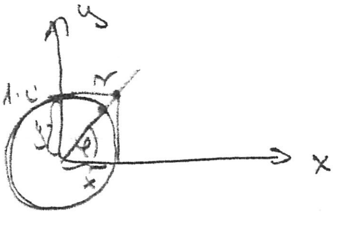
1. lap

Matematikai módszerek a fizikában - ea.

IX. 3.

Komplex számok - Emlékeztető



$$z = x+iy = \overbrace{r\,(\cos\varphi + i\sin\varphi)}^{r\cdot e^{i\varphi}}$$ exponenciális alak / trigonometrikus alak

$$r_1 e^{i\varphi_1}\cdot r_2 e^{i\varphi_2} = (r_1\cdot r_2)\,e^{i(\varphi_1+\varphi_2)}$$

$\sqrt{i} = z$

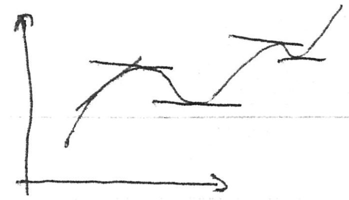
$zz = i = 1\,e^{i\frac{\pi}{2}}$

$z = 1\,e^{i\frac{\pi}{4}} = \frac{1+i}{\sqrt{2}} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$

deriválás + műveletei
+ fv.-ekkel

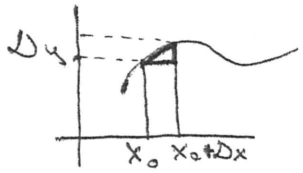
integrálás + műveletek
+ szabályai

Analízis - Emlékeztető



$y = f(x)$ $\quad f'(x) = \frac{dy}{dx}$

példa: $\sin(x^2)' = 2x\,\overset{\cos}{\cancel{\sin}}(x^2)$

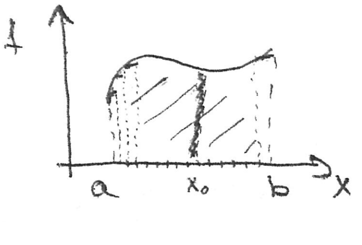


$$f' = \lim_{\Delta x\to 0}\frac{\overbrace{y(x_0+\Delta x)-y(x_0)}^{\Delta y}}{\Delta x}$$

$f' \approx \frac{\Delta y}{\Delta x}$ nem végtelen kicsi, hanem ~~nagyon~~ elég pici (mert az esetén 0-ra gondolnánk, és 0/0 = 0 és ennek nincs értelme)

$\underbrace{f[g(x)]'}_{??} = f'\cdot g'$ Lánc-szabály

$\frac{\Delta f}{\Delta x} = \frac{\Delta f}{\Delta g}\cdot\frac{\Delta g}{\Delta x}$ nem bizonyítás



$$\boxed{\int_a^b f(x)\,dx \approx \sum_i f(x_i)\Delta x}$$ közelítő összefüggés

↑ ez az amit az integrálás jelent

Parciális derivált

$\frac{\partial f(x,y)}{\partial x} = f_y'(x)$

görbe d-é $= \partial$
y-t gondolatban rögzítem

$\frac{\partial f}{\partial y}$

Vektoranalízis

Függvény

skalár → skalár ~~fv~~ skalár-skalár fv.

skalár változójú skalárértékű fv.

skalár → vektor

skalár változójú vektorértékű fv.

vektor → vektor

vektor → skalár

vektor változójú skalár

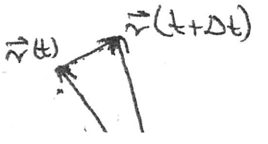
példák

- $T(l) = 2\pi\sqrt{\frac{l}{(9{,}81\frac{m}{s^2})}}$ inga lengésideje lengésidő fizikai képlete s.→s.
- $\vec{r}(t)$ s.→v.
- $T(\vec{r}), p(\vec{r})$ v.→s. → skalármező
hőm.
- $\vec{v}(\vec{r})$ folyadék seb.-e a helyfv.-ében v.→v. → mező
$\vec{E}(\vec{r})$
$\vec{B}(\vec{r})$
$\vec{g}(\vec{r})$

Derivált

$\vec{r}(t)$

$\vec{r}'(t) = \lim \frac{\Delta \vec{r}}{\Delta t}$



seb. a hely idő szerinti deriváltja

②

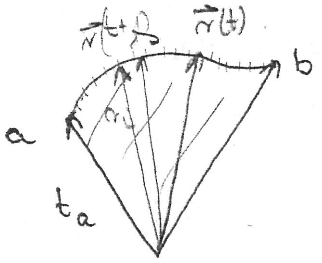
$\underline{r} = (x, y, z)$

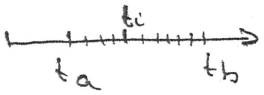
$\underline{r}(t) = (x(t), y(t), z(t))$

$\underline{r}'(t) = (x'(t), y'(t), z'(t))$

Ívhossz ~~kiszámítá~~sa

$\vec{r}(t) \qquad t_a \leq t \leq t_b$





poligon: egyenes szakaszokból álló

$|\Delta \vec{r}_i| \equiv |\vec{r}(t_{i+1}) - \vec{r}(t_i)|$ | | -et veszem mert a görbe hossza vagyok kíváncsi

azonosan egyenlő

$s(a \to b) \approx \sum_{i=1}^{N} |\Delta \vec{r}_i|$ az 1 közelítés

$s(a \to b) = \lim_{N \to 0} \sum_{i=1}^{N} |\Delta \vec{r}_i|$

$s(a \to b) = \lim_{N \to 0} \sum_{i=1}^{N} \left| \frac{\Delta \vec{r}_i}{\Delta t_i} \right| \cdot \Delta t_i \qquad \approx |\vec{r}'(t_i)| = f(t_i)$

kis csalás (helyettesítése) lehetséges
a lim - engedi
a csalás mértéke 0-hoz tart

