

Matematikai módszerek a fizikában

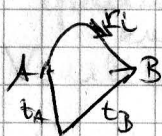
Csüdög Péter

Vektoranalízis:

1x9.

vektorváltozós vektorfüggvény: vektor

~~vektor~~ változós ~~vektor~~függvény: skalarvázó



$$s = \sum_{i=1}^n |\Delta \vec{r}_i| = \lim_{n \rightarrow \infty} \sum_{i=1}^n |\Delta \vec{r}_i| \frac{\Delta t_i}{\Delta t_i}$$

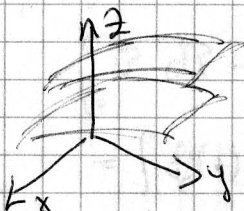
$$= \int_{t_A}^{t_B} |\vec{r}'(t)| dt$$

pl $\vec{r}(t) = (t, t^2, 0)$ $0 \leq t \leq 1$

$\vec{r}'(t) = (1, 2t, 0)$ $|\vec{r}'(t)| = \sqrt{1+4t^2}$ $2t := \sin u$

$$s = \int_0^1 \sqrt{1+4t^2} dt = \int_0^{\pi/2} \frac{1}{2} \cosh^2 u du$$

Skalarvázó: $\phi(\vec{r})$



szint felületek

pl: $\phi = \vec{u} \cdot \vec{r} + \phi_0 = u_1 x + u_2 y + u_3 z + \phi_0$

$\phi = \text{állandó} \rightarrow$ síkok

ϕ deriváltja: gradiens

$\text{grad } \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right)$ ϕ v. változói
űtemét adja meg

$$\phi'(x) = \lim_{\Delta x} \frac{\Delta \phi}{\Delta x} \Leftrightarrow \phi'(x) = \frac{\Delta \phi}{\Delta x} \Rightarrow \Delta \phi = \phi' \Delta x$$

$$\Delta\phi = \underbrace{\phi(\vec{r}_0 + \Delta\vec{r}) - \phi(\vec{r}_0)}_{\text{skalár}} \approx \underbrace{(\text{vártani})}_{\text{grad } \phi} (\Delta\vec{r})$$

ha $|\Delta\vec{r}|$ adott: grad ϕ iránya megegyezik a leggyorsabb változás irányával

$$|\text{grad } \phi| = \frac{\Delta\phi}{|\Delta\vec{r}|} \quad \text{milyen mértékben változik } \phi \text{ egy egységnyi hosszra}$$

$$\Delta\vec{r} := (\Delta x, 0, 0)$$

$$\Delta\phi \approx (\text{grad } \phi) \Delta\vec{r} = (\text{grad } \phi)_x \Delta x + () 0 + () 0$$

$$(\text{grad } \phi)_x \approx \left. \frac{\Delta\phi}{\Delta x} \right|_{\substack{\Delta y=0 \\ \Delta z=0}} \rightarrow \frac{\partial\phi}{\partial x}$$

legyen ϕ a potenciális energia!

$$\Delta\phi = \text{energia változás} = \text{általában végzett munka} =$$

$$\Delta\phi = \vec{F}_{\text{er}} \cdot \Delta\vec{r} = - \vec{F}_{\text{erőter}} \Delta\vec{r}$$

$$\text{grad } \phi \cdot \Delta\vec{r} = - \vec{F}_{\text{erőter}} \Delta\vec{r}$$

$$\underline{\underline{\text{grad } \phi = -\vec{F}}}$$

IX. 11.

$$\Delta\phi = \phi(r + \Delta r) - \phi(r) = \text{grad } \phi(r) \Delta r + \underbrace{C(r, \Delta r)}_{\text{másodrendűen kicsi}} \Delta r$$

$$\phi = \text{d} \rightarrow \text{szint felület / mint felület}$$

pl. $\phi = \vec{r}^2 = x^2 + y^2 + z^2$

$$\Delta\phi = (r + \Delta r)^2 - r^2 = \underbrace{2 \cdot r \cdot \Delta r}_{\text{grad } \phi} + \underbrace{\Delta r \cdot \Delta r}_C$$

$$\phi = r^2 \rightarrow \text{grad } \phi = 2 \cdot \vec{r}$$

$$\phi = \frac{1}{r} \rightarrow \text{grad } \phi = -\frac{1}{2} r^{-3} \cdot 2r = -\frac{\vec{r}}{r^3}$$

ha $\phi(r) \rightarrow$ szintfelületek gömbök

nem minden vektormező írható fel skalár fv. gradienseként

Vonalintegrál

emlékeztetőül $W = \vec{F} \cdot \vec{r}$

ha $\vec{F}(\vec{r})$ $\Delta W = \vec{F}(\vec{r}) \Delta\vec{r}$

$$W \approx \sum_{i=1}^N \vec{F}(\vec{r}_i) \Delta\vec{r}_i \xrightarrow{N \rightarrow \infty} \int_G \vec{F}(\vec{r}) d\vec{r}$$

pálya megadása pl $\vec{r}(t)$ paraméterezés

$\vec{r}(t)$ $\frac{d\vec{r}}{dt}$ G

$$\int_G \vec{r} d\vec{r} \approx \sum \vec{r}(\vec{r}_i) \Delta\vec{r}_i \cdot \frac{\Delta t_i}{\Delta t_i} \approx \sum (\vec{r}(\vec{r}_i) \cdot \vec{r}') \cdot \Delta t_i$$

$$= \int f(t) dt = \int_{t_0}^{t_1} \vec{r}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

pl. $\vec{r} = \vec{r}$ $\vec{r} \perp d\vec{r} \perp \vec{r}$



$$\vec{r} \cdot d\vec{r} = 0$$

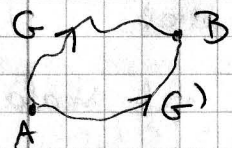


$$\vec{F}(\vec{r}) = \vec{e} \times \vec{r} \quad \vec{e} \cdot \vec{r} = 1 \quad \vec{F} \parallel d\vec{r}$$

$$\oint \vec{F} d\vec{r} = \oint \vec{F} \cdot d\vec{r} = F \oint dr = r \cdot 2\pi = 2\pi r$$

$$\vec{r} = \begin{pmatrix} r \cos t \\ r \sin t \\ 0 \end{pmatrix} \quad \vec{e} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{F} = \begin{pmatrix} -r \sin t \\ r \cos t \\ 0 \end{pmatrix} \quad \vec{r}' = \begin{pmatrix} -r \sin t \\ r \cos t \\ 0 \end{pmatrix}$$

$$\oint \vec{v} \cdot d\vec{r} = \int_0^{2\pi} \pm (r(t)) \cdot r'(t) dt = \int_0^{2\pi} \pm (2r^2 + r^2 + 1) dt = 2\pi r^2$$



általában: $\int_G \vec{v} \cdot d\vec{r} \neq \int_{-G} \vec{v} \cdot d\vec{r}$

$\Updownarrow$

$$\oint \vec{v} \cdot d\vec{r} \neq 0$$

Tétel:

Ha $\vec{v} = \text{grad } \phi$, akkor $\int_G \vec{v} \cdot d\vec{r} = \phi(\vec{r}_b) - \phi(\vec{r}_a)$

$\Rightarrow \oint \text{grad } \phi \cdot d\vec{r} = 0 \rightarrow$ I. gradiens tétel

Biz: $\int_G \text{grad } \phi \cdot d\vec{r} \approx \sum \text{grad } \phi \cdot \Delta \vec{r} = \sum \Delta \phi$

$\vec{v} = \text{grad } \phi \Leftrightarrow \oint \vec{v} \cdot d\vec{r} = 0 \Leftrightarrow \text{rot } \vec{v} = 0$

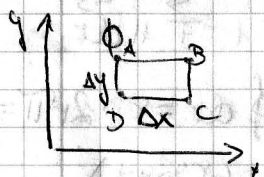
potenciális konzervatív örméyesmentes

Young-tétel:

$$\phi(x,y) \quad \frac{\partial}{\partial y} \left(\frac{\partial \phi}{\partial x} \right) = \frac{\partial}{\partial x} \left(\frac{\partial \phi}{\partial y} \right)$$

Ha mindkettő létezik,
és legalább egyik folytonos

Számításhoz:



$$\frac{\phi_B - \phi_A}{\Delta x} - \frac{\phi_C - \phi_D}{\Delta x} = \frac{\phi_D - \phi_A}{\Delta y} - \frac{\phi_C - \phi_B}{\Delta y}$$

Ha $\vec{v}(r)$ potenciális

Budaörs

2011-12-15

Bajcs 30 22 9 42 22

$$\vec{v}(r) = \text{grad } \phi$$

$$v_x = \frac{\partial \phi}{\partial x} \quad v_y = \frac{\partial \phi}{\partial y} \quad v_z = \frac{\partial \phi}{\partial z}$$

$$\frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x} \dots$$

$$\frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} = 0 = a_z$$

$$\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x} = 0 = a_y$$

$$\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z} = 0 = a_x$$

$$\underline{a} = \text{rot } \vec{v}$$

Mit jelent a rotáció, amikor nem 0?

1x25

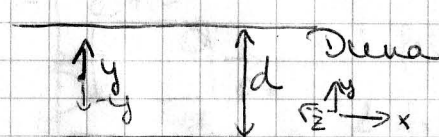
pl: $\vec{v} = \vec{\omega} \times \vec{r}$

$$v_x = \omega_y z - \omega_z y \quad v_y = \omega_z x - \omega_x z$$

$$v_z = \omega_x y - \omega_y x$$

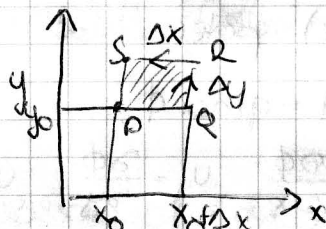
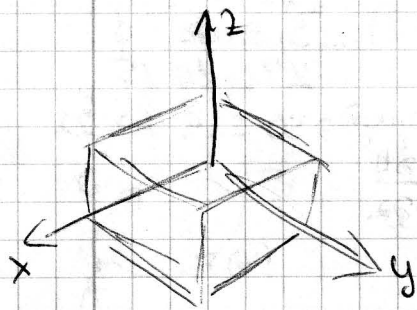
$$\text{rot } \vec{v} = 2\vec{\omega}$$

úszó farönk:



$$\vec{v}(r) = \begin{pmatrix} v_0(1 - \frac{4y^2}{d^2}) \\ 0 \\ 0 \end{pmatrix}$$

$$\text{rot } \vec{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{8v_0 y}{d^2} \end{pmatrix} \rightarrow$$



$$\oint_G \vec{v} d\vec{r} \approx \sum \underline{v} \cdot d\underline{r} = \int_P^Q \vec{v} d\vec{r} + \int_Q^R \vec{v} d\vec{r} + \int_R^S \vec{v} d\vec{r} + \int_S^P \vec{v} d\vec{r}$$

gradiens. $\Delta \phi \approx \text{grad } \phi \cdot \Delta \vec{r}$ x.2.
 $\oint \text{grad } \phi d\vec{r} = \phi(B) - \phi(A)$

rotáció. $\oint \vec{v} d\vec{r} \approx \text{rot } \vec{v}(r_0) \cdot \Delta \vec{F}$
 $\Delta F = \vec{e}_z \cdot |\Delta \vec{F}|$

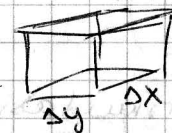
divergencia.

