec{q} = \frac{V(d)}{h^d} \int d^d p$$

$$\left(\frac{\varepsilon}{\alpha}\right)^{\frac{2}{\nu}} = p_1^2 + p_2^2 + \dots + p_d^2$$

fugyon: $\left(\frac{\varepsilon}{\alpha}\right)^{\frac{1}{\nu}}$

$\downarrow$

terfogat: $\frac{\pi^{\frac{d}{2}}}{\Gamma\left(1+\frac{d}{2}\right)} \cdot \left(\frac{\varepsilon}{\alpha}\right)^{\frac{d}{\nu}}$

$$\Sigma_0(\varepsilon) = g \cdot \frac{V(d)}{h^d} \cdot \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1+\frac{d}{2}\right)} \left(\frac{\varepsilon}{\alpha}\right)^{\frac{d}{\nu}}$$

$$g(\varepsilon) = \frac{d\Omega_d(\varepsilon)}{d\varepsilon} = \dots = \tilde{\mathcal{L}} \cdot \varepsilon^{\frac{d}{\nu}-1}$$

$$\tilde{\mathcal{L}} = g \cdot \frac{d}{\nu} \, V(d) \, \frac{\pi^{\frac{d}{2}}}{\Gamma\left(1+\frac{d}{2}\right)} \left(\frac{1}{\mathcal{L} R^{\nu}}\right)^{\frac{d}{\nu}}$$

$$\phi(T,V,\mu) = -pV = \mp k_B T \int_0^{\infty} g(\varepsilon) \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon$$

$$= \mp k_B T \int_0^{\infty} \tilde{\mathcal{L}} \, \varepsilon^{\frac{d}{\nu}-1} \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon$$

per-partes integrálás

$$= \mp k_B T \, \tilde{\mathcal{L}} \left\{ \left[\frac{\varepsilon^{\frac{d}{\nu}}}{\frac{d}{\nu}} \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right) \right]_0^{\infty} - \right.$$

$$\left. - \left(\mp 1\right) \beta \frac{\nu}{d} \int_0^{\infty} \frac{\varepsilon^{\frac{d}{\nu}} e^{-\beta(\varepsilon-\mu)}}{1 \pm e^{-\beta(\varepsilon-\mu)}} d\varepsilon \right\} =$$

$$= -\tilde{\mathcal{L}} \frac{\nu}{d} \int_0^{\infty} \frac{\varepsilon^{\frac{d}{\nu}}}{e^{\beta(\varepsilon-\mu)} \pm 1} d\varepsilon$$

$$E = \int g(\varepsilon) \, d\varepsilon \cdot \varepsilon \cdot n_{\pm}(\varepsilon) \longrightarrow n_{\pm}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

fels.: F

als.: B

$$= \tilde{\mathcal{L}} \int_0^{\infty} \frac{\varepsilon^{\frac{d}{\nu}-1} \cdot \varepsilon \, d\varepsilon}{e^{\beta(\varepsilon-\mu)} \pm 1}$$

$$E = \alpha \int_0^\infty \frac{\varepsilon^{\frac{d}{\nu}}}{e^{\beta(\varepsilon-\mu)} \pm 1} \, d\varepsilon$$

$$\uparrow V = \frac{\nu}{d} \cdot E$$

id. g. $\longrightarrow \uparrow V = \frac{2}{3} E$

ultrarel g. $\longrightarrow \uparrow V = \frac{1}{3} E$

grafén $\longrightarrow \uparrow V = \frac{1}{2} E$

$$N = \int_0^\infty g(\varepsilon) \, d\varepsilon \cdot n_\pm(\varepsilon)$$

Maxwellovo rozdelenie (kv. klasický)

tzn. klasick $e^{-\beta\mu} = \frac{Z_1}{N} = \frac{V}{N} \left(\frac{2\pi m k_B T}{h^2} \right)^{\frac{3}{2}}$

$e^{\beta\mu} \ll 1$

$$\frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1} = e^{-\beta(\varepsilon-\mu)} \frac{1}{1 \pm e^{-\beta(\varepsilon-\mu)}} \approx$$

$$\frac{1}{1 \pm x} = 1 \mp x + x^2 \mp x^3 + \dots = \sum_{\ell=0}^{\infty} (\mp 1)^\ell \cdot x^\ell \,, \quad x \ll 1$$

$$= e^{-\beta(\varepsilon-\mu)} \left[1 \mp e^{-\beta(\varepsilon-\mu)} \right]$$

id. g. $\varepsilon(p) = \frac{p^2}{2m}$ $\quad \nu = 2, \; d = 3$

$$g(\varepsilon) = \alpha \cdot \sqrt{\varepsilon}$$

$$N = \int_0^\infty g(\varepsilon)\, d\varepsilon \; n_\pm(\varepsilon) = \alpha \int_0^\infty \varepsilon^{\frac{1}{2}} \; e^{-\beta(\varepsilon-\mu)} \left(1 \mp e^{-\beta(\varepsilon-\mu)} \right) d\varepsilon$$

$$\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t}\, dt \qquad \Rightarrow \; \Gamma\left(\tfrac{3}{2}\right)$$

$$= \dots = g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2} \right)^{\frac{3}{2}} \cdot e^{+\beta\mu} \left[\frac{1}{\beta^{3/2}} \mp \frac{e^{\beta\mu}}{2^{\frac{3}{2}} \cdot \beta^{\frac{3}{2}}} \right] \Gamma\left(\tfrac{3}{2}\right)$$

$$\Gamma\left(\tfrac{3}{2}\right) = \tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) = \tfrac{1}{2}\sqrt{\pi}$$

$$E = \int_0^\infty g(\varepsilon)\, \varepsilon\, d\varepsilon \; n_\pm(\varepsilon) =$$

$$= \alpha \cdot \int_0^\infty \frac{\varepsilon^{\frac{1}{2}} \cdot \varepsilon \cdot d\varepsilon}{e^{\beta(\varepsilon-\mu)} \pm 1} \overset{\text{Int.}}{=} \dots =$$

$$= g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2} \right)^{\frac{3}{2}} \cdot \frac{e^{\beta\mu}}{\beta^{\frac{5}{2}}} \Gamma\left(\tfrac{5}{2}\right) \left(1 \mp \frac{e^{\beta\mu}}{2^{\frac{5}{2}}} \right) =$$

$$= N \cdot \underbrace{\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)}}_{\frac{3}{2}} \cdot \beta^{-1} \cdot \frac{1 \mp \frac{e^{\beta\Gamma}}{2^{5/2}}}{1 \mp \frac{e^{\beta\Gamma}}{2^{3/2}}} \approx$$

$$= N \cdot \frac{3}{2} \cdot k_B T \cdot \left[1 \mp \frac{e^{\beta\Gamma}}{2^{5/2}} \pm \frac{e^{\beta\Gamma}}{2^{3/2}} + \dots\right]$$

$$= \frac{3}{2} N k_B T \left[1 \pm \frac{e^{\beta\Gamma}}{2^{5/2}}\right]$$

$$N \text{ (vercela' nerdben)} \overset{\approx}{=} g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \cdot \frac{e^{\beta\Gamma}}{\beta^{3/2}}$$

$$\downarrow$$

E kifejtve

$$\downarrow$$

$$E = \frac{3}{2} N k_B T \left[1 \pm \frac{N}{gV} \frac{\beta^{\frac{3}{2}}}{2\pi\left(\frac{2m}{h^2}\right)^{\frac{3}{2}}} \cdot \frac{1}{2^{5/2}}\right]$$

$$= \frac{3}{2} N k_B T \left[1 \pm \frac{1}{2^{5/2}} \frac{N}{gV} \left(\frac{h^2}{2\pi m k_B T}\right)^{\frac{3}{2}}\right]$$

$$\searrow \quad \lambda_T^3$$

$$\lambda_T = \frac{h}{\sqrt{2\pi m k_B T}}$$

$$e^{\beta\Gamma} \ll 1$$

$$\frac{N \cdot \beta^{\frac{3}{2}}}{gV \cdot 2\pi\left(\frac{2m}{h^2}\right)^{\frac{3}{2}}} = e^{\beta\Gamma}$$

$$\frac{N}{g \cdot V} \lambda_T^3 \ll 1 \qquad \lambda_T^3 \ll \frac{g \cdot V}{N}$$

állapotegyenlet:

$$\uparrow V = \frac{2}{3} E = \frac{2}{3} \cdot \frac{3}{2} N k_B T \left(1 \pm \frac{1}{2^{5/2}} \frac{N}{gV} \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} \right)$$

vissza "nulla" $\uparrow V = N k_B T$

$$\uparrow V = N k_B T \left[1 \pm \frac{1}{2^{5/2}} \cdot \frac{\uparrow}{g k_B T} \cdot \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} \right]$$

$$C_V = \left. \frac{\partial E}{\partial T} \right|_{V,N} = \frac{3}{2} N k_B \pm \frac{3}{2} N k_B \frac{N}{gV} \frac{1}{2^{5/2}} \left(\frac{h^2}{2\pi m} \right)^{\frac{3}{2}} \left(-\frac{1}{2} \right) T^{-\frac{3}{2}}$$

$$= \frac{3}{2} N k_B \left[1 \mp \frac{N}{gV} \frac{1}{2^{7/2}} \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} + \dots \right]$$

$C_\uparrow = ?$

$$dH(S, \uparrow, N) = T\,dS + V\,d\uparrow + \mu\,dN$$

$$C_\uparrow = T \cdot \left. \frac{\partial S}{\partial T} \right|_{\uparrow,N} = \left. \frac{\partial H}{\partial T} \right|_{\uparrow,N} = \frac{5}{3} \left. \frac{\partial E}{\partial T} \right|_{\uparrow,N}$$

$$H = E + \uparrow V = E + \frac{2}{3} E = \frac{5}{3} E$$

E-t V helyett $\uparrow$-vel kell kifejezni

$$\uparrow V = N k_B T$$

vissza rend

$$\frac{N}{V} = \frac{\uparrow}{k_B T}$$

$$C_{\uparrow} = \dots = \frac{5}{2} N k_B \left[1 \mp \frac{3}{2^{5/2}} \frac{N}{g V} \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} + \dots \right]$$

$$\gamma = \frac{C_{\uparrow}}{C_V} = \frac{5}{3} \left[1 \mp \frac{1}{2^{5/2}} \cdot \frac{N}{g V} \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} \right]$$

$$\kappa_T = -\frac{1}{V} \left. \frac{\partial V}{\partial \uparrow} \right|_T$$

$$V = \frac{N k_B T}{\uparrow} \left(1 \pm \frac{1}{2^{5/2}} \left(\frac{N}{g V} \right) \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} \right)$$

$$\frac{\uparrow}{g k_B T}$$

$$\kappa_T = \frac{1}{\uparrow} \left(1 \mp \frac{1}{2^{5/2}} \cdot \frac{N}{g V} \cdot \left(\frac{h^2}{2\pi m k_B T} \right)^{\frac{3}{2}} + \dots \right)$$

$S = ?$ $\quad E = TS - \uparrow V + \mu N$ $\leftarrow$ G. d.