$s = \int_{t_a}^{t_b} |\vec{r}(t)|\, dt$

példa

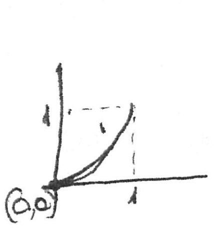
$r(t) = (t, t^2, 0)$

$0 \leq t \leq 1$

$x = t$

$y = t^2$ $z = 0$

$y = x^2$



$\underline{r}'(t) = (1, 2t, 0)$

$|\underline{r}'(t)| = f(t) = \sqrt{\underline{r}' \cdot \underline{r}'} = \sqrt{1+4t^2}$

$s = \int_0^1 \sqrt{1+4t^2}\, dt$ érvénybe lép a go to Bronstein

$$\text{sh}\,x = \frac{e^x - e^{-x}}{2}$$

$$\text{ch}\,x = \frac{e^x + e^{-x}}{2}$$

$$\text{ch}^2x - \text{sh}^2x = 1$$

$$s = \int_0^1 \sqrt{1+4t^2}\,dt = \int_0^1 \sqrt{\text{ch}^2u}\,du = \frac{1}{2}\int \text{ch}^2u\,du$$

$$2t = \text{sh}\,u$$

$$2dt = \frac{du\,\overset{c}{\cancel{s}}hu}{2}$$

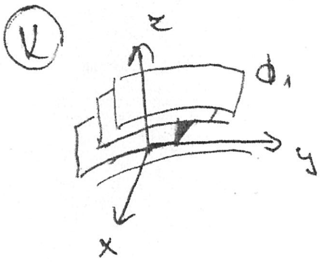
$$s = \text{sh}\,u$$

Vektor változós skalárfv.

$\underline{\Phi}(\vec{r})$: potenciálfv.

skalárfv. vektor változós

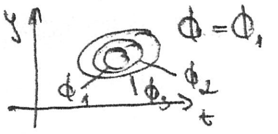
potenciál elmélet - Gauss dolgozta ki



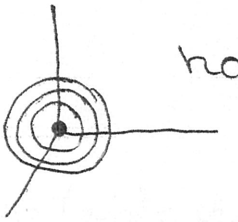
$\Phi(\underline{r}) = \Phi(x,y,z)$

pl.: $\Phi(\underline{r}) = |\underline{r}|$

$\Phi(x,y)$ szintvonalak



szemléletes, mert csak 2 koordináta van



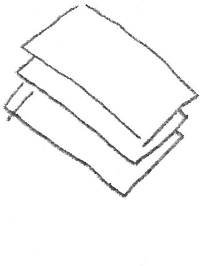
nagysága tarta, pontd: 0
gömb

$$\Phi = \vec{n}\cdot\vec{r} + \Phi_0 = \underbrace{n_1x + n_2y + n_3z}_{} + \Phi_0$$ milyen fv. az amire igaz

ez is konstansnak kell lennie

$\Phi = \text{áll} \Rightarrow$ síkok

a szintfelületek síkok



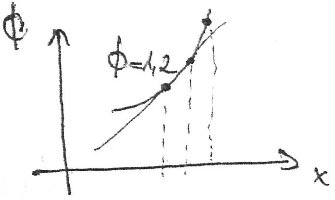
Gradiens

$$\cancel{\frac{\partial\phi}{\partial x},\partial}\ \underset{\overset{\Uparrow}{\Delta\phi}}{\operatorname{grad}\phi} = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

a fv. változását adja meg

$\phi(x)$

$$\phi'(x) = \lim\frac{\Delta\phi}{\Delta x} \underset{\text{egyenértékű}}{\Longleftrightarrow} \phi' \approx \frac{\Delta\phi}{\Delta x} \Rightarrow \boxed{\Delta\phi \approx \phi' \cdot \underset{\sim}{\Delta x}}$$



független változó megváltozása

$\underline{\Phi}(\vec r)$

$$\underset{s.}{\Delta\phi} = \underbrace{\phi(\vec r_0 + \Delta\vec r) - \phi(\vec r_0)}_{s.} \approx (\underbrace{\text{valami}}_{v.}) \cdot \underbrace{\Delta\vec r}_{v.}$$

vagy

$$= (\text{valami}) \cdot \Delta\vec r + \overset{v. \leftarrow v.}{\vec\varepsilon \cdot \Delta\vec r}$$

kis vacak magasabb renden kicsi, mint ez

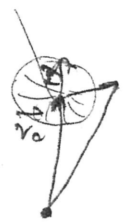
$\vec\varepsilon \to 0$-hoz

valami $\equiv$ grad ϕ

fv. gradiense megmutatja a fv. változását

még szemléletesebben megnézzük

$|\Delta\vec r|$ adott



a ϕ akkor nő a legnagyobbat, ha $\angle = 0°$

$\cos\angle = 1$,

grad ϕ vektor iránya megmutatja a legnagyobb növekedés irányát (ez a szemléletes jelentése) geometrikus

melyik irányba nő legnagyobb ütembe a ϕ

$\Delta\phi = |\operatorname{grad}\phi| \cdot |\Delta\vec r|$

~~milyen ütemben nő~~

Gradiens kiszámítása

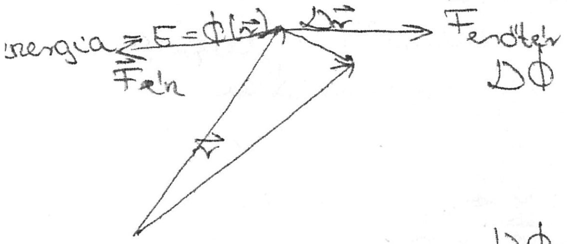
Legyen $\Delta r = (\Delta x, 0, 0)$

$$\Delta\phi = \underbrace{(\text{grad}\,\phi)}_{\text{ez mindig vektor}} \cdot \Delta r = (\text{grad}\,\phi)_x \cdot \Delta x + (\text{grad}\,\phi)_y \cdot 0 + (\text{grad}\,\phi)_z \cdot 0$$

$$(\text{grad}\,\phi)_x \approx \frac{\Delta\phi}{\Delta x}\Big|_{\substack{\Delta y \approx 0 \\ \Delta z \approx 0}} \rightarrow (\text{grad}\,\phi)_x = \frac{\partial\phi}{\partial x}$$