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \nabla \cdot \underline{v}$$

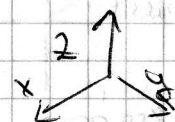
$$\vec{e}_z \cdot (\text{rot } \vec{v}) = \lim_{\Delta F \rightarrow 0} \frac{\oint \underline{v} d\vec{r}}{\Delta F}$$

alkalmazás és szemléltetés

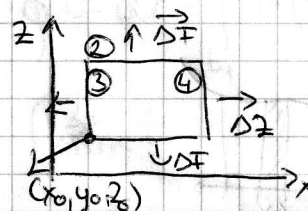
$\vec{v}(\vec{r})$ vízsebesség ($\rho = 1 \text{ kg/dm}^3$)
 adott p.



mennyi víz áramlik át rajta? vízfluxus



átáramló víz tömege/céló:
 $\vec{v} \cdot \Delta \vec{F} = |\vec{v}| \cdot |\Delta \vec{F}| \cdot \cos \alpha$



1. felületre. $\vec{v} \cdot \Delta \vec{F} \approx v_z(x_0, y_0, z_0) \Delta x \Delta y$

2. felületre. $\vec{v} \cdot \Delta \vec{F} \approx v_z(x_0, y_0, z_0 + \Delta z) \Delta x \Delta y$

$$\text{①②} = \frac{v_z(x_0, y_0, z_0) - v_z(x_0, y_0, z_0 + \Delta z)}{\Delta z} \cdot (\Delta x \cdot \Delta y \cdot \Delta z) =$$

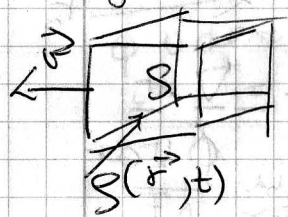
$$= \sum \vec{v} \cdot \vec{F} \approx \frac{\partial v_z}{\partial z} \cdot \Delta V$$

$$(3)+(4) \quad \sum \vec{r} \Delta \vec{r} \approx \frac{\partial u_x}{\partial x} \Delta V$$

$$(5)+(6) \quad \sum \vec{r} \Delta \vec{r} = \frac{\partial u_y}{\partial y} \Delta V$$

$$(7) \quad \sum \vec{r} \Delta \vec{r} \approx (\text{div } \vec{r}) \Delta V$$

Megmaradási tétel



hárommal Δt alatt:

$$\sum \vec{r} \cdot \Delta \vec{r} \Delta t$$

$$(\vec{r} \cdot \Delta V)_{t+\Delta t} - (\vec{r} \cdot \Delta V)_t$$

ha nincs szűletés:

$$\sum \vec{r} \cdot \Delta \vec{r} \Delta t = -\frac{((\vec{r} \cdot \Delta V)_{t+\Delta t} - (\vec{r} \cdot \Delta V)_t)}{\Delta t} \Delta t$$

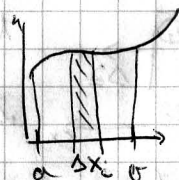
$$(\text{div } \vec{r}) \Delta V$$

$$\text{div } \vec{r} = -\frac{\partial \rho}{\partial t} \rightarrow \text{kontinuitási egyenlet}$$

$$\text{div } \vec{r} + \frac{\partial \rho}{\partial t} = f(\vec{r}, t) \leftarrow \text{forráserősség}$$

emlékeztető

$$\int_a^b f(x) dx \approx \sum f(x_i) \Delta x_i$$



vektoroknál (3dimenziós)

$$\int_V \vec{r}(\vec{r}) d\vec{r} \approx \sum \vec{r}(\vec{r}_i) \Delta \vec{r}_i \approx \sum \vec{r} \frac{\Delta \vec{r}}{\Delta t} \Delta t \approx$$

$$\approx \sum \vec{r}(\vec{r}_i(t)) \cdot \Delta t \approx \int_{t_1}^{t_2} (\vec{r} \cdot \vec{r}) dt$$

kisrészlet mod.



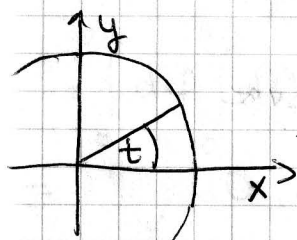
$$\text{pl. } \vec{r}(\vec{r}) = \vec{e} \times \vec{r} \quad |\vec{e}| = 1$$



$$\vec{r} \parallel \Delta \vec{r}$$

$$\oint \vec{r} d\vec{r} = ?$$

$$\oint \vec{r} \cdot d\vec{r} \approx \sum |\vec{r}| \Delta \vec{r} = 2 \sum |\Delta \vec{r}| = 4R\pi$$



$$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \cos t \\ 2 \sin t \\ 0 \end{pmatrix} \quad \vec{e} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \vec{u} = \begin{pmatrix} -2 \sin t \\ 2 \cos t \\ 0 \end{pmatrix}$$

$$\oint \vec{r} \cdot d\vec{r} = \int_{t_1}^{t_2} \vec{r}(\vec{r}(t)) \cdot \vec{r}'(t) dt = 4 \cdot 2\pi = 8\pi$$

$$\iint \vec{r} \cdot d\vec{r}$$

$\vec{r}$ vektormező felületi integrálja



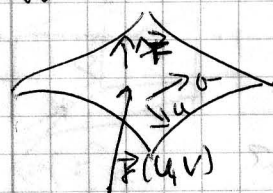
$$\approx \sum \vec{r}(\vec{r}_i) \Delta \vec{r}_i$$

pl. $\vec{r} = \frac{k \cdot Q}{|\vec{r}|^3} \vec{r} = \vec{E}(\vec{r})$

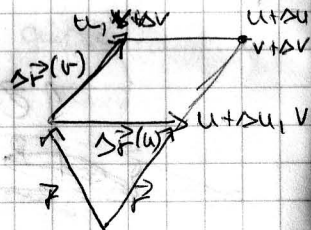
$\oint \vec{E} \cdot d\vec{F} \quad |\vec{r}| = R$

$\oint \vec{E} d\vec{F} = \sum |\vec{E}| |\Delta\vec{F}| = \frac{k \cdot Q}{R^2} 4\pi R^2 = 4\pi k Q$

gyakorlatban.



$u, u+du$
 $v, v+dv$



$\Delta\vec{r}(u) = \vec{r}(u+du, v) - \vec{r}(u, v) \approx \frac{\partial \vec{r}}{\partial u} du$

$\Delta\vec{r}(v) = \frac{\partial \vec{r}}{\partial v} dv$

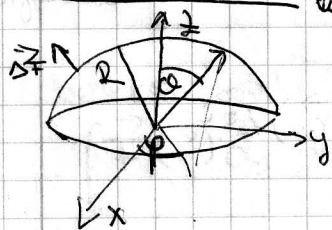
$\approx \sum_{u,v} \vec{r}(\vec{r}(u,v)) \left(\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right) du \cdot dv \approx$

$\approx \iint \left(\vec{r}, \frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v} \right) du dv$

$f(u,v)$

egy konkrét példa

napfény $\downarrow \downarrow \downarrow \downarrow$ beesés üveg



$0 \leq \varphi \leq 2\pi$
 $0 \leq \theta \leq \frac{\pi}{2}$

$f = \begin{pmatrix} 0 \\ 0 \\ 1+0,1 y^2 \end{pmatrix} \rightarrow$ egyenlő nagyságú felületen egyenlő nagyságú alatt

$\frac{\text{energia}}{\text{csd}} = - \iint_{\text{felgömb}} \vec{f} \cdot d\vec{F}$

$r(\theta, \varphi) = \begin{pmatrix} 2 \cdot \sin \theta \cdot \cos \varphi \\ 2 \cdot \sin \theta \cdot \sin \varphi \\ 2 \cdot \cos \theta \end{pmatrix} \quad \begin{matrix} u = \theta \\ v = \varphi \end{matrix}$

$f = \begin{pmatrix} 0 \\ 0 \\ -1+0,1 \cdot 4 \cdot \sin^2 \theta \cdot \sin^2 \varphi \end{pmatrix}$

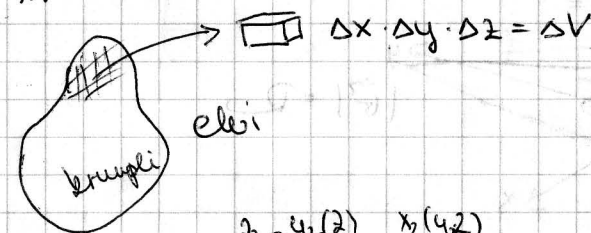
$\frac{\partial \vec{r}}{\partial \theta} = \begin{pmatrix} 2 \cos \theta \cos \varphi \\ 2 \cos \theta \sin \varphi \\ -2 \sin \theta \end{pmatrix} \quad \frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -2 \sin \theta \sin \varphi \\ 2 \sin \theta \cos \varphi \\ 0 \end{pmatrix}$

$= \int_0^{2\pi} \int_0^{\pi/2} \begin{vmatrix} 0 & 0 \\ 2 \cos \theta \cos \varphi & 2 \sin \theta \sin \varphi \\ 2 \cos \theta \sin \varphi & 2 \sin \theta \cos \varphi \end{vmatrix} d\varphi d\theta$

$= \int d\varphi \int d\theta 4(-1+0,4 \sin^2 \theta \sin^2 \varphi) \cos \theta \sin \theta =$

$= 4\pi \int_0^{\pi/2} d\theta (2\pi + 0,4 \sin^2 \theta \pi) \cos \theta \sin \theta = \dots$