① $\quad \frac{5}{3} E = T \cdot S + \mu N$

$$S = \frac{\frac{5}{3} E - \mu N}{T}$$

② $d\Phi(T,V,N) = -S\,dT - p\,dV - N\,d\mu$

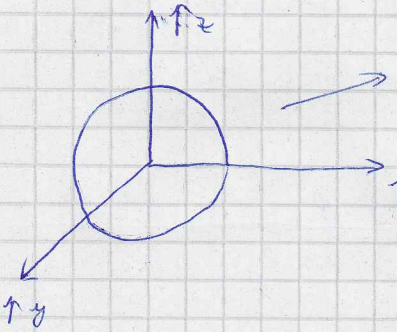
$$S = -\left.\frac{\partial \Phi}{\partial T}\right|_{V,\mu} = \frac{2}{3}\left.\frac{\partial E}{\partial T}\right|_{V,\mu}$$

II. ZH: Május 4. $12^{15} - 14^{00}$ $\dot{E} \cdot 0{,}81$ tesz

Hélis Fermi-gáz

$$g(\varepsilon) = \bar{\alpha}\sqrt{\varepsilon} \qquad \bar{\alpha} = g_s \cdot V \cdot 2\pi \cdot \left(\frac{2m}{\hbar^2}\right)^{\frac{3}{2}}$$

$$g(\varepsilon) = \frac{d\Phi_0(\varepsilon)}{d\varepsilon} \qquad \varepsilon = \frac{\hbar^2}{2m} = \frac{\hbar_x^2 + \hbar_y^2 + \hbar_z^2}{2m} \rightarrow \left(\sqrt{2m\varepsilon}\right)^2 = \hbar_x^2 + \hbar_y^2 + \hbar_z^2$$



sugara $\sqrt{2m\varepsilon}$

$$\Phi_0(\varepsilon) = \frac{g_s}{h^3}\int d_r\, dq = \frac{g_s V}{h^3}\,\frac{4\pi}{3}\left(\sqrt{2m\varepsilon}\right)^3$$

$$g(\varepsilon) = \underbrace{g_s \frac{V \cdot 2\pi (2m)^{\frac{3}{2}}}{h^3}}_{\bar{\alpha}} \sqrt{\varepsilon}$$

$$n_F(\varepsilon) = f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

$$N = \int_0^\infty g(\varepsilon)\, f(\varepsilon)\, d\varepsilon \quad \leftarrow \text{definíció } \mu\text{-re!}$$

$$E = \int_0^\infty \varepsilon \cdot g(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$\uparrow V = \frac{\nu}{\alpha} E \qquad \Leftarrow \qquad \varepsilon(\uparrow) = \alpha \cdot |\uparrow|^\nu$$

$\uparrow$: α dim vektor

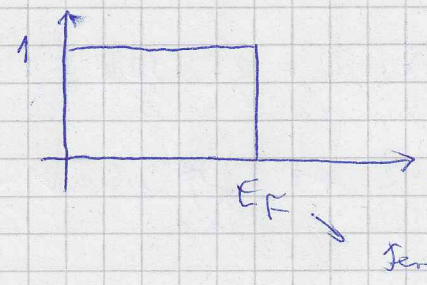
id. Fermi-gáz

$\nu = 2$

$\alpha = 3$

$$-\phi = \uparrow V = \frac{2}{3} E$$

$T = 0$



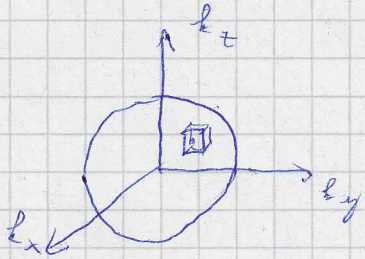
Fermi-energia

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon \overset{T=0}{=} \int_0^{E_F} g(\varepsilon)\, d\varepsilon = \tilde{\alpha} \int_0^{E_F} \sqrt{\varepsilon}\, d\varepsilon =$$

$$= \tilde{\alpha} \cdot \frac{2}{3} \cdot E_F^{3/2} = g_s \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{\frac{3}{2}} \cdot \frac{2}{3} E_F^{3/2}$$

$$E_F = \frac{\uparrow_F^2}{2m} \qquad \uparrow_F = \sqrt{2m E_F} \qquad \uparrow = \hbar \underline{k}$$

$$\uparrow_F = \hbar k_F$$



$$N = g_s \sum_{\underline{k}} 1 = g_s \cdot \frac{V}{(2\pi)^3} \int d^3k$$

k_F sugarú gömb

$$(\Delta k)^3 = \frac{(2\pi)^3}{V}$$

$$N = g_s \cdot \frac{V}{6\pi^2} k_F^3 \quad \rightarrow \quad k_F = \left(\frac{N}{g_s V} 6\pi^2 \right)^{\frac{1}{3}}$$

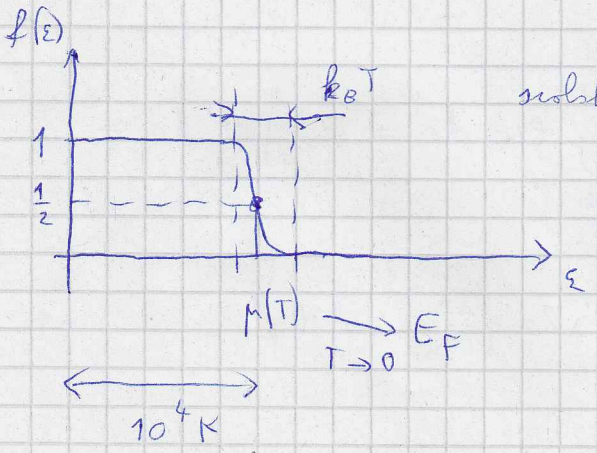
$$E_F = \frac{\hbar^2 k_F^2}{2m} = \frac{\hbar^2}{2m} \left(\frac{N}{g_s V} 6\pi^2 \right)^{\frac{2}{3}}$$

Sűrűségből: $E_F = \left[\frac{N}{g_s V} \cdot \frac{1}{2\pi \left(\frac{2m}{\hbar^2} \right)^{\frac{3}{2}}} \frac{3}{2} \right]^{\frac{2}{3}}$ → ugyanaz

Na: 1 vegyértékű $\frac{N}{V} = 2{,}5 \cdot 10^{28} \frac{1}{m^3}$ $g_s = 2$

$E_F = 2{,}4\,eV = k_B T_F \Rightarrow T_F = 27830\,K \approx 10^4\,K$

$m = 1{,}3\,m_e$, $g_s = 2 \Rightarrow E_F \approx 5{,}4\,eV \rightarrow T_F \sim 63000\,K$



szobában $T \sim 300\,K$

$$E = \int \varepsilon\, g(\varepsilon)\, f(\varepsilon)\, d\varepsilon$$

$T = 0$: $$E = \int_0^{E_F} \varepsilon \cdot g(\varepsilon)\, d\varepsilon$$

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon - \mu)} + 1} \qquad f(\mu) = \frac{1}{2}$$

$$E = \tilde{\lambda} \int_0^{E_F} \varepsilon \sqrt{\varepsilon}\, d\varepsilon = \tilde{\lambda} \cdot \frac{2}{5} E_F^{5/2} = g_s V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot \frac{2}{5} E_F^{5/2}$$

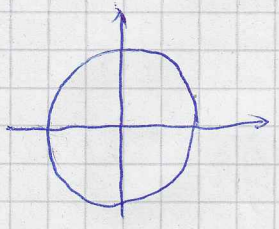
$$\uparrow V = \frac{2}{3} E \qquad n = \frac{2}{3}\frac{E}{V} = \frac{4}{15} g_s 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot E_F^{5/2}$$

E_F-et keresztül fejez $\frac{V}{N}$-tel

2 dimenzió, $\varepsilon(\uparrow) = \frac{\uparrow_x^2 + \uparrow_y^2}{2m}$

$g(\varepsilon) = ?$

$n(T) = ?$

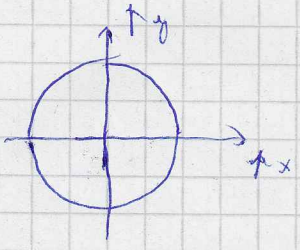


$$\left(\sqrt{2m\varepsilon}\right)^2 = \uparrow_x^2 + \uparrow_y^2$$

$$\Omega_0(E) = g_s \frac{1}{h^2} \int d^2\uparrow\, d^2q = g_s \frac{A}{h^2} \int d^2\uparrow = g_s \frac{A}{h^2} \cdot \left(\sqrt{2m\varepsilon}\right)^2 \pi$$

$$= g_s \frac{A}{h^2} 2m\varepsilon \cdot \pi \qquad g(\varepsilon) = \frac{d\Omega_0(\varepsilon)}{d\varepsilon} = g_s \frac{\pi A}{h^2} 2m = \text{konst}$$

Grafén $\varepsilon(\uparrow) = v \cdot |\uparrow| \qquad d = 2$



$$\frac{\varepsilon}{v} = |\uparrow| = \sqrt{\uparrow_x^2 + \uparrow_y^2}$$

$$g(\varepsilon) = 2D \qquad N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon = 2D \int_0^\infty \frac{d\varepsilon}{e^{\beta(\varepsilon-\mu)}+1}$$

$$x = e^{\beta(\varepsilon-\mu)} \rightarrow dx = \underbrace{e^{\beta(\varepsilon-\mu)}}_{x} \beta\, d\varepsilon \qquad d\varepsilon = \frac{dx}{\beta x}$$

$$N = \frac{2D}{\beta} \int_{x_0}^{\infty} \frac{dx}{x(x+1)} = \frac{2D}{\beta} \left[\int_{x_0}^{\infty} \frac{dx}{x} - \int_{x_0}^{\infty} \frac{dx}{x+1} \right]$$

$$x_0 = e^{-\beta\mu}$$

$$= \frac{2D}{\beta} \left(\ln x - \ln(x+1) \right]_{x_0}^{\infty} = \frac{2D}{\beta} \left[\ln \frac{x}{x+1} \right]_{x_0}^{\infty} =$$

$$= \frac{2D}{\beta} \left[\ln \left(\frac{1}{1+e^{-\beta(\varepsilon-\mu)}} \right) \right]_0^{\infty} = -\frac{2D}{\beta} \ln \left(\frac{1}{1+e^{\beta\mu}} \right)$$

$$= \frac{2D}{\beta} \ln \left(1 + e^{\beta\mu} \right)$$

$$\mu(T) = \frac{1}{\beta} \ln \left(e^{\frac{N}{k_B T \cdot 2D}} - 1 \right)$$

$$E = \int_0^{\infty} \varepsilon \cdot g(\varepsilon)\, d\varepsilon\, f(\varepsilon) = 2D \int_0^{\infty} \frac{\varepsilon \cdot d\varepsilon}{e^{\beta(\varepsilon-\mu)} + 1}$$