3 komponenst kell parciálisan deriválni

Legyen ϕ = potenciális energia



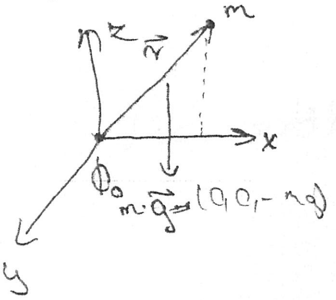
$\Delta\phi$ = energiaváltozás = általam végzett munka =

$= \vec{F}_{én} \cdot \Delta\vec{r} = - \vec{F}_{erőtér} \cdot \Delta\vec{r}$

$\Delta\phi \approx \text{grad}\,\phi \cdot \Delta\vec{r}$ ez így 1 matematikai összefüggés

$$\boxed{\vec{F}(\vec{r}) = -\,\text{grad}\,\phi}$$

példa



$\phi = \phi_0 + mgz$

$\text{grad}\,\phi = (0, 0, mg)$ 1 db fv.-t kaptunk

az erőtér által kifejtett erő a skalár fv. ~~rtg~~ grad · (−1)

példa

Coulomb-féle erőtér

$$\phi = k \overset{Q}{\cancel{\phi}} \frac{1}{\sqrt{x^2+y^2+z^2}}$$

1. lap Matematikai módszerek - ea. IX. 11.

$$\phi(\underline{r}) \equiv \phi(x,y,z)$$

$\big\downarrow$ deriválás

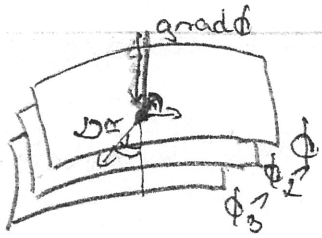
$$\underline{v}(\underline{r}) = \text{grad}\,\phi = \begin{pmatrix} v_x(x,y,z) \\ v_y(x,y,z) \\ v_z(x,y,z) \end{pmatrix}$$

$$\Delta\phi = \phi(\underline{r}+\Delta\underline{r}) - \phi(\underline{r}) \approx \text{grad}\,\phi \cdot \Delta\underline{r} = |\text{grad}\,\phi||\Delta\underline{r}|\cos\alpha$$

$$= \text{grad}\,\phi(\underline{r})\cdot\Delta\underline{r} + \underbrace{\underline{\varepsilon}(\underline{r},\Delta\underline{r})}\Delta\underline{r}$$

$$\underline{\varepsilon} \xrightarrow{|\Delta\underline{r}| \to 0} 0$$

$$\text{grad}\,\phi = \begin{pmatrix} \frac{\partial\phi}{\partial x} \\ \frac{\partial\phi}{\partial y} \\ \frac{\partial\phi}{\partial z} \end{pmatrix}$$



görbesereg ϕ = áll = ϕ_1

szintfelület (3 változó esetén)

grad ϕ iránya a leggyorsabb növekedés irányt adja meg, az erre merőleges irányban pedig 0 a növekedés

geotermikus gradiens a ↓ nagy

Gyakorlat:

1, fe

$$\phi = \vec{r}^2 \overset{f(u)\equiv u}{=} |\vec{r}|^2 = \underline{r}\cdot\underline{r} = x^2+y^2+z^2$$

$\vec{r}(x,y,z)$ - ugyanaz is!

$$\Delta\phi = (\vec{r}+\Delta\vec{r})^2 - \vec{r}^2 = \underbrace{2\vec{r}}_{\text{grad}\,\phi}\Delta\vec{r} + \underbrace{\Delta\vec{r}}_{\vec{\varepsilon}}\Delta\vec{r}$$

$$\left.\begin{aligned} (\text{grad}\,\phi)_x &= \frac{\partial\phi}{\partial x} = 2x \\ (\text{grad}\,\phi)_y &= \frac{\partial\phi}{\partial y} = 2y \end{aligned}\right\} \Rightarrow 2\underline{r}$$

2. feladat Coulomb potenciál

$\phi = k \cdot Q \frac{1}{|\underline{r}|}$ grad ϕ = ? $\phi = \frac{1}{|\underline{r}|} = \frac{1}{\sqrt{u}}$

$\phi = k \cdot Q \left(x^2 + y^2 + z^2\right)^{-1/2}$ $\underline{r} = |\underline{r}|$

$(\text{grad}\,\phi)_x = ? = kQ \underbrace{(x^2+y^2+z^2)^{-\frac{3}{2}}}_{(r^2)^{-\frac{3}{2}}} \cdot \left(-\frac{1}{2}\right) \cdot 2x = -kQ \frac{1}{r^3} x = \left(-\frac{kQ}{r^3}\underline{r}\right)_x$

$(\text{grad}\,\phi)_y = -kQ \frac{1}{r^3} y$

$(\text{grad}\,\phi)_z = -kQ \frac{1}{r^3} z$

-1-gyel beszorozva a Coulomb féle erőtörvényt kapom

3, feladat

$\phi = \frac{1}{2} D \underline{r}^2$

rugalmas energia

$-\text{grad}\,\phi = -D\underline{r}$ rugó által kifejtett erő

a potenciális erőtér ne. -1 - az ott ható erővel

4, f

Nap gravitációs

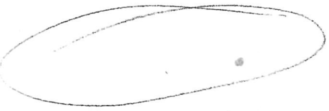
$-G \cdot \frac{M}{|\underline{r}|}$ $-\frac{1}{|\underline{r}|}$-rel arányos, ez a lényeg

Legyen $\phi(|\underline{r}|) = f(\overset{u}{\overset{\shortparallel}{|\underline{r}|^2}}) = f(u)$ grad ϕ = ?