$\iiint_S f(\vec{r}) dV$



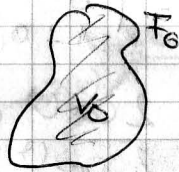
gyakorlat

$\int_{z_1}^{z_2} \int_{y_1(y,z)}^{y_2(y,z)} \int_{x_1(y,z)}^{x_2(y,z)} f(x,y,z) dx dy dz = \dots$

$$\left. \begin{aligned} \int \vec{v} \cdot d\vec{r} & \cdot f \\ \iint \vec{v} \cdot d\vec{F} & \cdot f \\ \iiint \rho \, dV \end{aligned} \right\} \text{2 tétel}$$

Gauss - Ostogradskij - tétel:

$$\iiint_V \operatorname{div} \vec{v} \cdot dV = \oint_{F_0} \vec{v} \cdot d\vec{F}$$



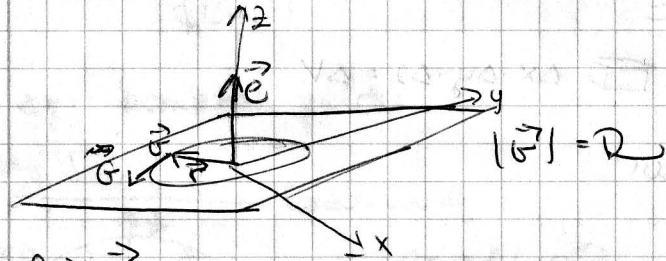
Stokes - tétel:

$$\iint_{F_0} \operatorname{rot} \vec{v} \cdot d\vec{F} = \oint_{G_0} \vec{v} \cdot d\vec{r}$$



$$\oint \vec{v} \cdot d\vec{r} \approx \operatorname{rot} \vec{v} \cdot \Delta \vec{F} \quad \Leftrightarrow \quad \text{kicsi felületre}$$

x.9
pl. ① $\vec{v} = \vec{e} \times \vec{r} \quad |\vec{e}| = 1$



$$\oint \vec{v} \cdot d\vec{r}$$

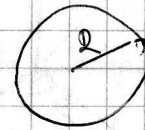
$$\iint \operatorname{rot} \vec{v} \cdot d\vec{F}$$

$$\operatorname{rot} \vec{v} = 2\vec{e}$$

$$\sum 2\vec{e} \cdot \Delta \vec{F} = 2\vec{e} \cdot \sum \Delta \vec{F} = 2\vec{e} \cdot (2^2 \pi \cdot \vec{e}) = 2 \cdot 2^2 \pi$$

$$\oint \vec{v} \cdot d\vec{r} \approx \sum (\vec{e} \times \vec{r}) \cdot \Delta \vec{F} = \sum \underbrace{(\vec{e} \times \vec{r}) \cdot \vec{r}}_2 \cdot \Delta r = 2 \cdot 2^2 \pi = 2 \cdot 2^2 \pi$$

② $\vec{v} = \vec{r} \rightarrow \operatorname{div} \vec{v} = 3$

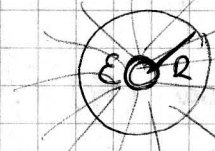


gömb

$$\iiint \operatorname{div} \vec{v} \, dV = \sum 3 \Delta V = 3 \cdot \frac{4 \cdot 2^3 \pi}{3} = 4 \cdot 2^3 \pi$$

$$\oint \vec{v} \cdot d\vec{F} = \sum |\vec{r}| |\Delta \vec{F}| = 2 \cdot 4 \cdot 2^2 \pi = 4 \cdot 2^3 \pi$$

③ $\vec{v} = k \cdot Q \frac{\vec{r}}{|\vec{r}|^3} = \vec{E}$



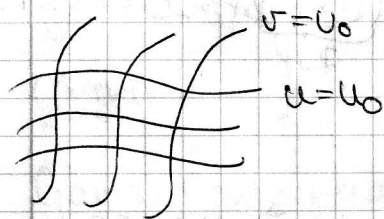
$$\operatorname{div} \vec{E} = 0$$

$$|\vec{r}| \neq 0$$

$$\iiint \operatorname{div} \vec{v} \, dV = 0 \quad \text{ha } \epsilon \neq \text{ körvonal}$$

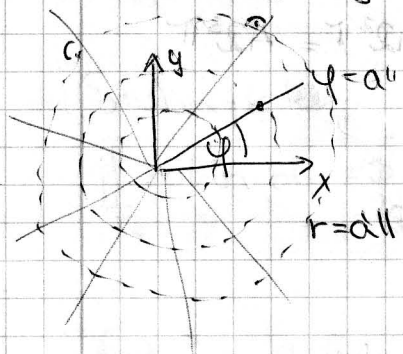
$$\begin{aligned} \oint \vec{E} \cdot d\vec{F} &= \sum |\vec{E}| \cdot |\Delta \vec{F}| \cdot 4 \cdot 2^2 \pi = 4 \cdot 2^2 \pi \cdot Q \rightarrow \text{kinenő} \\ &= - \text{bejövő} \\ &= 0 \end{aligned}$$

Vektoranalízis görbevonalú koordinátarendszerekben



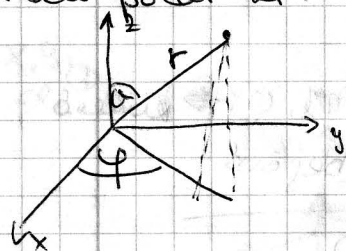
Polaris: $u=r$ $v=\varphi$

$$x=r \cdot \cos \varphi \quad y=r \sin \varphi \quad r=\sqrt{x^2+y^2} \quad \varphi=\arctg \frac{y}{x}$$



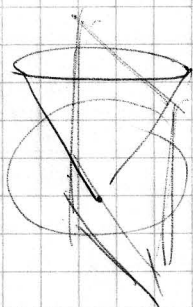
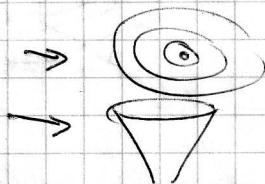
$\varphi = \text{all} \perp r = \text{all}$
 $\rightarrow$ derékszögű / ortogonális
 görbesíki koordinátarendszer

Térbeli-polaris kr:



$$\left. \begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \arccos \frac{z}{\sqrt{x^2 + y^2 + z^2}} \\ \varphi &= \arctg \frac{y}{x} \end{aligned} \right\} (r, \theta, \varphi)$$

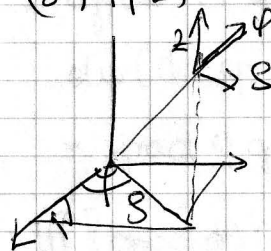
$$\begin{aligned} r &= r_0 = \\ \theta &= \theta_0 \\ \varphi &= \varphi_0 \\ 0 \leq \varphi \leq 2\pi \\ 0 \leq \theta \leq \pi \\ 0 \leq r \leq \infty \end{aligned}$$



$$x = r \cdot \sin \theta \cdot \cos \varphi \quad y = r \sin \theta \cdot \sin \varphi \quad z = r \cos \theta$$

Henger-kr

(s, φ, z) : síkbeli polaris + z



$$\begin{aligned} \phi(\vec{r}) &= \phi(r, \theta, \varphi) \\ \vec{e}(\vec{r}) &= \begin{pmatrix} e_r(r, \theta, \varphi) \\ e_\theta(r, \theta, \varphi) \\ e_\varphi(r, \theta, \varphi) \end{pmatrix} \end{aligned}$$

pl: $\vec{B}(\vec{r})$ mágneses indukció egyenes vezetékénél

$$B_s = B_z = 0$$

$$B_\varphi(s) = \frac{aI}{s}$$

$$\Delta \phi \approx \text{grad } \phi \cdot \Delta \vec{r} = (\text{grad } \phi)_s \Delta s + (\dots)_\varphi \Delta \varphi + (\dots)_z \Delta z$$

$$\phi(s_0 + \Delta s, \varphi_0 + \Delta \varphi, z_0 + \Delta z) - \phi(s_0, \varphi_0, z_0) \approx v_s \Delta s + v_\varphi \Delta \varphi + v_z \Delta z$$

$$\text{legyen } \Delta \varphi = 0 \quad \Delta z = 0 \quad \Delta s \neq 0$$

$$v_s = \frac{\Delta \phi}{\Delta s} \rightarrow \frac{\partial \phi}{\partial s} \quad \varphi = \text{all}, z = \text{all}$$

$$\text{legyen } s = 0 \quad \Delta z = 0 \quad \Delta \varphi \neq 0$$

$$v_\varphi = \frac{\Delta \phi}{s \Delta \varphi} = \frac{1}{s} \frac{\partial \phi}{\partial \varphi}$$

$$h_a(u, v, w) \rightarrow (u + \Delta u, v + \Delta v, w + \Delta w)$$

$$\text{te\u00edny le\u00edges elemek} \cdot \Delta s = \sqrt{h_u^2 \Delta u^2 + h_v^2 \Delta v^2 + h_w^2 \Delta w^2}$$

$$\Delta s = h_u \cdot \Delta u \quad \text{ha csak } u \text{ v\u00e1ltozik}$$

$$(\text{grad } \phi)_u = \frac{1}{h_u} \cdot \frac{\partial \phi}{\partial u} \dots$$

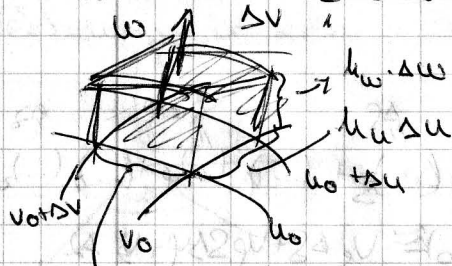
T\u00e9rbeli polar

$$h_r = 1 \quad h_\varphi = r \quad h_\psi = r \cdot \sin \varphi$$

$$\text{grad } \phi = \begin{pmatrix} \frac{\partial \phi}{\partial r} \\ \frac{1}{r} \frac{\partial \phi}{\partial \varphi} \\ \frac{1}{r \sin \varphi} \frac{\partial \phi}{\partial \psi} \end{pmatrix}$$

divergencia

$$\text{div } \vec{v} \approx \frac{1}{\Delta V} \sum \vec{v} \cdot \Delta \vec{r} \rightarrow 12 \cdot v_u(u_0 + \Delta u) - v_u(u_0, -) h_v h_w =$$



$$= \frac{\partial}{\partial u} (v_u \cdot h_v \cdot h_w) \cdot \frac{1}{h_u \cdot h_v \cdot h_w}$$

$$h_v = \Delta v$$

$$\Delta V = (h_u \cdot h_v \cdot h_w) (\Delta u \cdot \Delta v \cdot \Delta w)$$

$$\text{div } \vec{v} = \frac{1}{h_u h_v h_w} \left(\frac{\partial}{\partial u} (v_u \cdot h_v \cdot h_w) + \frac{\partial}{\partial v} (\dots) + \dots \right)$$