Bethe – Sommerfeld – sorfejtés (B-S)

$$\int_0^\infty g(\varepsilon)\, f(\varepsilon)\, d\varepsilon = \varphi(\mu) + \frac{\pi^2}{6}(k_B T)^2 \cdot \varphi''(\mu) + O(T^4)$$

tetszőleges $\qquad f(\varepsilon) = \dfrac{1}{e^{\beta(\varepsilon-\mu)}+1}$

$$\varphi(\mu) = \int_0^\mu g(\varepsilon)\, d\varepsilon$$

$$= \int_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \cdot \left.\frac{d\, g(\varepsilon)}{d\varepsilon}\right|_{\varepsilon=\mu}$$

thermodyn. sorfejtés (fermi-gázra): B-S

$$\varepsilon(\vec{k}) = \frac{\hbar^2 \vec{k}^2}{2m} \qquad \vec{k} = \begin{pmatrix} k_x \\ k_y \\ k_z \end{pmatrix} \qquad k V = \frac{2}{3} E$$

$$g(\varepsilon) = \underbrace{\alpha V \cdot g_s \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{\frac{3}{2}}}_{A} \cdot \sqrt{\varepsilon} = A \cdot V \cdot \sqrt{\varepsilon}$$

$$E = \int_0^\infty \varepsilon\, g(\varepsilon)\, f(\varepsilon)\, d\varepsilon = \int_0^\infty \varepsilon\, A V \sqrt{\varepsilon}\, f(\varepsilon)\, d\varepsilon =$$

$$= A \cdot V \cdot \int_0^\infty \underbrace{\varepsilon^{3/2}}_{g(\varepsilon)} \cdot f(\varepsilon)\, d\varepsilon \overset{B-5}{=}$$

$$= A \cdot V \left[\int_0^{\mu(T)} \varepsilon^{3/2}\, d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \left. \frac{d\varepsilon^{3/2}}{d\varepsilon} \right|_{\varepsilon=\mu} \right] =$$

$$= A \cdot V \left[\frac{2}{5} \mu^{5/2} + \frac{\pi^2}{6} (k_B T)^2 \cdot \frac{3}{2} \sqrt{\mu} \right] =$$

$$= \frac{2}{5} A \cdot V \cdot \mu^{5/2} \left[1 + \frac{5}{8} \pi^2 \left(\frac{k_B T}{\mu} \right)^2 \right] = E$$

$\mu(T) = ?$ $\quad$ N definíció

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon = \int_0^\infty A \cdot V \cdot \sqrt{\varepsilon}\, f(\varepsilon)\, d\varepsilon \overset{B-5}{=}$$

$$= A \cdot V \left[\int_0^\mu \sqrt{\varepsilon}\, d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \cdot \left. \frac{d\sqrt{\varepsilon}}{d\varepsilon} \right|_{\varepsilon=\mu} \right] =$$

$$= A \cdot V \left[\frac{2}{3} \mu^{3/2} + \frac{\pi^2}{6} (k_B T)^2 \cdot \frac{1}{2} \cdot \mu^{-\frac{1}{2}} \right] =$$

$$= \frac{2}{3} A \cdot V \cdot \mu^{3/2} \left[1 + \frac{\pi^2}{6} (k_B T)^2 \cdot \frac{3}{2} \cdot \frac{1}{\mu^2} \right] =$$

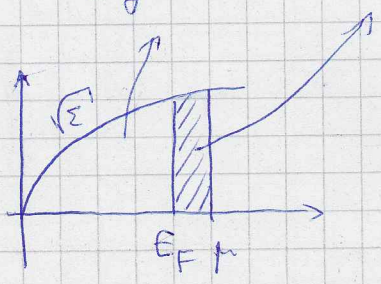
$$= \frac{2}{3} A \cdot V \cdot \mu^{3/2} \left[1 + \frac{\pi^2}{8} \cdot \left(\frac{k_B T}{\mu} \right)^2 \right]$$

Elolyett csúnyuk egy trükkel!

inner folyt

$$N \overset{\text{S-S}}{=} A \cdot V \left[\int_0^{E_F} \sqrt{\varepsilon}\, d\varepsilon + (\mu - E_F)\sqrt{E_F} + \frac{\pi^2}{12} \cdot \frac{(k_B T)^2}{\sqrt{\mu}} \right]$$

$\uparrow$ E_F



$$N = A \cdot V \left[\frac{2}{3} E_F^{3/2} + (\mu - E_F)\sqrt{E_F} + \frac{\pi^2}{12} \frac{(k_B T)^2}{\sqrt{E_F}} \right]$$

$\| N$

$$0 = (\mu - E_F)\sqrt{E_F} + \frac{\pi^2}{12}(k_B T)^2 \frac{1}{\sqrt{E_F}}$$

$T = 0$:

$$N = \int_0^{E_F} g(\varepsilon)\, d\varepsilon = A \cdot V \int_0^{E_F} \sqrt{\varepsilon}\, d\varepsilon$$

$$\boxed{\mu(T) = E_F \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2 \right]}$$

$$E = \frac{2}{5} A \cdot V \cdot \mu^{5/2} \left[1 + \frac{5\pi^2}{8} \left(\frac{k_B T}{\mu} \right)^2 \right]$$

$\uparrow$ $\mu = E_F$

$$E = \frac{2}{5} A \cdot V \cdot E_F^{5/2} \left[1 - \frac{\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2 \right]^{5/2} \cdot \left[1 + \frac{5\pi^2}{8} \left(\frac{k_B T}{E_F} \right)^2 \right]$$

$$E = \frac{2}{5} A \cdot V \cdot E_F^{5/2} \left[1 - \frac{5\pi^2}{24}\left(\frac{k_B T}{E_F}\right)^2\right] \cdot \left[1 + \frac{5\pi^2}{8}\cdot\left(\frac{k_B T}{E_F}\right)^2\right] =$$

$$= \frac{2}{5} A \cdot V \cdot E_F^{5/2} \left[1 + \frac{5\pi^2}{12}\cdot\left(\frac{k_B T}{E_F}\right)^2\right]$$

$$\boxed{N = \frac{2}{3} A \cdot V \cdot E_F^{3/2}}$$

$$\boxed{E = \frac{3}{5} N \cdot E_F \cdot \left[1 + \frac{5\pi^2}{12}\left(\frac{k_B T}{E_F}\right)^2\right]} \qquad k_B T_F = E_F$$

$$\uparrow V = \frac{2}{3} E = \frac{2}{3} N E_F \cdot \left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$

$$\boxed{\uparrow = \frac{2}{5} k_B T_F \cdot \frac{N}{V}\left[1 + \frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]}$$

Hőtágulás: $$\alpha = \frac{1}{V}\left.\frac{\partial V}{\partial T}\right|_{\uparrow, N} = \dots = \frac{5\pi^2}{6}\cdot\frac{T}{T_F^2}$$

Entrópia $$d\phi(T, V, \mu) = -S\,dT - \uparrow dV - N\,d\mu$$

$$\Rightarrow S = -\left.\frac{\partial \phi}{\partial T}\right|_{V,\mu} = \frac{2}{3}\left.\frac{\partial E}{\partial T}\right|_{V,\mu} = \dots = \frac{\pi^2}{2} N k_B \cdot \frac{T}{T_F}$$

$$\phi = -\uparrow V = -\frac{2}{3} E$$

Fajhő $$C_V = T \cdot \left.\frac{\partial S}{\partial T}\right|_{V,N} = \frac{\pi^2}{2} N k_B \cdot \frac{T}{T_F}$$

$$C_\uparrow = T \cdot \left.\frac{\partial S}{\partial T}\right|_{\uparrow, N} = \left.\frac{\partial H}{\partial T}\right|_{\uparrow, N}$$

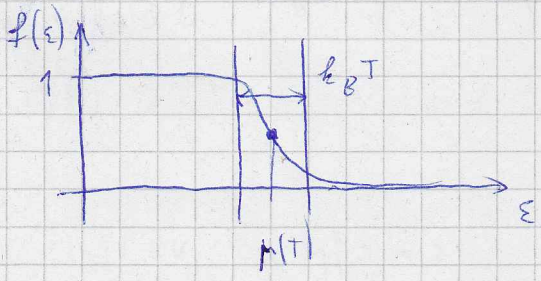
$$H = E + pV = \frac{5}{3}E$$

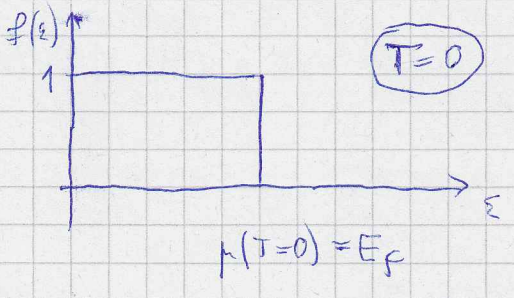
$$C_T = \frac{5}{3}\left.\frac{\partial E}{\partial T}\right|_{p,N} = \frac{5}{2}\cdot p\cdot\left.\frac{\partial V}{\partial T}\right|_{p,N} = \frac{5\pi^2}{6}\,N k_B\cdot\frac{T}{T_F}$$

$$\kappa_T = -\frac{1}{V}\cdot\left.\frac{\partial V}{\partial p}\right|_{T,N} = \dots = \frac{5}{2}\frac{V}{N}\cdot\frac{1}{E_F}\cdot\left[1-\frac{5\pi^2}{12}\left(\frac{T}{T_F}\right)^2\right]$$

Fermi - Dirac st.

$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$





$$N = \int_0^\infty g(\varepsilon)\,f(\varepsilon)\,d\varepsilon$$

B-S:

$$\int_0^\infty g(\varepsilon)\,f(\varepsilon)\,d\varepsilon = \int_0^\mu g(\varepsilon)\,d\varepsilon + \frac{\pi^2}{6}(k_B T)^2\cdot\left.\frac{dg}{d\varepsilon}\right|_{\varepsilon=\mu}$$

$$N \overset{B\text{-}S}{=\!=} \int_0^{\mu} g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \cdot g'(\mu) =$$

$$= \underbrace{\int_0^{E_F} g(\varepsilon)\, d\varepsilon}_{N} + (\mu - E_F)\, g(E_F) + \frac{\pi^2}{6}(k_B T)^2\, g'(E_F)$$

$$\mu = E_F - \frac{\pi^2}{6}(k_B T)^2 \cdot \frac{g'(E_F)}{g(E_F)} = E_F - \frac{\pi^2}{6}(k_B T)^2$$

$$E = \int \underbrace{\varepsilon g(\varepsilon)}_{g(\varepsilon)} f(\varepsilon)\, d\varepsilon \overset{B\text{-}S}{=\!=} \int_0^{\mu} \varepsilon g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \left.\frac{d(\varepsilon g(\varepsilon))}{d\varepsilon}\right|_{\varepsilon=\mu}$$