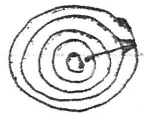
$\Delta\phi \approx \frac{f(u+\Delta u) - f(u)}{\Delta u} \cdot \Delta u \approx \underbrace{f'(u) \cdot 2\underline{r}}_{\text{grad}\,\phi} \Delta\underline{r}$

Ha $\phi = r^2$ akkor grad $\phi = 2\underline{r}$

Ha $\phi = \frac{1}{|\underline{r}|} = \frac{1}{\sqrt{u}}$ grad $\phi = -\cancel{\frac{1}{2}} \underset{\downarrow}{u^{-\frac{3}{2}}} \cdot \cancel{2}\underline{r} = -\frac{1}{r^3} \cdot \underline{r}$

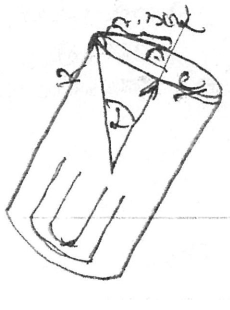


a szintfelületek koncentrikus gömbök



régi feladat legyen $\phi = |\vec{e} \times \vec{r}|$ ábrázoljuk ezen ϕ szintfelületeit

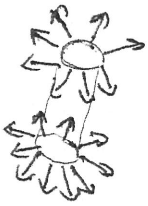
$\phi = |\underline{e}||\underline{r}| \sin\alpha$



más konstansnál a ⬭ kisebb

pl: póréhagyma

szintfelületek



a gradiens iránya

Példa Kérdés $\phi = x^2 + yz$

$$\operatorname{grad}\phi = ? = \begin{pmatrix} 2x \\ z \\ y \end{pmatrix}$$

$$v(\underline{r}) = \begin{pmatrix} 2x \\ z \\ -y \end{pmatrix} \overset{?}{=} \operatorname{grad}\phi$$ nincs megold.-a

nem minden vektormező írható fel skalár fv. gradienseként

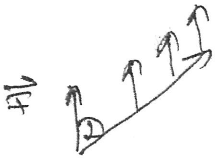
potenciálosnak nevezzük azt a vektormezőt (erőteret), amelyet származtatható egy fv gradienséből.

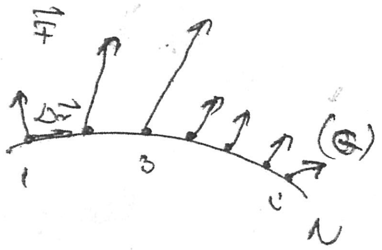
A vonalintegrál fogalma, kiszámítása, fizikai alkalmazásai

Emlékeztető (Fogalma)

$W = \text{munka} = \vec{F}\vec{r}$

Ha $\vec{F}(\vec{r})$





kicsiny munka $\Delta W = \vec{F}(\vec{r})\Delta\vec{r}$

kicsi munkákból össze tudom adni az egész munkát

általában: $W \approx \underbrace{\sum_{i=1}^{N} \vec{F}(\vec{r}_{(i)})\, \Delta\vec{r}_{(i)}}_{\Delta W} \xrightarrow[N\to\infty]{} \int_G \vec{F}(\vec{r})\, d\vec{r}$ ← arra utal, h. $\Delta\vec{r}$ vektor szerinti összeg

i edik helyen lévő Δr

itt még vonalak folytonosan megyek

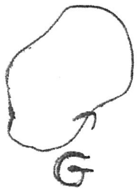
igazán mit jelent a vonalintegrál

munkakiszámítása

$g\sum_{i=1}^{N} \vec{F}(\vec{r}_{(i)})\, \Delta$

vektormező skalár típusú szorzását jelenti

$U = \int \vec{G}(\vec{r})\, d\vec{r}$

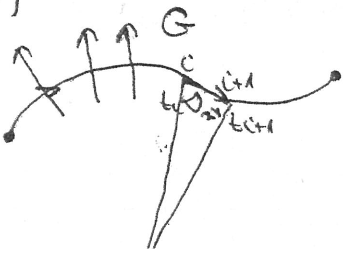


$\oint \vec{v}\, d\vec{r}$

zárt görbére integrálok
nemcsak kör lehet, akár négyzet is

Kiszámítása

$\vec{v}(\vec{r})$



légy pályáját hogyan adhatom meg?

pl.: $\vec{r}(t)$ paraméterezés

$\int \vec{v}\, d\vec{r} \approx \sum \vec{v}(\vec{r}_{(i)})\, \underline{\Delta\vec{r}_{(i)}} \cdot \Delta t_i$ $\quad \vec{r}_{(i)}$ helyen

3. lap Matematikai módszerek - ea. IX.11.

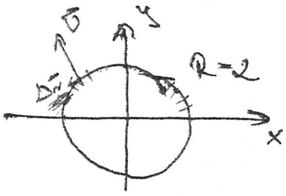
$$\int_{G(A\to B)} \vec{v}\,d\vec{r} \overset{=}{\cancel{\approx}} \sum \vec{v}(\vec{r}_{(i)}) \frac{\Delta \vec{r}_{(i)}}{\Delta t_i}\Delta t_i \approx \sum_i \overbrace{\left(\vec{v}(\vec{r}_{(i)})\cdot \vec{r}'(t)\Big|_{t=t_i}\right)}^{f(t_i)\text{ - skalár fv.}} \Delta t_i \to$$

ebbe még nincs 'idő' én vittem bele mesterségesen

$$\to \int_{t_A}^{t_B} \vec{v}(\vec{r}(t))\, r'(t)\,dt$$

ez a közelítés pontos jó-e?
igen, jó. ha a v folytonos, 1 sima görbe.

csak lehet VKL ?