Gauss-O. t\u00e9tel.

$$\iiint \text{div } \vec{v} dV = \oint \vec{v} \cdot d\vec{r}$$

$$\text{legyen } \vec{v} = \phi(\vec{r}) \text{grad } \psi(\vec{r})$$

$$\begin{aligned} \text{div}(\phi \cdot \text{grad } \psi) &= \nabla \cdot (\phi \cdot \nabla \psi) = (\nabla \phi) \cdot (\nabla \psi) + \phi \nabla^2 \psi = \\ &= \text{grad } \phi \cdot \text{grad } \psi + \phi \Delta \psi \end{aligned}$$

$$\iiint [\text{grad } \phi \cdot \text{grad } \psi + \phi \Delta \psi] dV = \oint \phi \text{grad } \psi \cdot d\vec{r} - \underbrace{(\dots)}_{\text{I. (szimmetrikus) Green-t\u00e9tel}}$$

$$\begin{aligned} &= \oint (\phi \text{grad } \psi - \psi \text{grad } \phi) \cdot d\vec{r} \\ &\quad \downarrow \text{szimmetrikus Green-t\u00e9tel} \end{aligned}$$

deriv\u00e1ci\u00f3i egyenletek

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} \quad x_i \rightarrow u_i$$

ortogon\u00e1lis g\u00f6r\u00e9sek

$$X(u, v, w) \leftrightarrow u(x, y, z)$$

$$\Delta x = \frac{\partial x}{\partial u} \Delta u + \frac{\partial x}{\partial v} \Delta v + \frac{\partial x}{\partial w} \Delta w$$

$$\begin{aligned} \Delta s^2 &= \Delta x^2 + \Delta y^2 + \Delta z^2 = (h_u)^2 \Delta u^2 + (h_v)^2 \Delta v^2 + (h_w)^2 \Delta w^2 + \\ &+ (\dots) \Delta u \Delta v + (\dots) \Delta u \Delta w + (\dots) \Delta v \Delta w \end{aligned}$$

$$u \perp v \perp w$$

$$\textcircled{1} u = s \quad v = \varphi \quad w = z^* \quad (\text{henger-kr})$$

$$x = s \cos \varphi \quad y = s \sin \varphi \quad z = z^*$$

$$\Delta x = \cos \varphi \Delta s - s \sin \varphi \Delta \varphi$$

$$\Delta y = \sin \varphi \Delta s + s \cos \varphi \Delta \varphi$$

$$\Delta z = \Delta z^*$$

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = \overset{\sin^2 \varphi \cos^2 \varphi}{1} \Delta s^2 + s^2 \Delta \varphi^2 + \Delta z^{*2}$$

$$h_s = 1 \quad h_\varphi = s \quad h_{z^*} = 1$$

$$\begin{aligned} \text{ha} \quad u &= r & h_r &= 1 \\ v &= \varphi & h_\varphi &= r \\ w &= z & h_z &= 1 \end{aligned}$$

$$\Delta \phi = \text{grad } \phi \cdot \Delta \vec{r} = (\text{grad } \phi)_u \cdot h_u \Delta u + (\text{grad } \phi)_v \cdot h_v \Delta v + (\text{grad } \phi)_w \cdot h_w \Delta w$$

$$(\text{grad } \phi)_u = \frac{\Delta \phi}{\Delta u \cdot h_u} \Big|_{v, w = \text{const}} = \frac{1}{h_u} \frac{\partial \phi}{\partial u}$$

$$\text{pe. } (\text{grad } \phi)_\varphi = \frac{1}{r} \frac{\partial \phi}{\partial \varphi}$$

$$(\text{grad } \phi)_z = \frac{1}{h_z} \frac{\partial \phi}{\partial z}$$

$$\text{div } \vec{a}$$



$$\text{div } \vec{a} \cdot \Delta V = \sum_i \vec{a}_i \cdot \vec{dF}_i \cdot \frac{1}{\Delta V}$$

$$\text{div } \vec{a} = \frac{1}{h_u h_v h_w} \left[\frac{\partial}{\partial u} (a_u \cdot h_v \cdot h_w) + \frac{\partial}{\partial v} (a_v \cdot h_u \cdot h_w) + \frac{\partial}{\partial w} (a_w \cdot h_u \cdot h_v) \right]$$

$$\text{div } \vec{a} = \frac{1}{\prod_i h_i} \sum_{i=1}^3 \frac{\partial}{\partial u_i} \left(a_i \frac{h_1 \cdot h_2 \cdot h_3}{h_i} \right)$$

pe. henger-kr.

$$\begin{aligned} \text{div } \vec{v} &= \frac{1}{s} \left(\frac{\partial}{\partial s} (v_s \cdot s) + \frac{\partial}{\partial \varphi} (v_\varphi) + \frac{\partial}{\partial z} (v_z \cdot s) \right) = \\ &= \frac{1}{s} \cdot \frac{\partial}{\partial s} (v_s \cdot s) + \frac{1}{s} \frac{\partial v_\varphi}{\partial \varphi} + \frac{\partial v_z}{\partial z} \\ &= \frac{\partial v_s}{\partial s} + \frac{v_s}{s} \end{aligned}$$

$$\textcircled{1} \vec{v} = \begin{pmatrix} v(s) \\ 0 \\ 0 \end{pmatrix} \quad \text{ha } \text{div } \vec{v} = 0$$

$$s \cdot f(s) = k$$

$$f(s) = \frac{k}{s}$$

$$\text{rot } \vec{a}$$

$$\text{rot } \vec{a} \cdot \Delta \vec{r} = \sum_i \vec{a}_i \cdot \vec{dF}_i$$

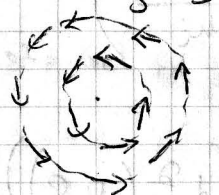
$$(\text{rot } \vec{a})_w \cdot h_u \Delta u \cdot h_v \Delta v = a_v \cdot h_v \Delta v \Big|_{u+\Delta u} - (\dots) \Big|_u + \dots$$

$$(\text{rot } \vec{a})_w = \frac{1}{h_u h_v} \left[\frac{\partial}{\partial u} (a_v h_v) - \frac{\partial}{\partial v} (a_u h_u) \right]$$

$$(\text{rot } \vec{a})_i = \frac{h_i}{h_1 h_2 h_3} \left(\sum_{k,l} \epsilon_{k,l,i} \frac{\partial}{\partial x_k} (v_l h_l) \right)$$

pl. hengerben

$$(\text{rot } \vec{v})_z = \frac{1}{h_\theta h_\varphi} \left(\frac{\partial}{\partial s} (v_\varphi h_\varphi) - \frac{\partial}{\partial \varphi} (v_s h_s) \right) = \frac{1}{s} \frac{\partial}{\partial s} (v_\varphi s) - \frac{1}{s} \frac{\partial v_s}{\partial \varphi}$$



$$v_s = 0$$

$$v_\varphi = f(s)$$

$$v_z = 0$$

pl. végtelen hosszú
mágneses tere

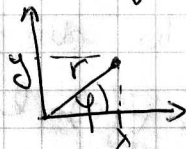
$$\text{rot} = \frac{1}{s} \frac{d}{ds} (s \cdot f(s)) \quad \text{ha } f(s) = \frac{k}{s} \rightarrow \text{rot} \vec{v} = 0 \quad s \neq 0$$

Komplex változós függvények

$$z = x + iy = r e^{i\varphi}$$

$$f(z) = w = u + iv$$

$$u(x, y) \quad v(x, y)$$



pl: $w = z^2 = \underbrace{x^2 - y^2}_u + i \underbrace{(2xy)}_v$

komplex számok leképezése

$$w = z + z_0 : z_0 \text{-al eltolás}$$

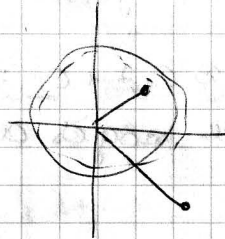
$$w = \lambda z : \text{nagyítás}$$

$$w = z_0 \cdot z = r_0 e^{i\varphi_0} \cdot r e^{i\varphi} = (r \cdot r_0) e^{i(\varphi + \varphi_0)} : \text{nyújt és forgat}$$

$$w = \frac{1}{z}$$

$$z = r \cdot e^{i\varphi}$$

$$w = \frac{1}{r} e^{-i\varphi} : \text{kör belsejét és külsőjét felcseréli}$$



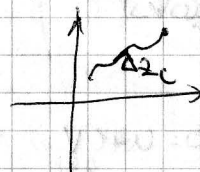
$$w = \ln z = \ln(r) + i\varphi$$

deriválás, integrálás

$$f'(z) \Big|_{z_0} = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

pl: $(z^2) \Big|_{z_0} = \lim_{\Delta z \rightarrow 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{2z_0 \Delta z + \Delta z^2}{\Delta z} = 2z_0$

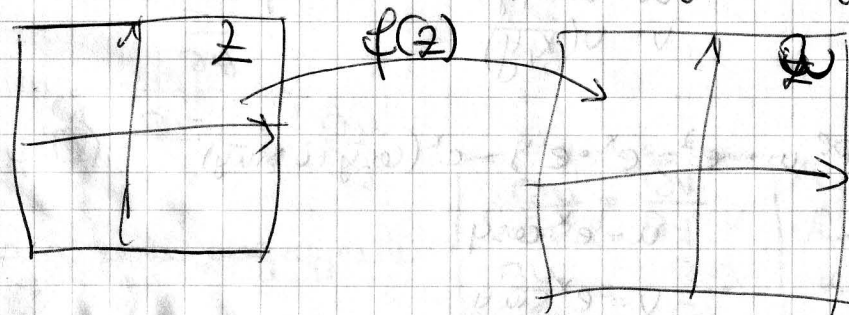
$$\int f(z) dz = \lim \sum f(z) \Delta z_i$$