$\frac{E}{N}$ — pl.: ultrarel. gáz $\quad \varepsilon = c|\vec{p}| \quad \nu = 1 \quad d = 3$

$$\uparrow V = \frac{\nu}{d} E \Rightarrow \uparrow V = \frac{1}{3} E$$

$$\Omega_0(E) = \frac{g}{h^3} \int d^3r\, d^3q = g \cdot \frac{V}{h^3} \int d^3\vec{p} =$$

$$= g \cdot \frac{V}{h^3} \cdot \frac{4\pi}{3} \left(\frac{\varepsilon}{c}\right)^3 \sim \varepsilon^3$$

$$g(\varepsilon) = \frac{d\Omega_0(\varepsilon)}{d\varepsilon} \sim \varepsilon^2$$

$$\Omega_0(\varepsilon) = g \sum_{\vec{k}} 1 = g \cdot \frac{V}{(2\pi)^3} \int d^3k = \qquad \vec{p} = \hbar \vec{k}$$

$$= g \cdot \frac{V}{\hbar^3} \int_{|\vec{p}| \le \varepsilon/c} d^3p = g \cdot \frac{V}{\hbar^3} \int_0^{\varepsilon/c} 4\pi p^2 \, dp =$$

$$= g \cdot \frac{V}{\hbar^3} \, 4\pi \cdot \frac{(\varepsilon/c)^3}{3}$$

$E_F \sim 10^4 K \sim eV$

$T = 300 K \approx 25 \, meV$

$$\Phi(T, V, \mu) = -pV = -k_B T \ln Q$$

↑ G-D

$N, E, pV = \frac{2}{3} E, S, C_V, \kappa_T$

↑

$\varepsilon = \frac{p^2}{2m}$ $\qquad d\Phi = -S\,dT - p\,dV - N\,d\mu$

$$S = -\left.\frac{\partial \Phi}{\partial T}\right|_{V,\mu} \qquad C_V = T \cdot \left.\frac{\partial S}{\partial T}\right|_{V,N}$$

$$\kappa_T = -\frac{1}{V} \cdot \left.\frac{\partial V}{\partial p}\right|_{T,N}$$

$$S = -\left.\frac{\partial(-pV)}{\partial T}\right|_{V,\mu} = \frac{2}{3} \left.\frac{\partial E}{\partial T}\right|_{V,\mu}$$

ELMF12 III. 13.12

$\mu(T_1) = 0$ Fermiokaasu, $T_1 = ?$

$$\varepsilon(p) = \frac{p^2}{2m} \rightarrow g(\varepsilon) = g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot \sqrt{\varepsilon}$$

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon = \int_0^\infty \frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}+1}\, d\varepsilon =$$

$$= \alpha \cdot V \int \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\beta\varepsilon}+1} =$$

$\mu(T_1) = 0$ $\qquad \beta = \frac{1}{k_B T_1}$

$$\beta\varepsilon = x^2$$

$$d\varepsilon = \frac{2x\, dx}{\beta}$$

$$= \alpha \cdot V \int \frac{x/\sqrt{\beta} \cdot 2x\, dx/\beta}{e^{x^2}+1} = \alpha \cdot V \cdot 2 \cdot (k_B T_1)^{3/2} \cdot \int_0^\infty \frac{x^2}{e^{x^2}+1} \cdot dx$$

$$N = \int_0^{E_F} g(\varepsilon)\, d\varepsilon = \alpha \cdot V \int_0^{E_F} \sqrt{\varepsilon}\, d\varepsilon = \alpha \cdot V \cdot \frac{2}{3} \cdot E_F^{3/2} =$$

$$= \alpha \cdot V \, \frac{2}{3} (k_B T_F)^{3/2}$$

$$1 = \left(\frac{T_F}{T_1}\right)^{3/2} \cdot \frac{1}{3} \cdot \frac{1}{\int_0^\infty \frac{x^2}{e^{x^2}+1}\, dx} \Rightarrow \frac{T_F}{T_1} = \left[3 \int_0^\infty \frac{x^2\, dx}{e^{x^2}+1}\right]^{\frac{2}{3}}$$

$$\frac{2-\sqrt{2}}{8}\sqrt{\pi} \cdot \zeta\left(\frac{3}{2}\right)$$

$$\zeta(x) = \sum_{k=1}^{\infty} \frac{1}{k^x}$$

$$\frac{T_F}{T_1} \simeq 1{,}01139$$

Bose - Einstein statistik

Fristår: $Q = \sum_{\{n_i\}} e^{-\beta\left(\sum_i \varepsilon_i n_i - \mu \sum_i n_i\right)} =$

Fermion $n_i = 0, 1$

Boson $n_i = 0, 1, 2, \ldots$

$$= \sum_{\{n_i\}} e^{-\beta \sum_i (\varepsilon_i - \mu) n_i} = \sum_{\{n_i\}} \prod_i e^{-\beta(\varepsilon_i - \mu)}$$

B-E

$$= \sum_{n_1=0}^{\infty} e^{-\beta(\varepsilon_1 - \mu) n_1} \cdot \sum_{n_2=0}^{\infty} e^{-\beta(\varepsilon_2 - \mu) n_2} \cdot \ldots =$$

$$= \prod_i \sum_{n_i=0}^{\infty} e^{-\beta(\varepsilon_i - \mu) n_i} = \prod_i \frac{1}{1 - e^{-\beta(\varepsilon_i - \mu)}}$$

F-D

$$= \prod_i \left(1 + e^{-\beta(\varepsilon_i - \mu)}\right)$$

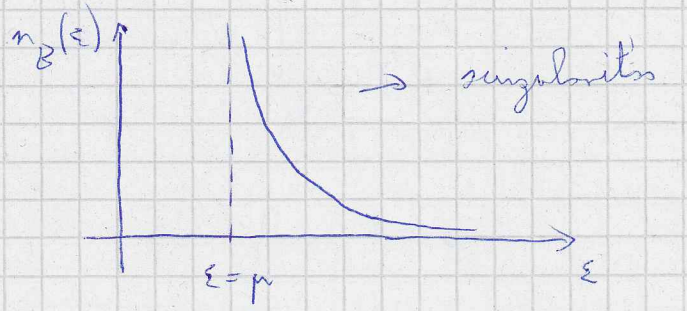
$$\Phi = \mp k_B T \sum_i \ln\left(1 \pm e^{-\beta(\varepsilon_i - \mu)}\right)$$

felső: Fermion

alsó: Bozon

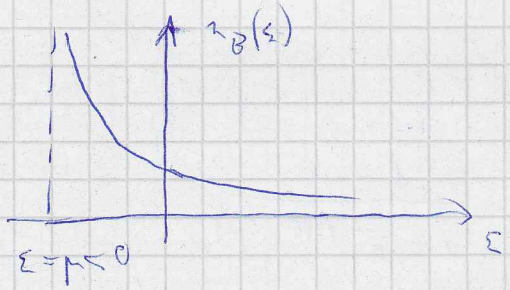
$$N = -\left.\frac{\partial \Phi}{\partial \mu}\right|_{T,V} = \dots = \sum_i \frac{1}{e^{\beta(\varepsilon_i - \mu)} \pm 1}$$

$$n_{B-E}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} \qquad n_{F-D}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}$$



Tfh. $\min \varepsilon(p) = 0$

Pl.: $\varepsilon(p) = \frac{p^2}{2m}$ $\quad \varepsilon(p=0) = 0$



$\mu < 0$ → elkerüljük a szingularitást

$$g(\varepsilon) = g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot \sqrt{\varepsilon}$$

$$= \alpha \cdot V \cdot \sqrt{\varepsilon}$$

$$N = \int_0^\infty g(\varepsilon)\, n_B(\varepsilon)\, d\varepsilon = \alpha \cdot V \int_0^\infty \frac{\sqrt{\varepsilon}\, d\varepsilon}{e^{\beta(\varepsilon-\mu)} - 1} =$$

$$= \alpha \cdot V \cdot \left(\frac{1}{\beta}\right)^{3/2} \int_0^\infty \frac{dx \cdot x^{1/2}}{e^{x+\alpha} - 1} = g \cdot V \cdot 2\pi \cdot \left(\frac{2m k_B T}{h^2}\right)^{3/2} \cdot \int_0^\infty \frac{x^{1/2}\, dx}{e^{x+\alpha} - 1}$$

$x = \beta\varepsilon \qquad \alpha = -\beta\mu > 0$

$dx = \beta\, d\varepsilon$

$$N = g \cdot V \cdot 2\pi \left(\frac{2m k_B T}{h^2}\right)^{3/2} \cdot \Gamma\left(\frac{3}{2}\right) F^{(-)}\left(\frac{3}{2}, \alpha\right), \text{ ahol}$$

$$F^{(\pm)}(\gamma, \alpha) = \frac{1}{\Gamma(\gamma)} \int_0^\infty \frac{x^{\gamma-1}\, dx}{e^{x+\alpha} \pm 1}$$

$-$: B-E, bozon

$+$: F-D, fermion

① $F^{(-)}\left(\frac{3}{2}, 0\right) = \zeta\left(\frac{3}{2}\right) = 2{,}612$

② $F^{(-)}(\gamma, \alpha)$ sorfejtése

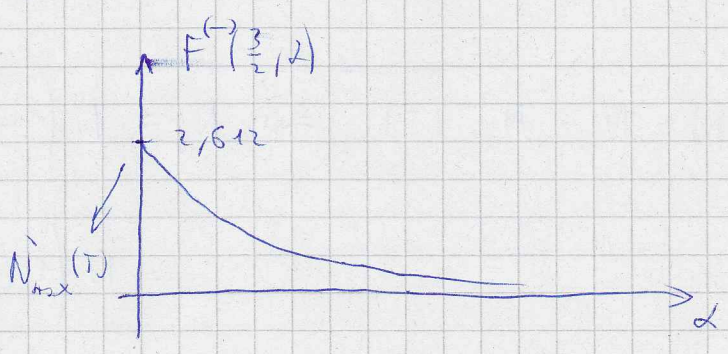
$$F^{(-)}(\gamma, \alpha) \cong \Gamma(1-\gamma)\, \alpha^{\gamma-1} + \zeta(\gamma)$$

γ nem pozitív egész

③ $$\frac{\partial^n F^{(\pm)}(\gamma, \alpha)}{\partial \alpha^n} = (-1)^n F^{(\pm)}(\gamma - n, \alpha)$$

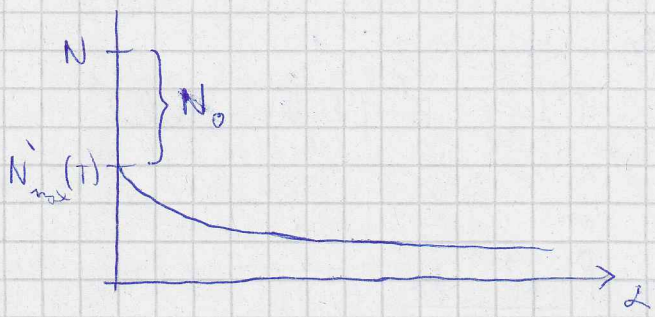
rögzített $T > T_c$

$\zeta\left(\frac{3}{2}\right) = F^{(-)}\left(\frac{3}{2}, 0\right)$



$N'_{max}(T)$: adott T-nél a $\varepsilon \neq 0$ állapotok max száma

*) → Bőv. alább



$N_0 = N - N'_{max}(T)$ $\varepsilon = 0$ állapotban lévő részecskék száma

$T = T_c \Rightarrow N_0(T = T_c) = 0 \Rightarrow N = N'_{max}(T_c) =$

$$= g \cdot V \cdot 2\pi \left(\frac{2m k_B T_c}{h^2}\right)^{3/2} \cdot \Gamma\left(\frac{3}{2}\right) \cdot \zeta\left(\frac{3}{2}\right)$$

$$\boxed{\frac{N}{V} = g \cdot 2\pi \cdot \left(\frac{2m k_B T_c}{h^2}\right)^{3/2} \cdot \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right)}$$