$\vec v = \vec r$

$\oint \vec v\,d\vec r = ? = 0$ $\quad ! \vec v \perp \vec r ! \Rightarrow \vec v\cdot\vec r = 0 \Rightarrow \oint\ldots = 0$

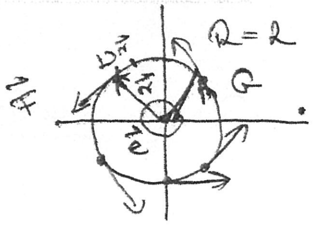
$\sum \vec v\,\Delta\vec r$

pl: a Föld Nap körüli körpályája

IKL $\vec F(\vec r) = \vec e\times\vec r$

$\oint_G \vec F\,d\vec r = ?$

$|\vec e| = 1$



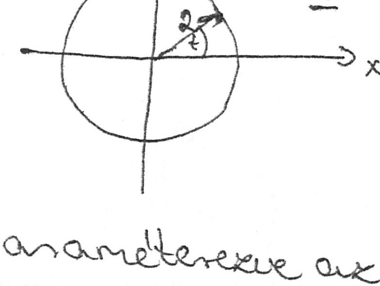
$\vec F$ és $\Delta\vec r$ iránya nem merőleges → W-t végez

$\sum |\vec F|\,|\Delta\vec r| = 2\cdot 4\pi = 8\pi$

ilyen erőteret csak úgy lehet létrehozni, hogyha folyamatosan tápláljuk

~~pld~~.

$|\vec e\times\vec r| = 1\cdot 2\cdot \sin 90° = 2$



$\vec r = \begin{pmatrix}2\cos t & =x\\ 2\sin t & =y\\ 0 & =z\end{pmatrix} \qquad 0\le t\le 2\pi$

paraméterezve az előző feladat

$\vec e = \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\vec F = \vec e\times\vec r$ kifejezve t-vel

$\vec F = \begin{pmatrix}0\\0\\1\end{pmatrix}\times\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}-y\\x\\0\end{pmatrix} = \begin{pmatrix}-2\sin t\\2\cos t\\0\end{pmatrix}$

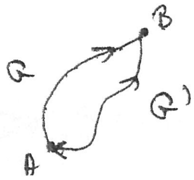
$$\oint \vec F\,d\vec r = \int_0^{2\pi} F[r(t)]\,r'(t)\,dt$$

$$r'(t) = \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix}$$

$$\oint \vec F d\vec r = \int_0^{2\pi} \underbrace{F[r(t)]\cdot r'(t)\,dt}_{\underbrace{(4\sin^2 t + 4\cos^2 t + 0)}_{4}dt} = 4\cdot\int_0^{2\pi} dt = 8\pi$$

a fizikai alkalmazások nem vizsga követelmény.

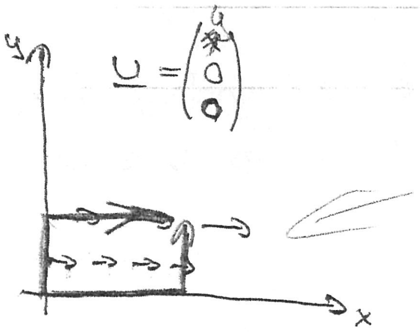
$$\int_{\substack{A\to B\\(G)}} \vec v(\vec r)d\vec r \neq \int_{G'} \vec v d\vec r$$



$\Downarrow$

$$\oint \vec v d\vec r \neq 0 \quad \text{általában}$$

$$\underline{u} = \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}$$



a víz is még örvényes

Tétel: Ha $\vec v = \text{grad}\,\phi(\vec r)$ $\quad \int_{\substack{G\\A\to B}} \vec v d\vec r = \phi(\vec r_B) - \phi(\vec r_A)$

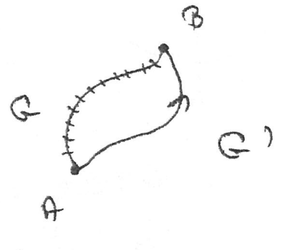
$\Downarrow$

1. gradiens tétel

$$\oint (\text{grad}\,\phi)d\vec r = 0$$

Biz.:

$$\int_G \text{grad}\,\phi\, d\vec r \approx \sum \underbrace{\text{grad}\,\phi\,\Delta\vec r}_{\Delta\phi} = \sum \Delta\phi$$



a potenciálos vektormezők körintegrálja 0.

$\oint \vec v d\vec r = 0$ konzervatív $\Longleftrightarrow$ $\vec v = \text{grad}\,\phi$ potenciálos $\quad$ a 2 uazt. jelenti

Young-tétel mélyesek segítés

$\phi(x,y)$

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) \overset{?}{=} \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)$$

példa

$$\phi = x\sin y$$

$$\frac{\partial\phi}{\partial x} = \sin y \rightarrow \frac{\partial\cancel{\phi}}{\partial y}\sin y = \cos y$$

$$\parallel$$

$$\frac{\partial\phi}{\partial y} = x\cos y \rightarrow \frac{\partial}{\partial x}x\cos y = \cos y$$

ellenpélda

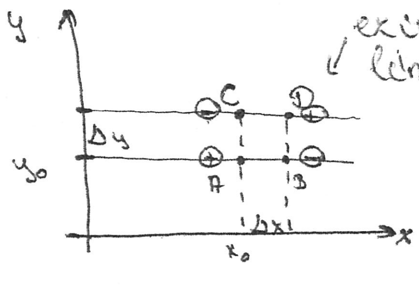
$$\phi = |x| + y$$

$$\frac{\partial\phi}{\partial y} = 1 \qquad \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = 0$$

$\cancel{\frac{\partial\phi}{\partial x}}$ nem létezik #

a sima (folytonos) fv.-ekre érvényes az, hogy a parciális deriválás sorrendje felcserélhető.