~ vektormező vonalintegrálja

x1.6

$$w(z) = C_1 z + C_2 : \text{lineáris fv: arány és eltolás}$$



valós fv: $y(x \approx x_0) \approx y(x_0) + y'(x_0)(x - x_0) = c_1 x + c_2$

ha $w(z)$ "sima": $w(z) \approx c_1 z + c_2$
 kicsiben szög és aránytartó

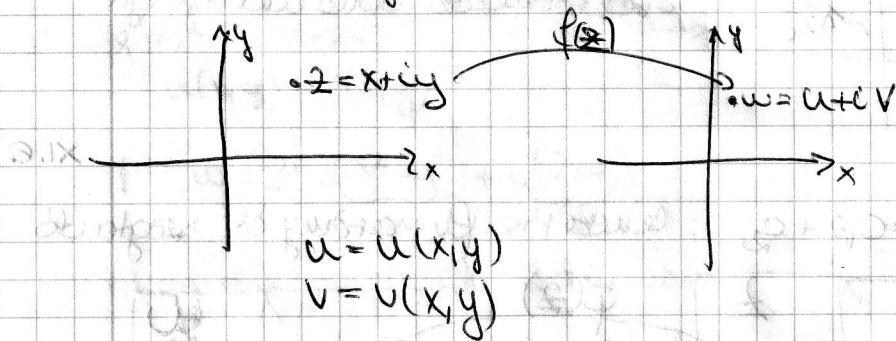
deriválható:

$$w'(z) \Big|_{z_0} = \lim_{z \rightarrow z_0} \frac{w(z) - w(z_0)}{z - z_0} \rightarrow \text{iránytól független}$$

komplex változós függvények vagy egyrendszerű vagy végteleneszer deriválható (= analitikus)

mert a deriválhatónak lennie van, hogy minden irányból tudnunk kell (valóban csak 2 irányban kell megvizsgálni)

Deriválás a komplex számokon
 (Cauchy-Riemann relációk)



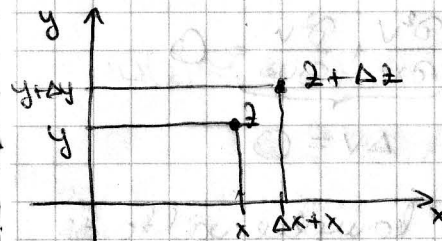
$$e^z w = e^z = e^x \cdot e^{iy} = e^x (\cos y + i \sin y)$$

$$u = e^x \cos y$$

$$v = e^x \sin y$$

$$f'(z) \Big|_{z_0} = \lim_{\Delta z} \frac{\Delta u + \Delta v \cdot i}{\Delta x + \Delta y \cdot i}$$

$$\text{ahol } \Delta u = u(x + \Delta x, y + \Delta y) - u(x, y)$$



melyik irányból megyek?

• legyen $\Delta y = 0$

$$f'(z) = \underbrace{\lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x}}_{\frac{\partial u}{\partial x}} + i \underbrace{\lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x}}_{\frac{\partial v}{\partial x}} \Big|_{\Delta y = 0}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

• legyen $\Delta x = 0$

$$f'(z) = \underbrace{\lim_{\Delta y \rightarrow 0} \frac{\Delta v}{\Delta y}}_{\frac{\partial v}{\partial y}} + i \underbrace{\lim_{\Delta y \rightarrow 0} \frac{\Delta u}{\Delta y}}_{\frac{\partial u}{\partial y}}$$

$$f'(z) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

ha a kettő megegyezik:

$$\left[\begin{array}{l} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{array} \right] \text{ Cauchy-Riemann reláció}$$

könvetkezmények:

$$\frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = - \frac{\partial^2 v}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

$$\Delta u = 0 \quad \text{és}$$

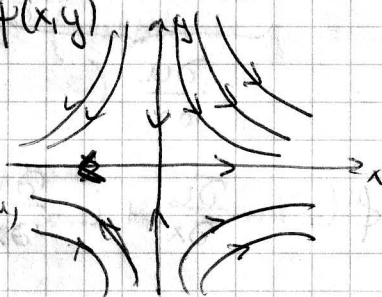
$$\Delta v = 0$$

vagyis egy deriválható komplex változós függvény valós és képzetes része automatikusan teljesíti a Laplace-egyenletet

pl: $w = z^2 = \underbrace{(x^2 - y^2)}_{\phi(x,y)} + i \underbrace{(2xy)}_{\psi(x,y)}$

$$\vec{v} = \text{grad } \phi = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$$

pl: áramlás vektor, kvadrupól tér



$$w = A \ln z = A(\ln r + i\phi)$$

$$u(x,y) = A \ln \sqrt{x^2 + y^2}, \quad v(x,y) = A \arctan \frac{y}{x}$$

$$\phi(r) = A \ln r$$



$$(\text{grad } \phi)_r = \frac{\partial \phi}{\partial r} = \frac{A}{r}, \quad (\text{grad } \phi)_\phi = \frac{1}{r} \frac{\partial \phi}{\partial \phi} = 0$$

$$\psi(x,y) = A \arctan \frac{y}{x} = A\phi$$

$$(\text{grad } \psi)_r = \frac{\partial \psi}{\partial r} = 0, \quad (\text{grad } \psi)_\phi = \frac{1}{r} \frac{\partial \psi}{\partial \phi} = \frac{A}{r}$$

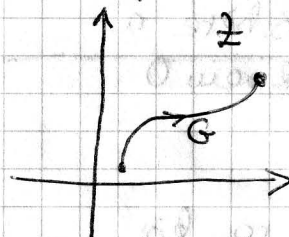
Ha $w = f(z)$ deriválható, akkor

$$\frac{w(z) - w(z_0)}{z - z_0} \approx f'(z_0)$$

$$w(z) \approx w(z_0) + f'(z_0)(z - z_0) = C_1 z + C_0$$

ha $f'(z_0) \neq 0 \rightarrow$ ha mégis: mindig több-mennyi

Komplex vonalintegrál



$$w = f(z)$$

$$\int_{\Gamma} f(z) dz = \lim_{|\Delta z| \rightarrow 0} \sum_{i=1}^n f(z_i) \Delta z_i =$$

$$= \underbrace{\lim \sum (u \Delta x - v \Delta y)}_{I_1} + i \underbrace{\lim \sum (v \Delta x + u \Delta y)}_{I_2}$$

$$I_1 = \int_{\Gamma} \vec{a}(\vec{r}) \cdot d\vec{r}, \quad \vec{a} = \begin{pmatrix} u \\ -v \end{pmatrix}$$

$$I_2 = \int_{\Gamma} \vec{b}(\vec{r}) \cdot d\vec{r}, \quad \vec{b} = \begin{pmatrix} v \\ u \end{pmatrix}$$

akkor nem függ utvonalról, ha $\vec{a}$ és $\vec{b}$ irrotációmentesek:

$$(\text{rot } \vec{a})_x = 0 - 0 = 0, \quad (\text{rot } \vec{a})_y = 0$$

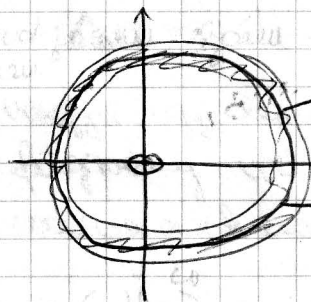
$$(\text{rot } \vec{a})_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (C-2)$$

$$(\text{rot } \vec{b}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

következő:

$\oint f(z) dz = 0$ - Stokes tétel következménye:
egyszeresen összefüggő tartomány

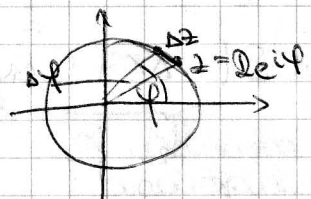
pl: $f(z) = \frac{1}{z}$ - origóban nem jó



ez a tartomány nem egyszeresen összefüggő
mire a görbét a körintegrál nem 0

$$\oint \frac{1}{z} dz:$$

$|z|=R$: görbe egy kör

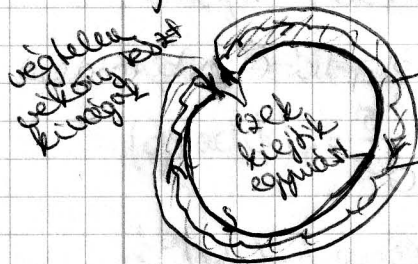


$$z = R \cdot e^{i\phi}$$

$$\Delta z = \frac{dz}{d\phi} \cdot \Delta\phi = i R e^{i\phi} \Delta\phi$$

$$\sum \frac{1}{z} \cdot \Delta z = \sum \frac{1}{R e^{i\phi}} \cdot i R e^{i\phi} \Delta\phi = \sum i \Delta\phi = i \sum \Delta\phi = 2\pi i$$

$$\oint \frac{1}{z} dz = 2\pi i \quad \text{Cauchy tétel}$$



ez már egyszeresen összefüggő

ez az a görbe odavittük,

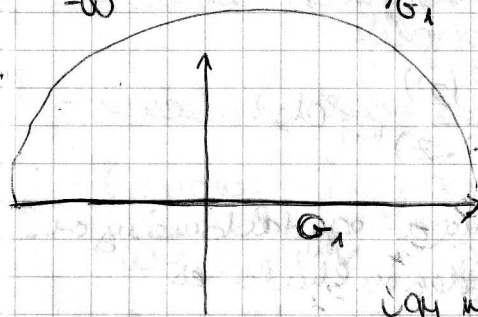
$$\oint f(z) dz = 0$$

minden görbét $2\pi i$

$$\oint \frac{1}{z^n} dz = 0, \text{ ha } n \neq 1$$

Alkalmazás:

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \int_{G_1} \frac{1}{1+z^2} dz, \text{ ahol } G_1 \text{ a valós tengely}$$



0-hoz tart ennek a félkörnek az integrálja ($R \rightarrow \infty$)

így már csak görbe

$$\oint \frac{1}{z} dz = 2\pi i$$

3.12.

$$z = R e^{it}$$

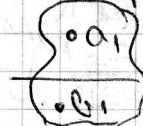
$t \in [0, 2\pi]$ spec. eset körre

$$\oint \frac{1}{z} dz = \int_0^{2\pi} \frac{dz}{dt} \frac{1}{z(t)} dt = \int_0^{2\pi} R i e^{it} \frac{1}{R e^{it}} dt = \int_0^{2\pi} i dt = 2\pi i$$

$$\oint \frac{1}{z^n} dz = \int_0^{2\pi} i R e^{it} \frac{1}{(R e^{it})^n} dt = \int_0^{2\pi} i R^{1-n} e^{it(1-n)} dt = 0$$

$$\frac{a_1}{z-z_0} = f(z)$$

elsőrendű pólus z_0 -ban



$$\oint \frac{a_1}{z-z_0} dz = 2\pi i a_1$$

$$\oint f(z) dz = \sum_{\text{pólusok}} 2\pi i \cdot \text{pólusresz}$$

XI. 13.

$$\oint \frac{f(z)}{z-z_0} dz = 2\pi i \cdot f(z_0)$$

$$\frac{1}{2\pi i} \oint \frac{f'(z)}{z-z} dz = f(z)$$

$$f'(z) = \frac{1}{2\pi i} \oint \frac{f(z)}{(z'-z)^2} dz$$

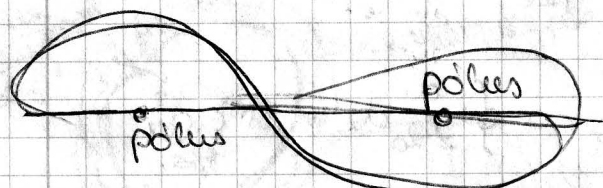
$$f^{(n)}(z) = \frac{1}{2\pi i} n! \oint \frac{f(z)}{(z'-z)^{n+1}} dz$$

vagyis ha egy fv. sima egy tartományon
akkor akárhány-szor deriválható és
előadható a tartomány szélén futó
körintegrálból

$$\oint \frac{e^z}{z^2} dz \approx \oint \frac{1 + \frac{z^2}{2!} + \frac{z^3}{3!}}{z^2} dz = 1 \cdot 2\pi i$$