$T < T_c$ $N_0 > 0$ makroszkopikus

$\varepsilon = 0$ állapotban N_0 részecske kondenzálódik!

$$T_c\left(\frac{V}{N}, m\right)$$

$$N = N_0 + N'_{max}(T) \Rightarrow N_0 = N\left(1 - \frac{N'_{max}(T)}{N}\right) =$$

$$= N\left(1 - \left(\frac{T}{T_c}\right)^{3/2}\right)$$

*) $N_{ex}(T) = g \cdot V \cdot 2\pi \cdot \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \cdot \Gamma\left(\frac{3}{2}\right) \varphi\left(\frac{3}{2}\right)$

Hogyan változik μ T_c körül

$T < T_c \Rightarrow \mu = 0$

$T \geq T_c \Rightarrow \mu < 0$

$$N = g \cdot V \cdot 2\pi \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \cdot F^{(-)}\left(\frac{3}{2}, \alpha\right) \Gamma\left(\frac{3}{2}\right)$$

$$N = g \cdot V \cdot 2\pi \left(\frac{2\pi m k_B T_c}{h^2}\right)^{3/2} F^{(-)}\left(\frac{3}{2}, 0\right) \Gamma\left(\frac{3}{2}\right)$$

$$\left(\frac{T}{T_c}\right)^{\frac{3}{2}} = \frac{F^{(-)}\left(\frac{3}{2}, 0\right)}{F^{(-)}\left(\frac{3}{2}, \alpha\right)}$$

$$\left(\frac{T_c}{T}\right)^{\frac{3}{2}} = \frac{F^{(-)}\left(\frac{3}{2}, \alpha\right)}{F^{(-)}\left(\frac{3}{2}, 0\right)} \underset{\text{sorfejtés}}{\overset{\alpha \ll 1}{=}} \frac{\Gamma(1-s)\alpha^{1-1} + \varphi(1)}{\varphi\left(\frac{3}{2}\right)} = \quad \left(1 = \frac{3}{2}\right)$$

$$= 1 + \Gamma\left(-\frac{1}{2}\right) \alpha^{1/2}$$

$\Gamma\left(-\frac{1}{2}\right) = ?$ $\Gamma(1+x) = x\,\Gamma(x)$

$x = -\frac{1}{2}$

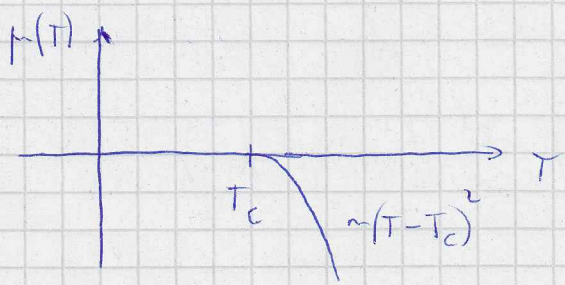
$$\frac{\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}} = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right) \qquad \Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$\frac{T_c}{T} = \left[1 - \frac{2\sqrt{\pi}\sqrt{\alpha}}{\zeta\left(\frac{3}{2}\right)}\right]^{2/3} \approx 1 - \frac{4\sqrt{\pi}\sqrt{\alpha}}{3\,\zeta\left(\frac{3}{2}\right)}$$

$$1 - \frac{T}{T_c} = \frac{4\sqrt{\pi}\sqrt{\alpha}}{3\,\zeta\left(\frac{3}{2}\right)} \Rightarrow \mu = -\frac{\alpha}{\beta} = k_B T_c \left(\frac{3\,\zeta\left(\frac{3}{2}\right)}{4\sqrt{\pi}}\right)^2 \cdot \left(1 - \frac{T}{T_c}\right)^2$$

$$\beta \approx \frac{1}{k_B T_c}$$

$$\mu \sim -(T - T_c)^2$$



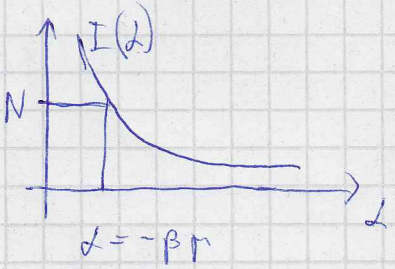
$d = 1$
$d = 2$
} nincs BE-kondenzáció

$d = 1$ $\quad g(\varepsilon) \sim \frac{1}{\sqrt{\varepsilon}}$

$d = 2$ $\quad g(\varepsilon) \sim$ const.

$$d = 1: \quad \frac{N}{L} = \left(\frac{2\pi k_B T}{h^2}\right)^{1/2} \int_0^\infty \frac{x^{-\frac{1}{2}}\,dx}{e^{x+\alpha} - 1}$$

$$d = 2: \quad \frac{N}{A} = 2\pi \cdot \frac{m k_B T}{h^2} \int_0^\infty \frac{dx}{e^{x+\alpha} - 1}$$



$I(\alpha)$ divergál,
nincs kondenzáció

$$\Phi = -pV = -k_B T \ln Q = \mp g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot k_B T$$

$$\cdot \int_0^\infty \sqrt{\varepsilon}\, \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right) d\varepsilon$$

$x = \beta\varepsilon \qquad \alpha = -\beta\mu$

$$pV = \pm g \cdot k_B T \cdot V \cdot 2\pi \cdot \left(\frac{2m k_B T}{h^2}\right)^{3/2} \cdot \int_0^\infty dx \cdot x^{1/2} \cdot \ln\left(1 \pm e^{-x-\alpha}\right)$$

$$= \cdots \overset{\text{parc. int}}{=\!=} \frac{2}{3} g V \cdot k_B T \cdot 2\pi \left(\frac{2m k_B T}{h^2}\right)^{3/2} \cdot \int_0^\infty \frac{x^{3/2}\, dx}{e^{x+\alpha} \pm 1}$$

$$= \frac{2}{3} g \cdot V \cdot 2\pi \cdot (k_B T)^{3/2} \left(\frac{2m}{h^2}\right)^{3/2} \cdot \Gamma\left(\frac{5}{2}\right) F^{(\pm)}\left(\frac{5}{2}, \alpha\right)$$

$$N = gV \cdot 2\pi (k_B T)^{3/2}\, \Gamma\left(\frac{3}{2}\right) F^{(\pm)}\left(\frac{3}{2}, \alpha\right) \cdot \left(\frac{2m}{h^2}\right)^{3/2}$$

$$\frac{E}{N} = \frac{\frac{3}{2} pV}{N} = \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} \frac{F^{(\pm)}\left(\frac{5}{2}, \alpha\right)}{F^{(\pm)}\left(\frac{3}{2}, \alpha\right)} k_B T =$$

$$= \frac{3}{2} k_B T \cdot \frac{F^{(\pm)}\left(\frac{5}{2}, \alpha\right)}{F^{(\pm)}\left(\frac{3}{2}, \alpha\right)}$$

$\alpha \to \infty \qquad$ Langados $\to 1$

$$pV = \frac{2}{3} E \Rightarrow pV = N k_B T \cdot \frac{F^{(\pm)}\left(\frac{5}{2}, \alpha\right)}{F^{(\pm)}\left(\frac{3}{2}, \alpha\right)}$$

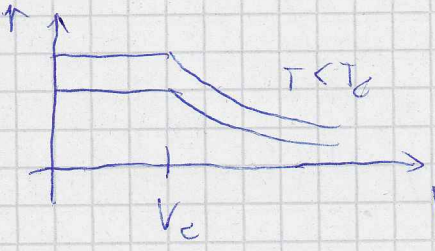
Ha Bose és $T < T_c$, $\mu = 0 \Rightarrow \alpha = 0$

$$F^{(-)}\left(\frac{5}{2}, 0\right) = \varphi\left(\frac{5}{2}\right)$$

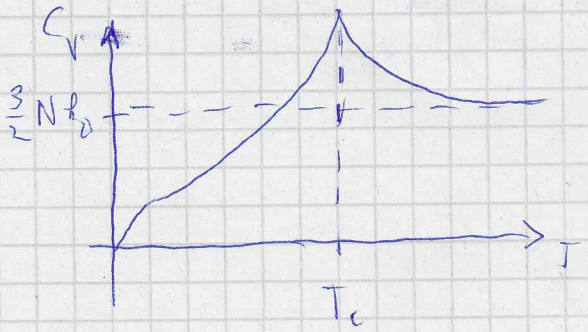
$$pV = N k_B T \cdot \frac{F^{(-)}\left(\frac{5}{2}, 0\right)}{F^{(-)}\left(\frac{3}{2}, 0\right)} \sim V$$

$$N = g \cdot V \cdot 2\pi \left(\frac{2m k_B T_c}{h^2}\right)^{3/2} \cdot F^{(-)}\left(\frac{3}{2}, 0\right)$$

p = const V-hez, ha $T < T_c$



$$S = -\left.\frac{\partial \Phi}{\partial T}\right|_{V,N} = \left.\frac{\partial (pV)}{\partial T}\right|_{V,N} = \dots$$



II. rendű fázisátalakulás

$\frac{\partial^2 \Phi}{\partial T^2}$ szingulális

Ismétlés: $\Phi(T,V,\mu) = -k_B T \ln Q = \dots =$

$-pV$

$$= \mp k_B T \cdot \sum_i \ln\left(1 \pm e^{-\beta(\varepsilon_i - \mu)}\right)$$

felső: fermion

alsó: boson

$$\doteq \mp k_B T \cdot \sum_i \int g(\varepsilon)\, d\varepsilon \ln\left(1 \pm e^{-\beta(\varepsilon-\mu)}\right)$$

a) Magasabb. sorfejtés (kv. korrekció)

$T \to \infty$, $e^{\beta\mu} \ll 1$ sorfejtési paraméter

$$\frac{1}{e^{\beta(\varepsilon-\mu)} \pm 1} = \frac{e^{-\beta(\varepsilon-\mu)}}{1 \pm e^{-\beta(\varepsilon-\mu)}} \approx$$

$$\approx e^{-\beta(\varepsilon-\mu)} \cdot \sum_{\ell=1}^{\infty} (\mp 1)^{\ell} \cdot e^{\beta\ell(\varepsilon-\mu)}$$