Szemléltetés:



ez is nagyon lényeges!!! $\phi(x,y)$

$$\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)$$

$$\parallel$$

ha sima a fv. akkor jó ez a közelítés

$$\approx \frac{\frac{\phi(D)-\phi(C)}{\Delta x} - \frac{\phi(B)-\phi(A)}{\Delta x}}{\Delta y} = \frac{\phi(D)+\phi(A)-\phi(B)-\phi(C)}{\Delta x\cdot\Delta y}$$

Ha $\underline{v}(\underline{r}) = \operatorname{grad}\phi$ (potenciálos a mező)

$$v_x = \frac{\partial \phi}{\partial x} \quad \rightarrow \quad \frac{\partial v_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)$$

$$v_y = \frac{\partial \phi}{\partial y} \quad \rightarrow \quad \frac{\partial v_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) \qquad \frac{\partial v_x}{\partial y} = \frac{\partial v_y}{\partial x}$$

$$v_z = \frac{\partial \phi}{\partial z}$$

$$\frac{\partial v_i}{\partial x_k} = \frac{\partial v_k}{\partial x_i} \qquad \begin{matrix} i = 1,2,3 \\ k = 1,2,3 \end{matrix}$$

→ 3 db összefüggés, nem 9!

példa

$$\underline{v}(\underline{r}) = \begin{pmatrix} 2x \\ z \\ y \end{pmatrix} \qquad \underline{v}(\underline{r}) = \begin{pmatrix} 2x \\ z \\ -y \end{pmatrix}$$

potenciálos nem potenciálos

jelölés

$$\left.\begin{aligned} \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} &= a_3(\underline{r}) \\ -\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} &= a_2(\underline{r}) \\ \frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} &= a_1(\underline{r}) \end{aligned}\right\} \underline{a}(\underline{r}) = \operatorname{rot} \underline{v}$$

vektormező, aminek a tagjait úgy kell kiszámolni

$$\begin{tikzcd}
 & \boxed{\begin{matrix}\underline{v} = \operatorname{grad}\phi \\ \text{potenciálos}\end{matrix}} \arrow[dl, Leftrightarrow] \arrow[dr, Leftrightarrow] & \\
\boxed{\begin{matrix}\oint \underline{v}\,d\underline{r} = 0 \\ \text{konzervatív}\end{matrix}} \arrow[rr, Rightarrow, shift right] & & \boxed{\begin{matrix}\operatorname{rot}\underline{v} = 0 \\ \text{örvénymentes}\end{matrix}} \arrow[ll, Rightarrow, shift right]
\end{tikzcd}$$

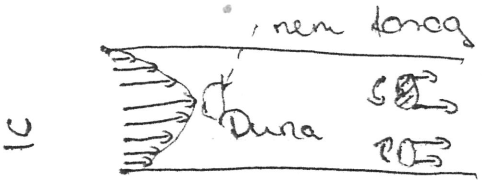
← forgása 0-a

a 3 tul. valamelyikéből le lehet vezetni a másik kettőt.

alkalmazás a fizika tv.-ei közül sok. megfogalmazható

1. lap Matematikai módszerek - ea. IX.11.

Rotáció



→ mozog, de forog is a jégtábla

áramlási seb. ideális esetben

$$2\cdot\underline{\omega} = \text{rot}\,\underline{v}$$

3. óra Matematikai módszerek a fiz.-ban. IX. 25.

Elvi kiszámítás, szemléletesjelentés ; Gyakorlati kisz.

$\phi(\vec r) \Rightarrow$ gradiens : $\Delta\phi(\vec r) \approx \text{grad}\,\phi \cdot \Delta\vec r$ $\qquad (\text{grad}\,\phi)_x = \dfrac{\partial\phi(x,y,z)}{\partial x}$ stb.

$v(\vec r) \Rightarrow$ rotáció $= \vec\omega(\vec r)$ vektor $\qquad$ Helyi (lokális) "örvényerősség"

$$\omega_x = \frac{\partial}{\partial y} v_z - \frac{\partial}{\partial z} v_y \text{ stb}$$

$$\omega_y = \frac{\partial}{\partial z} v_x - \frac{\partial}{\partial x} v_z$$

$$\omega_z = \dots$$

Mire jó?

pl. Ha $\vec v = \text{grad}\,\phi \Rightarrow \vec\omega = 0$

pl: Fizikai törvények

Ha pl $\vec v(\vec r) = \vec\omega_0 \times \vec r \Rightarrow$

$$\Rightarrow v_x = \omega_{0y} z - \omega_{0z} y$$

$$v_y = \omega_{0z} x - \omega_{0x} z$$

$$v_z = \omega_{0x} y - \omega_{0y} x$$

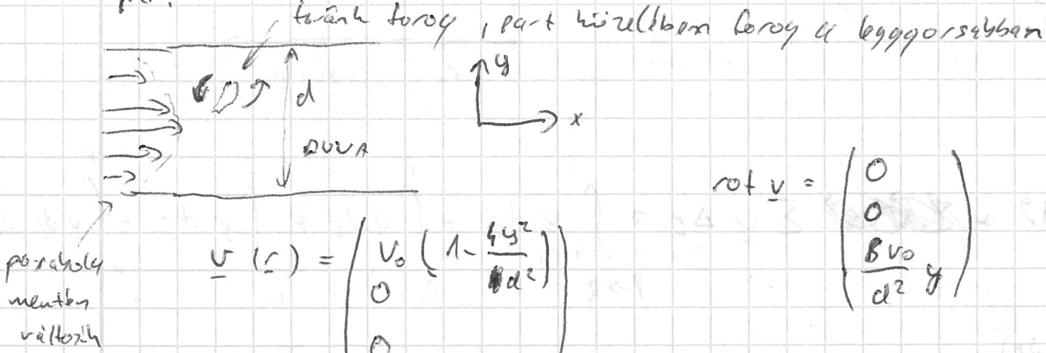
$$(\text{rot}\,\vec v)_x = \omega_{0x} - (-\omega_{0x}) = 2\omega_{0x}$$