0-ban másodrendű pólus: ha e^z konstans lenne, akkor 0 lenne

$$\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \int_{-\infty}^{\infty} \frac{\sin z}{z} dz = \int_{-\infty}^{\infty} \frac{e^{iz}}{2iz} dz - \int_{-\infty}^{\infty} \frac{e^{-iz}}{2iz} dz$$



$$= \frac{1}{2i} \cdot 2\pi i \cdot 1 \cdot (-1) = -\pi$$

korrekció

Singularitások osztályozása

Ha $\sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k}$: lényeges szingularitás : végtelen sok tag

Ha $pe = \lim_{z \rightarrow z_0} (z-z_0)f(z)$: vágás

Analitikus (komplex változós, deriválható) XI.20. függvények

- lokálisan szög és aránytartó ha $w(z) \neq 0$

- Cauchy-Riemann relációk

$$\left. \begin{aligned} w &= u + iv \\ z &= x + iy \end{aligned} \right\} \begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \text{ és } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{aligned}$$

$$\text{div-grad } u = \Delta u = 0$$

$$\Delta v = 0$$

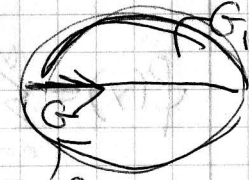
- $\oint w(z) dz = 0$ egyszerűen összefüggő tartományban

ha $w(z)$ analitikus (kivéve pólusok)
izolált szing. pólus $\frac{f(z)}{z-z_0}$

$$\oint w(z) dz = 2\pi i \sum \text{résztartó pólus erősség}$$

$$f = 2\pi i f(z_0)$$

$$pe \int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \int_{-\infty}^{\infty} \frac{\cos z}{1+z^2} dz =$$



$$= \oint_{G_1 \cup G_2} \frac{e^{iz}}{2(2+i)(2-i)} dz = \oint_{G_1 \cup G_2} \frac{e^{-iz}}{2(2+i)(2-i)} dz = \oint_{G_2}$$

$$= 2\pi i \left(\frac{e^{iz}}{2(2+i)} \Big|_{z=i} - \frac{e^{-iz}}{2(2-i)} \Big|_{z=-i} \right) = 2\pi i \frac{1}{4e} + 2\pi i \frac{1}{4e} = \frac{\pi}{e}$$

Fourier-analízis

Sor:

$$f(x) = f(x - 2\pi)$$

Fourier-sorozat (követelés, hullámmód)

$$f(x) = () \cos x + () \sin x \dots$$

$$= \sum a_n \cos(nx) + b_n \sin(nx)$$

$$\int_0^{2\pi} f(x) \sin mx = b_m \int_0^{2\pi} \sin^2 mx dx = \pi \cdot b_m$$

$$b_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin mx dx$$

$$a_m = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos mx dx \quad (m \neq 0)$$

$$a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x) \cos mx dx \quad (m = 0)$$

$$f(x) = a_0 + a_1 \frac{e^{ix} + e^{-ix}}{2} + \dots + b_1 \frac{e^{ix} - e^{-ix}}{2} \dots$$

$$= a_0 + e^{ix} \left(\frac{a_1}{2} + \frac{b_1}{2i} \right) + e^{2ix} \left(\frac{a_2}{2} + \frac{b_2}{2i} \right) + \dots$$

$$+ e^{-ix} \left(\frac{a_1}{2} - \frac{b_1}{2i} \right) \dots$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$\text{formálva: } \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx = c_n$$

$$\left[\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n \cdot e^{inx} \\ c_n &= \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \end{aligned} \right]$$

$$f(x) \rightarrow c = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \end{pmatrix} \text{ végtelen komponensű vektor}$$

$$f(x) \leftrightarrow c_n \quad n = -\infty, \dots, \infty$$

$$f(x) \leftrightarrow d_n = i n \cdot c_n$$

Fourier-transzformáció:
(nem periodikus függvények)

$$f(x) \text{ periódusa: } L \rightarrow \infty$$

$$f(x) = f(x+L) \quad x = \frac{L}{2\pi} \cdot u$$

$$(x, x+L) \rightarrow (u, u+2\pi)$$

$$f(x = \frac{L}{2\pi} \cdot u) \equiv g(u)$$

$$g(u) = g(u+2\pi) = \sum_{n=-\infty}^{\infty} c_n e^{inu}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} g(u) e^{-inu} du$$

$$L \rightarrow \infty$$

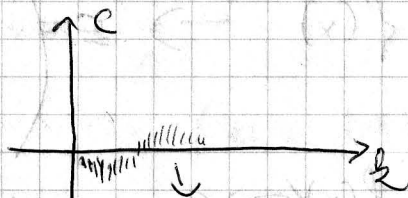
$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(u) e^{-inu} du = \frac{1}{2\pi} \int_{-L/2}^{L/2} f(x) e^{-in \frac{2\pi}{L} x} \frac{2\pi}{L} dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in \frac{2\pi}{L} x}$$

jelölés $n \cdot \frac{2\pi}{L} = k_n$

$$c_n = \frac{1}{L} \int f(x) e^{-ik_n x} dx$$

$$f(x) = \sum_n c_n e^{ik_n x}$$



$L \rightarrow \infty$: besűrűsödnek a pontok az tengelyen

együttessel
majdnem ugyan-
akkorok

$$\tilde{f}(k) = L \cdot c_n = \int_{-L/2}^{L/2} f(x) e^{-ik_n x} dx$$

$$L \rightarrow \infty$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{2\pi} \sum_n c_n e^{ik_n x} \Delta k \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-ikx} dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk$$

lineáris transzformáció a függvények térben

$$\widehat{\lambda f} = \lambda \cdot \widehat{f}$$

$$\widehat{f+g} = \widehat{f} + \widehat{g}$$

$$\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 \dots = \sum_{i=1}^n a_i b_i$$

komplex: $a^* = b^*$

$$f \cdot g = \int_{-\infty}^{\infty} f^*(x) \cdot g(x) dx$$

$f \cdot g = \widehat{f} \cdot \widehat{g}$ Fourier-transz a
végtelen dim. térben
a forgatás
(skalár szorzat megtartó)

$$f(x) \xrightarrow{F} \tilde{f}(k)$$

$$f'(x) \rightarrow \frac{1}{2\pi i} \int dk \tilde{f}(k) \frac{d}{dx} (e^{-ikx}) = -ik \tilde{f}(k)$$

diffegyenlet:

$$f(x) + 3 f(x) = e^{-x^2}$$

$$\widehat{f(x) + 3 \cdot f} = \widehat{e^{-x^2}}$$

$$\widehat{f(x)} [1+3] = \widehat{e^{-x^2}} = e^{-k^2/4}$$

Gauss = Gauss

$$e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{x^2}{2}} e^{ikx}$$

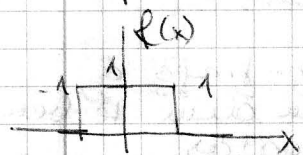
$$-\frac{x^2}{2} + ikx = -\frac{1}{2}(x-ik)^2 - \frac{k^2}{2}$$

$$= e^{-\frac{k^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} \int dx e^{-\frac{1}{2}(x-ik)^2} = e^{-\frac{k^2}{2}}$$

$$e^{-\frac{ax^2}{2}} = e^{-\frac{k^2}{2a^2}}$$

XI.24. $f(x) \in L^2$, wogis $\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$

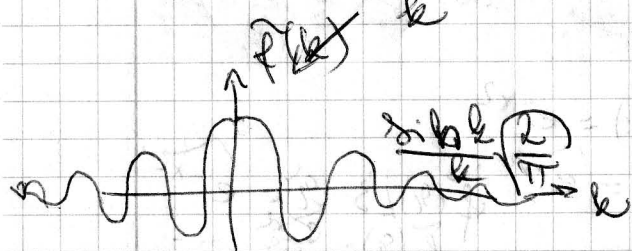
$\hat{f} \in L^2$



$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^1 1 e^{ikx} dx =$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{\sin kx}{k} \Big|_{-1}^1$$

$\cos kx$ + $\sin kx$
paros. + $\sin kx$
paros. $\hat{0}$



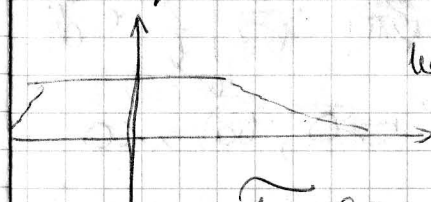
$$f(x) = e^{-\frac{x^2}{2a^2}} \Rightarrow \hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int e^{-\frac{x^2}{2a^2} + ikx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2 a^2}{4}} \int e^{-\frac{(x - \frac{ika^2}{2})^2}{2a^2}} dx$$

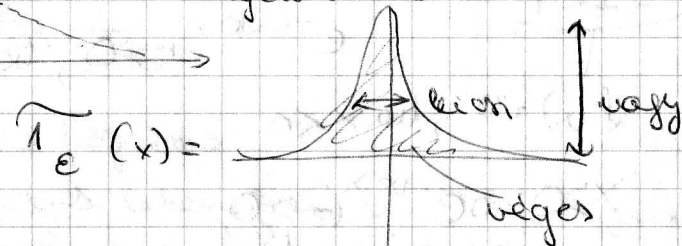
$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2 a^2}{4}} \int e^{-\frac{(x - \frac{ika^2}{2})^2}{2a^2}} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2 a^2}{4}} \underbrace{\int e^{-\frac{x^2}{2a^2}} dx}_{\sqrt{\pi} \cdot a^2} = \frac{a}{2} e^{-\frac{k^2 a^2}{4}}$$