$\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m}$ szabad id. kv. gáz

$$g(\varepsilon) = g \cdot V \cdot \left(\frac{2\pi m}{h^2}\right)^{3/2} \cdot \sqrt{\varepsilon}$$

alt $\varepsilon(\vec{p}) = |\vec{p}|^{\nu}$ $\vec{p} = (p_1, \dots, p_d)$

$pV = \frac{\nu}{d} E$ $g(\varepsilon) = C \cdot \varepsilon^{\frac{d}{\nu}-1}$

$$N = \int_0^{\infty} g(\varepsilon)\, n(\varepsilon)\, d\varepsilon \longrightarrow \mu(T)$$

$$E = \int_0^{\infty} g(\varepsilon)\, \varepsilon \cdot n(\varepsilon)\, d\varepsilon$$

$$n(\varepsilon) = \begin{cases} \dfrac{1}{e^{\beta(\varepsilon-\mu)} - 1}, & \text{boson} \\ \dfrac{1}{e^{\beta(\varepsilon-\mu)} + 1}, & \text{fermion} \end{cases}$$

$$\frac{E}{N} = \frac{3}{2} k_B T + \dots$$

$$pV = \frac{2}{3} E \quad \text{Zust. gg.}$$

$$d\Phi = -S\,dT - p\,dV - N\,d\mu$$

$$S = ? = -\left.\frac{\partial \Phi}{\partial T}\right|_{V,\mu} = \frac{2}{3}\left.\frac{\partial E}{\partial T}\right|_{V,\mu}$$

$$\Phi = -pV = -\frac{2}{3}E$$

$$E = TS - pV + \mu N \rightarrow S = \frac{E + pV - \mu N}{T} =$$

$$= \frac{\frac{5}{2}pV - \mu N}{T} = \frac{\frac{5}{3}E - \mu N}{T}$$

$$C_V = T \cdot \left.\frac{\partial S}{\partial T}\right|_{V,N} \qquad C_p = T \cdot \left.\frac{\partial S}{\partial T}\right|_{p,N}$$

$$C_V = \left.\frac{\partial E}{\partial T}\right|_{V,N}$$

$$C_p = T \cdot \left.\frac{\partial S}{\partial T}\right|_{p,N} = \left.\frac{\partial H}{\partial T}\right|_{p,N} = \frac{5}{3}\left.\frac{\partial E}{\partial T}\right|_{p,N}$$

$$H(S,p,N) = E + pV$$

$$\chi_T = -\frac{1}{V}\left.\frac{\partial V}{\partial p}\right|_{T,N}$$

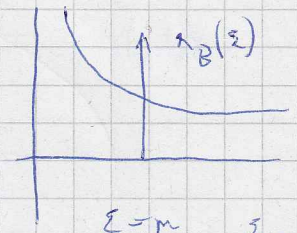
$$V = \frac{1}{p} \cdot \frac{2}{3} E(p,N,T)$$

b) klasický limit

fermion ↙ ↘ boson

B-S rozd. — Bose-Einstein kondenzace

$\mu < 0$

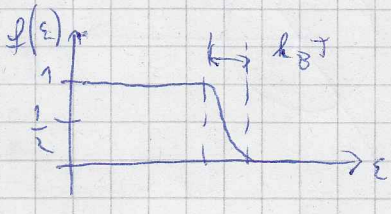


$$\int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon \overset{B\text{-}S}{=} \int_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \cdot \left.\frac{dg}{d\varepsilon}\right|_{\varepsilon=\mu(T)}$$

$g(\varepsilon)$ → derivabilní

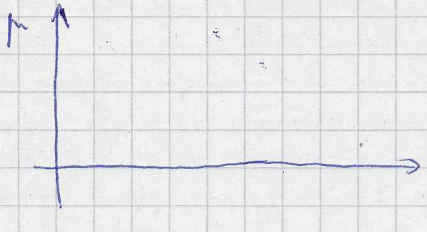
$F^{(\pm)}$

$N_0 \sim$ ~ pravděpodobnost



$$f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1} \qquad E = k_B T\, N \cdot \frac{k_B T}{E_F}$$

$$N = g \cdot V \cdot z\pi \left(\frac{2\pi k_B T_c}{h^2}\right)^{3/2} \cdot \Gamma\left(\frac{3}{2}\right) \zeta\left(\frac{3}{2}\right)$$



B-S

$$N = \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon \qquad f(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

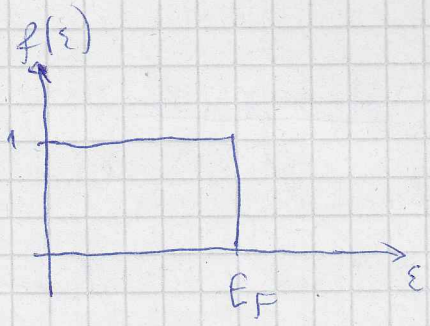
$$\overset{B\text{-}S}{=} \int_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \cdot \left.\frac{dg}{d\varepsilon}\right|_{\varepsilon=\mu(T)} =$$

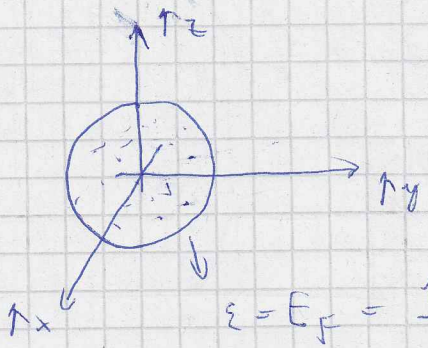
$\mu(T) \approx E_F$

$$= \int_0^{E_F} g(\varepsilon)\, d\varepsilon + g(E_F)(\mu - E_F) + \frac{\pi^2}{6}(k_B T)^2\, g'(\varepsilon = E_F)$$

$\int_0^{E_F} g(\varepsilon)\, d\varepsilon = N$ ↖ $T = 0$







$$\varepsilon = E_F = \frac{p_x^2 + p_y^2 + p_z^2}{2m}$$

$$\mu - E_F = -\frac{\pi^2}{6}(k_B T)^2 \cdot \frac{g'(E_F)}{g(E_F)}$$

$$\mu = E_F - \frac{\pi^2}{6}(k_B T)^2 \cdot \frac{d}{d\varepsilon} \ln g \Big|_{\varepsilon = E_F}$$

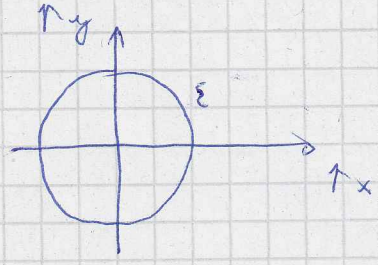
Pl.: 2 dim. el. gáz

$$\varepsilon(\vec{p}) = \frac{p_x^2 + p_y^2}{2m} = \varepsilon$$

$$\Downarrow$$

$$p_x^2 + p_y^2 = \left(\sqrt{2m\varepsilon}\right)^2$$

$g(\varepsilon) = ?$



$$N(\varepsilon) = \Omega_0(\varepsilon) = \frac{2}{h^2}\int d^2p\, d^2q = \frac{2 \cdot A}{h^2}\int d^2p = \frac{2A}{h^2}\cdot\left(\sqrt{2m\varepsilon}\right)^2 \pi$$

$\varepsilon = \varepsilon(\vec{p})$

$$g(\varepsilon) = \frac{dN(\varepsilon)}{d\varepsilon} \sim \text{const} = 2D$$

$$N = \sum_{\underline{k}} 1 = 2 \cdot \frac{L}{(2\pi)^2} \int d^2\underline{k} = \qquad \underline{p} = \hbar \underline{k}$$

$$\left(3d \rightarrow \left(\right) \frac{V}{(2\pi)^3} \int d^3 k \right)$$

$$\varepsilon = \frac{\underline{p}^2}{2m} = \frac{\hbar^2 \underline{k}^2}{2m}$$

$$k_x^2 + k_y^2 = \left(\sqrt{\frac{2m\varepsilon}{\hbar^2}} \right)^2$$

$$= \frac{2L}{(2\pi)^2} \cdot \frac{2m}{\hbar^2} \cdot \varepsilon \cdot \pi$$

---

2 dim $\qquad \varepsilon = \frac{\underline{p}^2}{2m}$

$$g(\varepsilon) = 2D \qquad \frac{2D}{\beta} \ln\left(1 + e^{\beta\mu}\right)$$

$$N = \int g(\varepsilon)\, f(\varepsilon)\, d\varepsilon \overset{\text{ergibt}}{=} \left(\right)$$

$$E = \int_0^\infty g(\varepsilon)\, \varepsilon \cdot f(\varepsilon)\, d\varepsilon = 2D \int \varepsilon\, f(\varepsilon)\, d\varepsilon =$$

$$\overset{B-5}{=} 2D \left[\int_0^\mu \varepsilon\, d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \cdot 1 \right] \left(\right)$$

$$E = 2D \cdot \left[\frac{\mu^2}{2} + \frac{\pi^2}{6} (k_B T)^2 \right]$$

$$N \overset{B\text{-}S}{=} \int_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 \; \underbrace{g'(\varepsilon)\Big|_{\varepsilon=\mu(T)}}_{\neq 0} =$$

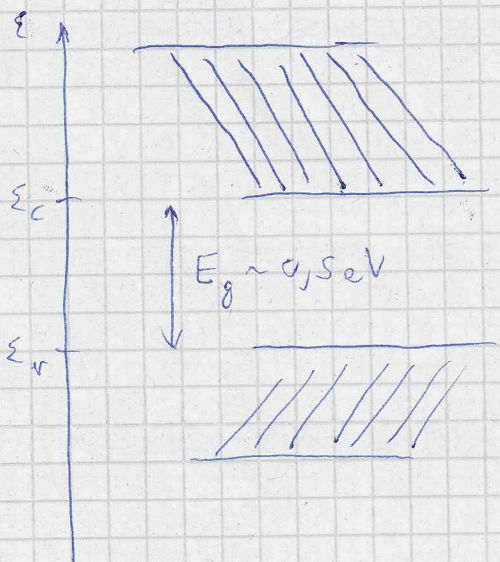
$$= \int_0^\mu g(\varepsilon)\, d\varepsilon = 2D \cdot \int_0^\mu d\varepsilon = 2D\,\mu \quad \rightarrow \quad \mu = \frac{N}{2D}$$

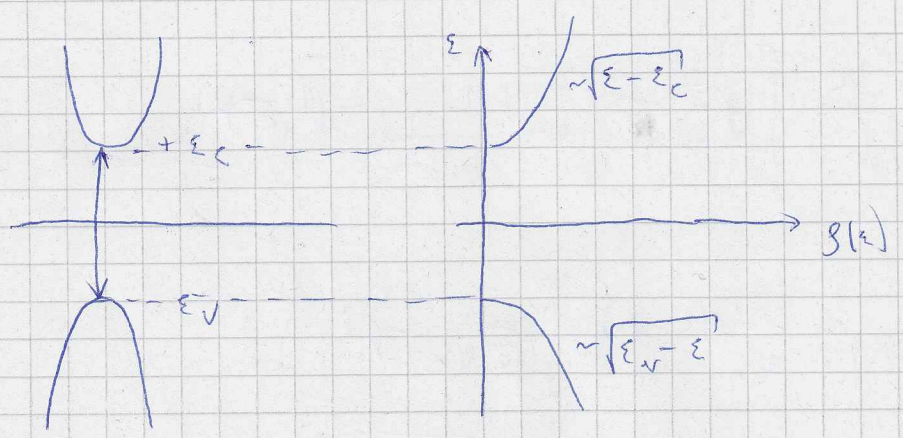
$$E = D\mu^2 \left[1 + \frac{\pi^2}{3} \cdot \left(\frac{k_B T}{\mu} \right)^2 \right] = D\mu^2 \left[1 + \frac{\pi^2}{3} \left(\frac{k_B T}{E_F} \right)^2 \right] =$$