~~Itt~~ $\text{rot}\,\vec v = 2\vec\omega_0$ merev testnél

$\boxed{\text{rot: helyi forgási seb. ad meg}}$

~~[illegible] a kerületi seb. = rot $\vec v$ [illegible]~~

pl.:



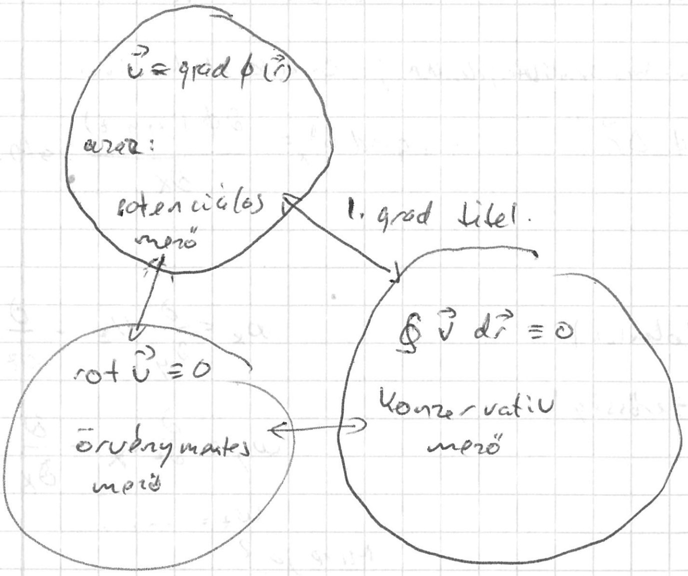
$$\underline{v}(\underline{r}) = \begin{pmatrix} v_0\left(1 - \frac{4y^2}{d^2}\right) \\ 0 \\ 0 \end{pmatrix}$$

$$\text{rot}\,\underline{v} = \begin{pmatrix} 0 \\ 0 \\ \frac{8v_0}{d^2} y \end{pmatrix}$$

$$\omega_z = \frac{\partial}{\partial x} v_y - \frac{\partial}{\partial y} v_x$$

Fáránk forgási seb. kétszerese a rot v

~~[illegible]~~ szokás örvénylésnek is mondani.

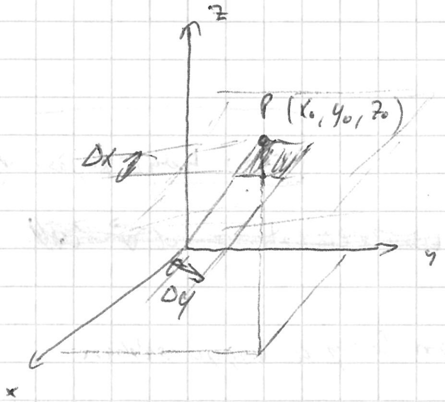


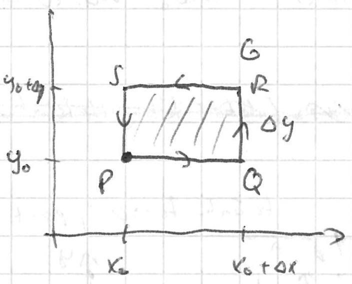
$$\int \vec v\, d\vec r \approx \overset{\text{Ha grad }\phi}{\sum} \vec v\, \Delta \vec r =$$

$$= \sum \Delta\phi = \phi(B) - \phi(A)$$

konzervatív vektormező

Legyen G „kicsiny" $\overset{\text{sík}}{\text{görbe}}$, pl téglalap





$$\oint \vec v\, d\vec r \approx \sum \underline{v}\,\Delta \underline{r} \approx \int_{P\to Q} \underline{v}\, d\underline{r} + \int_Q^R \underline{v}\, d\underline{r} + \int_R^S \underline{v}\, d\underline{r} + \int_S^P \underline{v}\, d\underline{r} \approx$$

$$\approx \underbrace{\underline{v}\begin{pmatrix}\Delta x\\ 0\\ 0\end{pmatrix}} +$$

$$v_x(x_0,y_0,z_0)\,\Delta x + v_y(\overset{x_0+\Delta x}{\downarrow})\,\Delta y - v_x(\overset{x_0,\,y_0+\Delta y,\,z_0}{\downarrow})\,\Delta_x - v_y(x_0\quad)\,\Delta y \approx$$

$$\approx \Delta x \Delta y \left[\frac{v_x(x_0,y_0,z_0) - v_x(x_0,y_0+\Delta y,z_0)}{\Delta y} + \frac{v_y(x_0+\Delta x,y_0,z_0) - v_y(x_0,y_0,z_0)}{\Delta x}\right) \approx$$

$$\simeq \underbrace{\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)}_{(\mathrm{rot}\,\underline{v})_z} \underbrace{\Delta x \Delta y}_{} = (\mathrm{rot}\,\underline{v})\ \Delta \underline{F}$$

terület felület $= (\Delta \underline{F})_z$ $\qquad \Delta \underline{F} = \begin{pmatrix} 0 \\ 0 \\ \Delta x \Delta y \end{pmatrix}$