$f(x) \notin L^2$



kegyelteszük $e^{-\varepsilon x^2}$ $\varepsilon \ll 1$



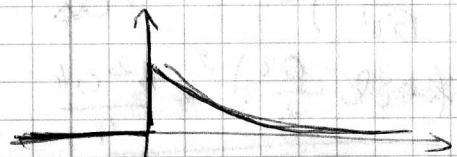
$$\int_{-\infty}^{\infty} \hat{f}_\varepsilon(x) dx = 1$$

de: $\hat{f}_\varepsilon(x) \approx 0$
kisebbe $x \rightarrow 0$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(k) e^{ikx} dk = \frac{1}{\sqrt{2\pi}} \int \hat{f}_\varepsilon(k) dk$$

$$\hat{f} = \frac{1}{2\pi} \hat{f}_\varepsilon(k)$$

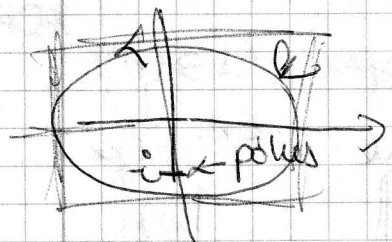
$$\widehat{\delta x} = [\delta(k-1) + \delta(k+1)]$$



$$f(x) = \begin{cases} 0, & \text{ha } x < 0 \\ e^{-x}, & \text{ha } x \geq 0 \end{cases}$$

$$\begin{aligned} \tilde{f}(k) &= \frac{1}{2\pi} \int_0^{\infty} e^{-x} e^{ikx} dx = \frac{1}{2\pi} \cdot \frac{1}{1-ik} \cdot \frac{1}{i} \\ &= \frac{1}{2\pi i} \cdot \frac{1}{k+i} \end{aligned}$$

fordítva: $\tilde{f}(k) = \frac{1}{2\pi i} \cdot \frac{1}{k+i} \Rightarrow f(x) = ?$



$$f(x) = \frac{1}{2\pi i} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{k+i} e^{-ikx} dk$$

$e^{-i\omega x} \rightarrow 0, \text{ ha } x < 0$

$$f(x) = 0, \text{ ha } x < 0$$

$$x > 0 \quad e^{-\omega x} \rightarrow 0$$

$$f(x) = 2\pi i \cdot \frac{1}{2\pi} \cdot \frac{1}{2\pi} e^x (-1) = e^x$$

$$\widehat{f'(x)} = -ik \tilde{f}(k)$$

$$f'(x) = \frac{1}{2\pi i} \int \tilde{f}(k) (-ik) e^{-ikx} dk$$

1. Differenciál

$$y'(x) + y(x) = h(x)$$

$$\widehat{y' + y} = \widehat{h}$$

$$\tilde{y}' + \tilde{y} = -ik \cdot \tilde{y} + \tilde{y} = \tilde{h}$$

$$(1-ik) \tilde{y} = \tilde{h}$$

$$\tilde{y}(k) = \frac{\tilde{h}(k)}{1-ik}$$

inverz Fourier-transzformáció

$$v'(t) + v(t) = F(t)$$

legyen $F(t) = \delta(t)$ erőösszeg

$$\tilde{\delta}_x = \frac{1}{2\pi i}$$

$$\tilde{G}(k) = \frac{1}{2\pi i} \cdot \frac{1}{1-ik}$$

Green-fü. fogalma

(közösre, inhomogén) lin. diff. egyenletre van Green-fü., ami δ -s inhomogén taghoz tartozik.

$$G'(x) + G(x) = \delta(x)$$

$$G'(x, x_0) + G(x, x_0) = \delta(x - x_0)$$

$$\rightarrow G(x, x_0) = G(x - x_0)$$

$$h(x) = \int h(x_0) \delta(x - x_0) dx_0$$

$$y(x) = \int G(x, x_0) h(x_0) dx_0$$

$$x \cdot y'(x) + 3y = h(x)$$

úgy

$$y'(x) + x y(x) = h(x) = \text{---}$$

$$\tilde{y}'(x) + x \tilde{y} = \tilde{h} = \frac{\sin k}{k} \cdot \text{all}$$

$$-ik \tilde{y}(k)$$

$$-i \tilde{f}(k) = \frac{\sin k}{k} \int dx x e^{ikx} f(x)$$

$$x f(x) = -i \tilde{f}'(k)$$

$$-ik \tilde{y}(k) - i \tilde{y}'(k) = \frac{\sin k}{k} \cdot \text{all}$$

$$y''' + x y = h$$

$$(-ik)^3 \tilde{y}(k) - i \tilde{y}'(k) = \tilde{h}$$

Varianciacímítás



$$\text{pl. } I = \int_a^b y(x) dx \quad \left. \vphantom{\int_a^b} \right\} \text{lin.}$$

$$I = y(0)$$

$$I = \int_1^3 |y(x)|^2 dx \rightarrow \text{nem lin.}$$

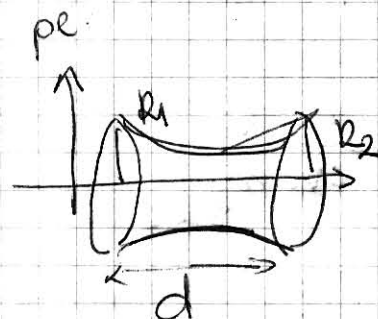
lokális

$$I = \int_{-10}^{10} y(x) y(x-x_0) dx \rightarrow \text{korrelációs integrál}$$

nem lokális

általában adott 3 változó

$$I = \int_{x_1}^{x_2} L(y(x), y'(x), x) dx \quad \begin{array}{l} \text{melyik } y \text{-től van} \\ \text{szélsőséges} \\ \text{alap feladat} \end{array}$$



$$E_{\text{felület}} = \sigma \cdot A =$$

$$= \sigma \cdot 2 \cdot \pi \cdot \int_0^d \sqrt{1 + y'^2(x)} y(x) dx$$

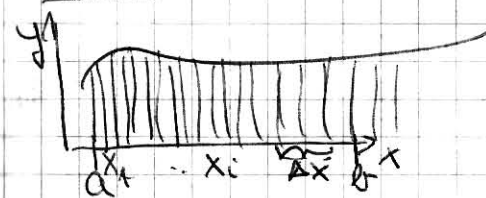
$$L(y, y', x)$$

$$I = \int_{a_1}^{a_2} L(y(x), y'(x), x) dx \quad \begin{array}{l} \text{extremális} \\ \text{adott} \end{array}$$

x1, x2

első körbe, ha nem változik.

Euler módszer



$$\int f(x) dx \rightarrow \sum_i L(y(x_i), \frac{y(x_{i+1}) - y(x_i)}{\Delta x}, x_i) \Delta x$$

$$I(y_1, \dots, y_n) = \text{extremum},$$

$$\text{ha } \frac{\partial I}{\partial y_i} = 0$$

$$I = L(y_1, \frac{y_2 - y_1}{\Delta x}, x_1) \Delta x + L(y_2, \frac{y_3 - y_2}{\Delta x}, x_2) \Delta x \dots$$

$$\text{pl. } \frac{\partial I}{\partial p} = 0 \Rightarrow \frac{\partial L}{\partial y'} \Big|_{i=1} \frac{1}{\Delta x} + \frac{\partial L}{\partial y} \Big|_{i=2} + \frac{\partial L}{\partial y'} \Big|_{i=2} \frac{1}{\Delta x} = 0$$

$$\frac{\partial L}{\partial y} \Big|_{i=1} \frac{1}{\Delta x} + \frac{\partial L}{\partial y} \Big|_{i=2} + \frac{\partial L}{\partial y'} \Big|_{i=2} \frac{1}{\Delta x} = 0$$

$$\frac{\partial L}{\partial y} \Big|_{x_i} = \frac{\frac{\partial L}{\partial y} \Big|_{x_{i+1}} - \frac{\partial L}{\partial y} \Big|_{x_{i-1}}}{\Delta x} = \frac{d}{dx} \frac{\partial L}{\partial y'} \Big|_{x_i}$$

$$I(y(x)) = I(y(x) + \varepsilon h(x)) = I(\varepsilon) \quad \begin{matrix} \text{teljes } \varepsilon \text{ rögz.} \\ \text{de } h(a) = 0 \\ h(b) = 0 \end{matrix}$$

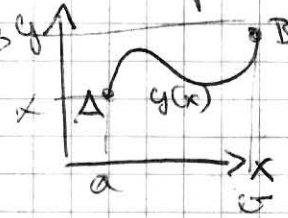
$$\frac{dI}{d\varepsilon} = 0$$

$$I = \int L(y_0 + \varepsilon h, y_0' + \varepsilon h'), x) dx$$

$$\frac{dI}{d\varepsilon} = \int \left[\frac{\partial L}{\partial y} h(x) + \frac{\partial L}{\partial y'} h'(x) \right] dx = \int \left[\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} \right] h(x) dx \stackrel{=0}{=} 0$$

$$\boxed{\frac{\partial L}{\partial y} - \frac{d}{dx} \left(\frac{\partial L}{\partial y'} \right)} \quad \text{Euler-Lagrange-egyenlet}$$

mi két pont között a legrövidebb út?



$$y(a) = \alpha \quad y(b) = \beta$$

$$\text{úthossz} = I = \int_a^b \sqrt{1+y'^2} dx \quad L = \sqrt{1+y'^2}$$

$$\frac{\partial L}{\partial y} = 0 \quad \frac{\partial L}{\partial y'} = \frac{y'}{\sqrt{1+y'^2}} \rightarrow \frac{d}{dx} \frac{\partial L}{\partial y'} = \frac{y''}{\sqrt{1+y'^2}} + \frac{y' y''}{(1+y'^2)^{3/2}}$$

$$y'' \left(\frac{1}{\sqrt{1+y'^2}} - \frac{y'^2}{(1+y'^2)^{3/2}} \right) = 0$$

$$(y')^2 = (1+y'^2) \quad \downarrow$$

$$y'' = 0 \rightarrow y' = c \rightarrow y = cx + a$$



fergősfelület / minimális felület

$$L = y \sqrt{1+y'^2}$$

$$\frac{\partial L}{\partial y} = \sqrt{1+y'^2} \quad \frac{\partial L}{\partial y'} = y \frac{y'}{\sqrt{1+y'^2}}$$

$$y'(1+y'^2)^{\frac{1}{2}} y' + y(1+y'^2)^{\frac{1}{2}} y'' + y y' \left(-\frac{1}{2}\right) (1+y'^2)^{\frac{3}{2}} 2y' y'' = (1+y'^2)^{\frac{1}{2}}$$