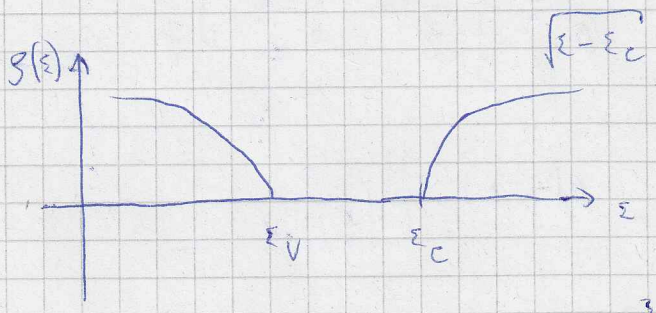
($\mu \rightarrow E_F$)

$$= D \left(\frac{N}{2D} \right)^2 \left[1 + \frac{\pi^2}{3} \left(\frac{T}{T_F} \right)^2 \right]$$

Feltevések: (Nem alkalmazható B-S sorfejtés)







$$g_c(\varepsilon) = g\sqrt{2}\pi\cdot\left(\frac{2m_e}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon-\mu}$$

z

$$g_v(\varepsilon) = g\sqrt{2}\pi\cdot\left(\frac{2m_h}{\hbar^2}\right)^{3/2}\sqrt{\varepsilon_v-\varepsilon}$$

$$N_e^c = 4\pi V\cdot\left(\frac{2m_e}{\hbar^2}\right)^{3/2}\int_{\varepsilon_c}^{\infty}\sqrt{\varepsilon-\varepsilon_c}\;\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon =$$

$$\tilde{\varepsilon} = \varepsilon-\varepsilon_c \qquad d\tilde{\varepsilon} = d\varepsilon$$

$$= 4\pi\cdot V\cdot\left(\frac{2m_e}{\hbar^2}\right)^{3/2}\int_0^{\infty}\sqrt{\tilde{\varepsilon}}\cdot\frac{1}{e^{\beta\tilde{\varepsilon}+\beta(\varepsilon_c-\mu)}+1}$$

$$\varepsilon_c-\mu \approx \frac{E_g}{2} \leftarrow \text{gap} \sim 0{,}25\,eV \gg 25\,meV = 300\,K$$

$\varepsilon_c-\mu \gg k_B T$ $\qquad \beta(\varepsilon_c-\mu) \gg 1$

$$\approx 4\pi V \cdot \left(\frac{2m_e}{h^2}\right)^{3/2} \cdot \int_0^\infty d\tilde{\varepsilon}\ \sqrt{\tilde{\varepsilon}} \cdot e^{-\beta\tilde{\varepsilon}} \cdot e^{-\beta(\varepsilon_c-\mu)}$$

$$= 4\pi V \cdot \left(\frac{2m_e}{h^2}\right)^{3/2} \cdot e^{-\beta(\varepsilon_c-\mu)} \cdot \underbrace{\int_0^\infty \sqrt{\tilde{\varepsilon}} \cdot e^{-\beta\tilde{\varepsilon}} \cdot d\tilde{\varepsilon}}$$

$$y = \beta \cdot \tilde{\varepsilon} \qquad \beta^{-\frac{3}{2}} \int_0^\infty y^{1/2} \cdot e^{-y}\, dy$$

$$= 4\pi V \cdot \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \cdot e^{-\beta(\varepsilon_c-\mu)} \cdot \Gamma\left(\frac{3}{2}\right)$$

$$= 2V \cdot \left(\frac{2\pi m_e k_B T}{h^2}\right)^{3/2} \cdot e^{-\beta(\varepsilon_c-\mu)}$$

$$N_e^v = \int_{-\infty}^{\varepsilon_v} g_v(\varepsilon)\, f(\varepsilon)\, d\varepsilon = \dots$$

$\mu - \varepsilon_v \gg k_B T$

$$N_h(T) = N - N_e^v = 2 \cdot V \cdot \left(\frac{2\pi m_h k_B T}{h^2}\right)^{3/2} \cdot e^{-\beta(\mu-\varepsilon_v)}$$

$$N = N_e^c + N_e^v \quad \Rightarrow \quad n(T) = n(T)$$

$$n(T) = \frac{N_e^c}{N}$$

$$n(T) = p(T) \Rightarrow 2 \cdot V \cdot \left(\frac{2\pi m_e k_B T}{h^2} \right)^{3/2} \cdot e^{-\beta(\varepsilon_c - \mu)} =$$

$$= 2V \cdot \left(\frac{2\pi m_p k_B T}{h^2} \right)^{3/2} \cdot e^{-\beta(\mu - \varepsilon_v)}$$

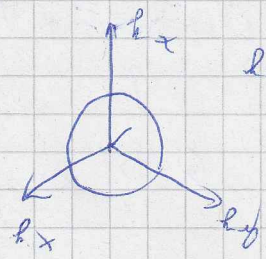
$$m_e^{3/2} \cdot e^{-\beta(\varepsilon_c - \mu)} = m_p^{3/2} \cdot e^{-\beta(\mu - \varepsilon_v)}$$

$$\frac{3}{2} \ln m_e - \beta(\varepsilon_c - \mu) = \frac{3}{2} \ln m_p - \beta(\mu - \varepsilon_v)$$

$$\boxed{\mu = \frac{\varepsilon_c + \varepsilon_v}{2} + \frac{3}{4} k_B T \ln \frac{m_p}{m_e}}$$

Ideális kvantumgázok eloszlásfüggvénye

$$\varepsilon(\vec{p}) = \frac{\vec{p}^2}{2m} \rightarrow g(\varepsilon) = ?$$



$$k_x^2 + k_y^2 + k_z^2 = \left(\sqrt{\frac{2m\varepsilon}{\hbar^2}} \right)^2$$

$$N(\varepsilon) = \sum_{\vec{k}} 1 = \frac{V}{(2\pi)^3} g \int d^3k = \frac{V}{(2\pi)^3} g \cdot \frac{4\pi}{3} \left(\frac{2m\varepsilon}{\hbar^2} \right)^{3/2}$$

$$|\vec{k}| < \sqrt{\frac{2m\varepsilon}{\hbar^2}}$$

$$g(\varepsilon) = \frac{dN}{d\varepsilon} = \frac{V g 2\pi}{8\pi^3} \left(\frac{2m}{\hbar^2} \right)^{3/2} \cdot \varepsilon^{1/2} = g V \cdot 2\pi \left(\frac{2m}{\hbar^2} \right)^{3/2} \cdot \sqrt{\varepsilon}$$

$$E = \int \varepsilon\, g(\varepsilon)\, d\varepsilon \cdot n(\varepsilon) = g V \cdot 2\pi \left(\frac{2m}{h^2}\right)^{3/2} \int_0^\infty \frac{\varepsilon^{3/2}}{e^{\beta(\varepsilon - \mu)} \pm 1}\, d\varepsilon$$

$$x = \beta\varepsilon \qquad \alpha = -\beta\mu$$

$$= g V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot \frac{1}{\beta^{3/2}} \cdot \frac{1}{\beta} \cdot \int_0^\infty \frac{x^{3/2}}{e^{x+\alpha} \pm 1}\, dx =$$

$$= g \cdot V \cdot 2\pi \cdot \left(\frac{2m}{h^2}\right)^{3/2} \cdot (k_B T)^{5/2} \int_0^\infty \frac{x^{3/2}\, dx}{e^{x+\alpha} \pm 1} = V \cdot T^{5/2} f(\alpha)$$

$$\Phi(T, V, \mu) = -pV = -\frac{2}{3} E$$

↑ G.-D. ↑ stat. fi

$$d\Phi = -S\, dT - p\, dV - N\, d\mu$$

$$N = -\left.\frac{\partial \Phi}{\partial \mu}\right|_{T,V} = \frac{2}{3} \cdot \left.\frac{\partial E}{\partial \mu}\right|_{T,V}$$

$$N = \frac{2}{3} V \cdot T^{5/2} \cdot \frac{\partial f(\alpha)}{\partial \mu} = \frac{2}{3} V T^{5/2} \cdot \underbrace{\frac{\partial f(\alpha)}{\partial \alpha}}_{f'(\alpha)} \underbrace{\frac{\partial \alpha}{\partial \mu}}_{-\beta} =$$

$$= \frac{2}{3} V \cdot T^{5/2} \cdot (-\beta) f'(\alpha) = V \cdot T^{3/2} \cdot \tilde{f}(\alpha)$$

$$S = -\left.\frac{\partial \Phi}{\partial T}\right|_{V,T} = \frac{2}{3}\left.\frac{\partial E}{\partial T}\right|_{V,T} =$$

$$= \frac{2}{3} V \cdot \left(\frac{5}{2} T^{3/2} f(\alpha) + V f'(\alpha) \cdot \frac{\partial \alpha}{\partial T} \right)$$

$$\frac{\partial \alpha}{\partial T} = \frac{\partial}{\partial T}\left(-\frac{\mu}{k_B T}\right) = -\frac{\mu}{k_B}\left(-\frac{1}{T^2}\right) = \frac{\mu}{k_B T^2} = -\frac{\alpha}{T}$$

$$= \frac{2}{3} V \left(\frac{5}{2} T^{3/2} f(\alpha) + T^{3/2} f'(\alpha)(-\alpha) \right) = V \cdot T^{3/2} \cdot \tilde{f}(\alpha)$$

<u>tudjuk</u>: $S = \text{áll}$, $N = \text{áll}$ $\Rightarrow \frac{S}{N} = \frac{\tilde{f}(\alpha)}{\tilde{f}(\alpha)} = \text{áll} \Rightarrow \alpha = \text{állandó}$

$$N = V \cdot T^{3/2} \cdot \tilde{f}(\alpha) \Rightarrow \boxed{V \cdot T^{3/2} = \text{áll}}$$

(N: áll; $\tilde{f}(\alpha)$: áll)

$$p V = \frac{2}{3} E = \frac{2}{3} V T^{5/2} \cdot f(\alpha) \Rightarrow \boxed{\frac{p}{T^{5/2}} = \text{áll}}$$

$$\Rightarrow \boxed{p V^{5/3} = \text{áll}}$$

$\alpha = -\beta p$ $\quad \frac{\partial \alpha}{\partial p} = -\beta$

adiabatikus $p V^{\frac{c_p}{c_V}} = \text{áll}$ klasszikus ideális gáz

$c_V = \frac{3}{2} k_B$ $\quad c_p = \frac{5}{2} k_B$ $\quad \frac{c_p}{c_V} = \frac{5}{3}$ vert. löketű állandó

$$\Phi = \mp k_B T \sum_i \ln\left(1 \pm e^{-\beta(\varepsilon_i - \mu)}\right)$$

fermion: felso

boson: also

$$n_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} \pm 1}$$

$$\frac{1}{n_i} \mp 1 = e^{+\beta(\varepsilon_i - \mu)}$$

$$e^{-\beta(\varepsilon_i - \mu)} = \frac{1}{\frac{1}{n_i} \mp 1} = \frac{n_i}{1 \mp n_i}$$

$$1 \pm e^{-\beta(\varepsilon_i - \mu)} = 1 \pm \frac{n_i}{1 \mp n_i} = \frac{1}{1 \mp n_i}$$

$$\Phi = \pm k_B T \cdot \sum_i \ln(1 \mp n_i)$$

$$S = -\left.\frac{\partial \Phi}{\partial T}\right|_{V,\mu} = \mp k_B \sum_i \ln(1 \mp n_i) \pm$$

$$\pm k_B T \sum_i \frac{1}{1 \mp n_i} (\mp 1) \frac{\partial n_i}{\partial T}$$

$$\frac{\partial n_i}{\partial T} = \frac{\partial n_i}{\partial \beta} \cdot \frac{\partial \beta}{\partial T} = \frac{-1}{\left(e^{\beta(\varepsilon_i - \mu)} \pm 1\right)^2} e^{\beta(\varepsilon_i - \mu)} \cdot (\varepsilon_i - \mu) \frac{1}{k_B} \left(-\frac{1}{T^2}\right)$$