$$(y')^2 + y'' y - (y')^2 \frac{y y''}{(1+y'^2)^2} = (1+y'^2)^{\frac{1}{2}}$$

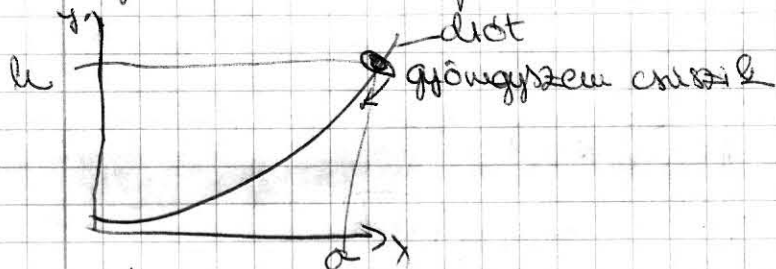
$$\frac{y y''}{1+y'^2} = 1 \quad \text{pl } y = chx$$

$$\frac{chx \cdot chx}{1+sh^2x} = 1 \quad \checkmark$$

alt. m.o:

$$y(x) = k \cdot ch \frac{x-x_0}{b}$$

Legrövidítés probléma



$$\frac{1}{2} m v^2 = m g (h-y)$$

$$v(x) = \sqrt{2g(h-y(x))}$$

$$\frac{dy}{dt} = |v_g| = \sqrt{2g(h-y(x))}$$

$$dt = \frac{dy}{\sqrt{2g} \sqrt{h-y(x)}} \cdot \frac{\sqrt{1+y'^2}}{y'} = \frac{dy}{dx} \frac{\sqrt{1+y'^2}}{\sqrt{2g} \sqrt{h-y(x)}} dx$$

$$\sqrt{2g} T = \min = \int_0^a dx \frac{\sqrt{1+y'^2}}{\sqrt{h-y}}$$

Euler-Lagrange megoldás: célfüggvény

Kiinduló L-függvény

$$\text{I. } L(y', x) \quad \frac{\partial L}{\partial y} = 0 \Rightarrow \frac{\partial L}{\partial y'} = \text{const}$$

$$\text{II. } L(y, x) \quad \frac{\partial L}{\partial y} = 0 : \text{ennek nincs semmi értelme}$$

$$\text{III. } L(y, y') \Rightarrow \frac{\partial L}{\partial y} = \frac{d}{dx} \frac{\partial L}{\partial y'}$$

de (mivel):

$$\text{I} = \int L dx = \int \frac{dx}{dy} L dy$$

$$\text{extr.} = \int L^*(\dots) dy \rightarrow$$

$$\frac{\partial L^*}{\partial x} = \frac{d}{dy} \left(\frac{\partial L^*}{\partial x'} \right) = \text{const}$$

$$L^* = x' L(y, \frac{1}{x'} x)$$

$$\frac{\partial L}{\partial x} = 1 \cdot L(\dots) + x \cdot \frac{\partial L}{\partial y} \left(-\frac{1}{x} \right) = \text{all}$$

$$-L + y \cdot \frac{\partial L}{\partial y} = \text{all}$$

H (ha L ~~is~~ energia állandós)

ha több ismeretünk van

$$\text{pl } I = \int L(q_1, q_2, \dot{q}_1, \dot{q}_2, t)$$

$$\text{akkor } \frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$$

$$\text{ha } I = \int L(y, y', y'', \dots) dx$$

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial L}{\partial y''} = 0$$



$$\text{energia} = \text{munka} = \int (p(x)y(x) + k(y'')^2) dx$$

XII.11.

Összefoglalás

Vektoralízis

gradiens: skálár $\rightarrow$ vektor

irányja: legnagyobb változás felé mutat

merőleges: optimális odébbmenésre
menés $\rightarrow$ változik

$$\Delta \phi \approx \text{grad } \phi \cdot \Delta \vec{r}$$

$$(\text{grad } \phi)_i = \frac{\partial \phi}{\partial x_i}$$

$$\text{pl: } \phi = |x^2 + y^2 + z^2|$$

$$(\text{grad } \phi)_x = \frac{\partial \phi}{\partial x} = 2(x^2 + y^2 + z^2) \cdot 2x = 4r^2 x$$

$$\text{grad } \phi = 4r^2 \vec{r}$$

$$\text{pl: } \phi(a, b^2) \text{ v. } (a, b) \text{ (} b \text{)} \\ \text{v. } (a, b)^2$$

divergencia: vektor $\rightarrow$ skálár
forrás erősség

$$\text{divergens: } \nabla \cdot \vec{v} = \text{const}$$

$$\vec{v} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$$

$$\text{nem divergens } \vec{v} = \text{konstans} \\ \vec{v} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \vec{v} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\text{div } \underline{v} = \sum_i \partial_i v_i$$

rotáció
örvényerősség

$$(\text{rot } \underline{v})_x = \frac{\partial}{\partial y} v_z - \frac{\partial}{\partial z} v_y$$

Integrálok

vonalmenti integrál

$$\int_G \underline{v} \cdot d\underline{r} \approx \sum \underline{v} \cdot \Delta \underline{r}$$

felületi integrál

$$\int_F \underline{v} \cdot d\underline{F} \approx \sum \underline{v} \cdot \Delta \underline{F}$$

terfogat integrál

$$\iiint_V \phi(\underline{r}) dV \approx \sum (\phi) \Delta V$$

2. tétel:

$$G \rightarrow O: \iiint \text{div } \underline{v} dV = \oint \underline{v} \cdot d\underline{F}$$

$$S: \iint \text{rot } \underline{v} \cdot d\underline{F} = \oint \underline{v} \cdot d\underline{r}$$

$$\int_A^B \text{grad } \phi d\underline{r} = \phi(B) - \phi(A) : 1. \text{ gradient-tétel}$$

pl:



$$\begin{aligned} \int_G \underline{r} \cdot d\underline{r} &= \int_G \text{grad} \left(\frac{r^2}{2} \right) \cdot d\underline{r} \\ &= \left. \frac{r^2}{2} \right|_0^1 = \frac{1}{2} - 0 = \frac{1}{2} \end{aligned}$$

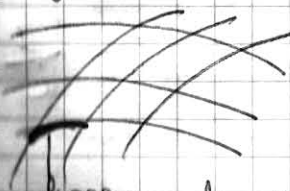
pl:

$$\oint (\underline{e} \times \underline{r}) \cdot d\underline{r} = 2 \cdot 2\pi \cdot 2 = 8\pi$$



$$\oint (\underline{e} \times \underline{r}) \cdot d\underline{F} = 0$$

Görbemenet koordinátatranszformációk



hossz: $h_1 \cdot \Delta u_1$

$$\Delta s^2 = (h_1 \Delta u_1)^2 + (h_2 \Delta u_2)^2 + (h_3 \Delta u_3)^2$$

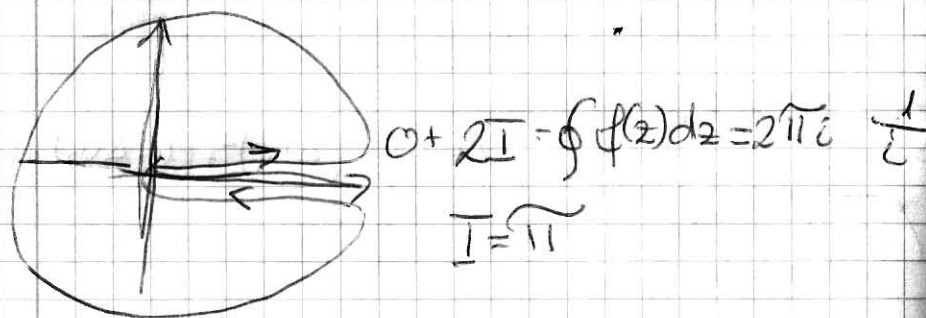
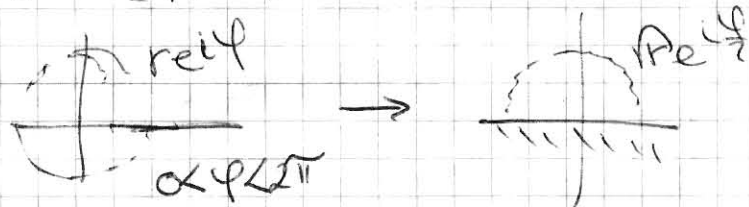
Komplex változós függvények

$$C-D: \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\oint f(z) dz = \begin{cases} 0 & \text{ha analitikus} \\ 2\pi i \cdot \text{pólusok összege} \end{cases}$$

$$pe \int_0^{\infty} \frac{1}{(x+i)^2} dx = \begin{matrix} -1 \text{ udl pólus} \\ 0 \text{ udl végpont} \end{matrix}$$

$$\sqrt{z} (re^{i\varphi})^{\frac{1}{2}} = \sqrt{r} e^{i\frac{\varphi}{2}}$$



Fourier-tr.

Fourier-sorfejtés általánosítása

$$f(x) \rightarrow \tilde{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{ixy} dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(y) e^{-ixy} dy$$

Művejő?

$$-\tilde{f}'(x) = -iy \tilde{f}(y)$$

$$\tilde{\delta} = \frac{1}{\sqrt{2\pi}}$$

Green függvény.

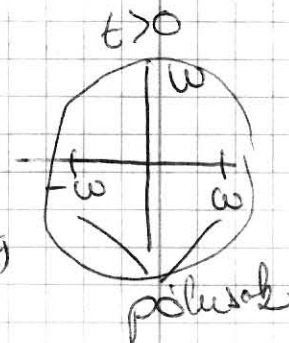
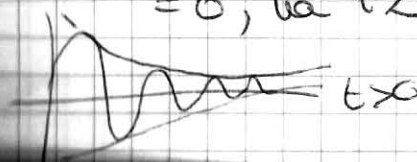
$$(\text{deriválások}) \cdot G(\frac{1}{t}) = \tilde{\delta}(\frac{1}{t})$$

$$pe: \left(\frac{d^2}{dt^2} + \gamma \frac{d}{dt} + \omega_0^2 \right) G(t) = \delta(t)$$

$$(-\omega^2 - i\gamma\omega + \omega_0^2) \tilde{G}(\omega) = \frac{1}{\sqrt{2\pi}}$$

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{G}(\omega) e^{-i\omega t} d\omega$$

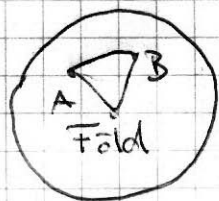
$$= 0, \text{ ha } t < 0$$



Variationsrechnung

$\int_{x_1}^{x_2} L(x, y(x), y'(x)) dx = \text{extremales}$
 bei $y(x_1)$ & $y(x_2)$ vorgegeben

$$\frac{\partial L}{\partial y} = + \frac{d}{dx} \frac{\partial L}{\partial y'}$$



← geodetisches
 gitter

$$ds^2 = r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$$

$$ds = \sqrt{r^2(\varphi)^2 + r^2 \sin^2 \theta} d\varphi$$

← L
 ebene geodetisches gitter erzeugt