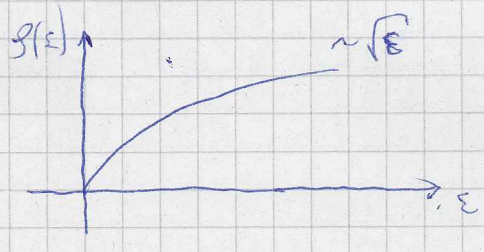
$\downarrow$ n_i^2 ; $\downarrow$ $\frac{1}{n_i} \mp 1$; $\downarrow$ $\ln\left(\frac{1}{n_i} \mp 1\right)$

$$S = \mp k_B \sum_i \left((1 \mp \bar{n}_i) \ln (1 \mp \bar{n}_i) \pm \bar{n}_i \ln \bar{n}_i \right)$$

$$\bar{n}_i = \frac{1}{e^{\beta(\varepsilon_i - \mu)} \pm 1}$$

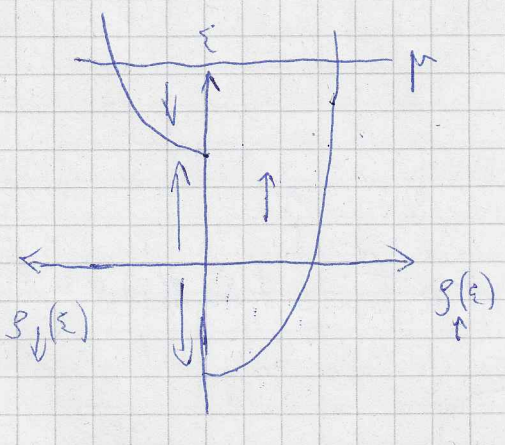
$$\overline{\Delta n_i^2} = k_B T \cdot \frac{\partial \bar{n}_i}{\partial \mu} = \dots = \bar{n}_i \mp \bar{n}_i^2 = \bar{n}_i (1 \mp \bar{n}_i)$$

$\frac{1}{2}$ spinű Fermi-gáz spin-szuszceptibilitása (Pauli-szuszceptibilitás)



$$\varepsilon(\pm) = \frac{\hbar^2}{2m}$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + \mu_B \hat{\underline{S}} \cdot \underline{B}$$



$$\varepsilon(\pm) = \frac{\hbar^2}{2m} \pm \mu_B B$$

$$\uparrow$$

$$\approx H$$

$$M = \mu_B \int_0^\infty \frac{g(\varepsilon)}{2} f(\varepsilon - \mu_B H)\, d\varepsilon - \int_0^\infty \frac{g(\varepsilon)}{2} f(\varepsilon + \mu_B H)\, d\varepsilon$$

$$= \mu_B \int_0^\infty \left(-\frac{\partial f}{\partial \varepsilon} \right) 2\mu_B H \cdot \frac{g(\varepsilon)}{2}\, d\varepsilon =$$

$$= \mu_B^2 H \int_0^\infty \left(-\frac{\partial f}{\partial \varepsilon}\right) g(\varepsilon)\, d\varepsilon$$

① $$\boxed{\chi = \frac{\partial M}{\partial H} = \mu_B^2 \int_0^\infty \left(-\frac{\partial f}{\partial \varepsilon}\right) g(\varepsilon)\, d\varepsilon}$$

~~part.~~ int.

② $$\chi = -\mu_B^2 \Big[g(\varepsilon) f(\varepsilon) \Big]_0^\infty + \mu_B^2 \int_0^\infty f(\varepsilon) \frac{\partial g(\varepsilon)}{\partial \varepsilon}\, d\varepsilon =$$

$g(\varepsilon) f(\varepsilon) = 0$

$$= -\mu_B^2 \cdot \int_0^\infty \frac{dg}{d\varepsilon}(\varepsilon)\, f(\varepsilon)\, d\varepsilon$$

↓ B-5

③ $$\boxed{\chi = \mu_B^2 \int_0^\infty \frac{\partial f(\varepsilon)}{\partial \mu_c} g(\varepsilon)\, d\varepsilon} = \mu_B^2 \frac{\partial}{\partial \mu_c} \int_0^\infty f(\varepsilon) g(\varepsilon)\, d\varepsilon =$$

$$\boxed{= \mu_B^2 \cdot \left.\frac{\partial N_c}{\partial \mu_c}\right|_{T,V}}$$

(...) T mag. Stoff:

$$N = gV \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} e^{\beta\mu} \left(1 - \frac{1}{2^{3/2}} e^{\beta\mu} + \dots\right)$$

$g = 2 \quad \frac{1}{2^{3/2}} e^{2\beta\mu}$

$$\left.\frac{\partial N}{\partial \mu}\right|_{T,V} = g \cdot V \cdot \left(\frac{2\pi m k_B T}{h^2}\right)^{3/2} \cdot \left[e^{\beta\mu} \cdot \beta - \frac{1}{2^{3/2}} e^{\beta\mu} \cdot \beta + \dots\right]$$

$$\chi = \mu_B^2 \cdot \left.\frac{\partial N}{\partial \mu}\right|_{T,V}$$

$$\frac{\chi}{N} = \mu_B^2 \beta \cdot \frac{1 - \frac{1}{2^{3/2}} e^{\beta\mu}}{1 - \frac{1}{2^{3/2}} e^{\beta\mu}} = \frac{\mu_B^2}{k_B T}\left(1 - \frac{1}{2^{3/2}} e^{\beta\mu}\right)$$

$$\Rightarrow \quad \chi = N \cdot \frac{\mu_B^2}{k_B T}\left[1 - \frac{1}{2^{3/2}} \cdot \frac{N}{gV}\left(\frac{h^2}{2\pi m k_B T}\right)^{3/2} + \dots\right]$$

$g(\varepsilon) = c\sqrt{\varepsilon} \qquad c = gV \cdot 2\pi \left(\frac{2m}{h^2}\right)^{3/2}$ T klein

$$\chi = \mu_B^2 \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon = \mu_B^2 \cdot \frac{c}{2} \cdot \int_0^\infty \varepsilon^{-\frac{1}{2}} \cdot f(\varepsilon)\, d\varepsilon \overset{\text{B-S}}{=}$$

$$= \mu_B^2 \cdot \frac{c}{2}\left[\int_0^\mu \varepsilon^{-\frac{1}{2}} d\varepsilon + \frac{\pi^2}{6}(k_B T)^2 \cdot \left.\frac{d\,\varepsilon^{-\frac{1}{2}}}{d\varepsilon}\right|_{\varepsilon=\mu}\right]$$

$$= \mu_B^2 \cdot \frac{c}{2}\left[2 \cdot \mu^{1/2} + \frac{\pi^2}{6}(k_B T)^2 \left(-\frac{1}{2}\right) \mu^{-\frac{3}{2}}\right]$$

$$= \mu_B^2 \cdot c \cdot \mu^{1/2} \left[1 - \frac{\pi^2}{24} \left(\frac{k_B T}{\mu} \right)^2 \right]$$

$\mu \approx k_B T_F$

$$\frac{\mu}{E_F} = 1 - \frac{\pi^2}{12} \left(\frac{k_B T}{E_F} \right)^2$$

$$N = \int_0^{E_F} g(\varepsilon)\, d\varepsilon = c \cdot \int_0^{E_F} \varepsilon^{1/2}\, d\varepsilon = c \cdot \frac{E_F^{3/2}}{3/2}$$

$\uparrow$ $T = 0$

$$\frac{\chi}{N} = \frac{3}{2} \frac{\mu_B^2 c \cdot \mu^{1/2}}{c \cdot E_F^{3/2}} \left[1 - \frac{\pi^2}{24} \left(\frac{T}{T_F} \right)^2 \right] =$$

$$\left(\frac{\mu}{E_F} \right)^{1/2} \approx 1 - \frac{\pi^2}{24} \left(\frac{T}{T_F} \right)^2$$

$$= \frac{3}{2} \mu_B^2 \cdot \frac{1}{E_F} \cdot \left[1 - \frac{\pi^2}{24} \left(\frac{T}{T_F} \right)^2 \right] \left[1 - \frac{\pi^2}{24} \left(\frac{T}{T_F} \right)^2 \right]$$

$$\rightarrow \boxed{\chi = \chi_0 \left(1 - \frac{\pi^2}{12} \left(\frac{T}{T_F} \right)^2 \right) \qquad \chi_0 = \frac{3}{2} \mu_B^2 \cdot \frac{N}{E_F}}$$

tetszőleges $g(\varepsilon)$-ra

$$\chi = \mu_B^2 \int_0^\infty g(\varepsilon) f(\varepsilon)\, d\varepsilon \overset{\text{B-S}}{=} \mu_B^2 \left[\int_0^\mu g(\varepsilon)\, d\varepsilon + \frac{\pi^2}{6} (k_B T)^2 g'(\mu) \right] =$$

$\uparrow$ konf

$$\approx \mu_B^2 \left[\int_0^{E_F} g(\varepsilon)\, d\varepsilon + (\mu - E_F)\, g'(E_F) + \frac{\pi^2}{6} (k_B T)^2\, g''(E_F) \right]$$

$$\int_0^{E_F} g(\varepsilon)\, d\varepsilon = g(\varepsilon)\Big|_0^{E_F}$$

$$\mu = E_F - \frac{\pi^2}{6} (k_B T)^2 \frac{g'(E_F)}{g(E_F)}$$

$g(0) = 0$

$$= \mu_B^2 \left[g(E_F) - \frac{\pi^2}{6} (k_B T)^2 \cdot \frac{g'(E_F)}{g(E_F)}\, g'(E_F) + \frac{\pi^2}{6} (k_B T)^2 \cdot g''(E_F) \right] =$$

$$= \mu_B^2\, g(E_F) \left[1 + \frac{\pi^2}{6} (k_B T)^2 \left(\frac{g''}{g} - \frac{g'^2}{g^2} \right) \right]$$

$$\left(\frac{g'}{g} \right)' = \left(\frac{d}{d\varepsilon} \ln g(\varepsilon) \right)' = \frac{d^2}{d\varepsilon^2} \left(\ln g(\varepsilon) \right)$$

$$= \mu_B^2\, g(E_F) \left[1 + \frac{\pi^2}{6} (k_B T)^2 \cdot \frac{d^2 \ln g(\varepsilon)}{d\varepsilon^2}\Big|_{\varepsilon = E_F} \right]$$