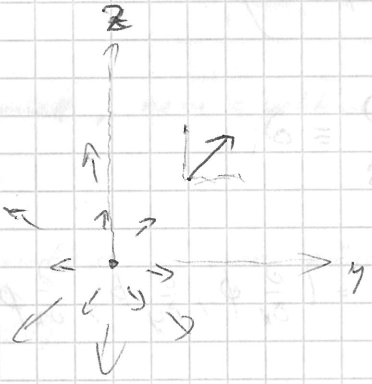
$$\boxed{\oint \vec{v}\, d\vec{r} \approx \mathrm{rot}\,\vec{v}\ \Delta\vec{F}}$$

kicsiny sík felületre

körüljárt felület vektora

$$\mathrm{rot}\, v_x = \frac{\partial}{\partial y} v_z - \frac{\partial}{\partial z} v_y$$

pl: $\vec{v}(\vec{r}) = \vec{r} = \mathrm{grad}\left(\frac{1}{2} r^2\right)$



$$(\mathrm{rot}\,\underline{v})_i = \sum_{jk} \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k$$

$$(\mathrm{rot}\,\underline{v})_1 = \underset{\substack{\| \\ +1}}{\varepsilon_{123}} \frac{\partial}{\partial x_2} v_3 + \underset{\substack{\| \\ -1}}{\varepsilon_{132}} \frac{\partial}{x_3} v_2$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{r} \equiv \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

pl: $\underline{v} = (\overset{\substack{\text{áll.} \\ \|}}{\underline{a}} \times \underline{r})$ $\qquad \mathrm{rot}\,\underline{v} = 2\underline{a}$

~~$v_1 = \sum_{jk} \varepsilon_{ijk} a_j x_k$~~

$$(\mathrm{rot}\,v)_i = \sum_{ijklm=1}^{3} \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm} a_l \cancel{x_m}\ \cancel{\delta_{jm}} =$$

(lm, a vektori szorzat ijk, a rot miatt)

$$\frac{\partial x_m}{\partial x_j} = \delta_{mj}$$

$$= \sum_{klm} \overbrace{\varepsilon_{imk} \varepsilon_{klm}}^{\delta_{il}} a_l = 2a_i$$

$$\sum_{k,m} \varepsilon_{imk} \varepsilon_{lmk} = \delta_{il} \, 2$$

Jelölés: $\frac{\partial}{\partial x_i} = \partial_i$

pl: $\partial_x \equiv \frac{\partial}{\partial x}$; $\partial_1 = \frac{\partial}{\partial x_1}$

Jelölés:

$$\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \text{„vektor"} = \overset{\text{nabla}}{\nabla}$$

differenciál operátor

Rhirotrén op. vektor és operátor egyben

$$\nabla_i = \frac{\partial}{\partial x_i} \equiv \partial_i$$

$$\text{grad}\,\phi = \left(\frac{\partial}{\partial x}\phi, \frac{\partial}{\partial y}\phi, \frac{\partial}{\partial z}\phi\right) = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\phi = \nabla\phi$$

$$\text{rot}\,\underline{v} = \underline{\nabla} \times \underline{v}$$

erre hat a deriválás

$$\text{rot}\left(\phi(\underline{r})\,\underline{v}(\underline{r})\right) = \underline{\nabla} \times (\phi \cdot \underline{v}) = \nabla_\phi \times \phi\,\underline{v} + \nabla \times \phi\,\underline{v} =$$

$$= \underbrace{\phi(\underline{\nabla} \times \underline{v})}_{\phi\ \text{rot}\,v} + \underbrace{(\nabla\phi)}_{\text{grad}\,\phi} \times \underline{v} = \phi\ \text{rot}\,\underline{v} + \text{grad}\,\phi \times \underline{v}$$

$\text{grad}\,\phi \equiv \nabla\phi$

$\text{rot}\,\underline{v} = \nabla \times \underline{v}$

Ha $\underline{v} = \text{grad}\,\phi \overset{\rightarrow}{\cancel{\Longleftrightarrow}} \text{rot}\,\underline{v} \equiv 0$

$$\underline{\nabla} \times (\nabla\phi) \equiv 0$$

$\nabla_1\nabla_2\phi - \nabla_2\nabla_1\phi = 0$ Young tétel miatt igaz

$$\operatorname{div} \underline{v} = \underline{\nabla} \cdot \underline{v} = \nabla_1 v_1 + \nabla_2 v_2 + \nabla_3 v_3 = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

(divergencia)

Mikor írható fel adott $\underline{v}(\underline{r})$ mező $\underline{v} = \operatorname{rot} \underline{u}(\underline{r})$ alakban?

pl. 1. $\underline{v} = \begin{pmatrix} 0 \\ 0 \\ y \end{pmatrix}$ $\quad \underline{u} = \begin{pmatrix} -\frac{y^2}{2} \\ 0 \\ 0 \end{pmatrix}$ ilyenkor igaz

2. $\underline{v} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix}$ Ha

$$v_x = \partial_y u_z - \partial_z u_y \qquad \partial_x v_x = \cancel{\partial_x \partial_y u_z} - \cancel{\partial_x \partial_z u_y}$$

$$v_y = \partial_z u_x - \partial_x u_z \qquad \partial_y v_y = \cancel{\partial_y \partial_z u_x} - \cancel{\partial_y \partial_x u_z}$$

$$v_z = \partial_x u_y - \partial_y u_x \qquad + \quad \partial_z v_z = \cancel{} \quad \cancel{}$$

$$\overline{\qquad 0 \qquad}$$

Ha $\underline{v} = \operatorname{rot} \underline{u} \Rightarrow \operatorname{div} \underline{v} \equiv 0$

$\underline{\nabla}(\underline{\nabla} \times \underline{u}) \equiv 0$ Young tétel miatt, nem a vektorok párhuzamossága miatt

A divergencia szemléletes jelentése (forráserősség)

(forrásmentes a mező ha div = 0)

pl: $\underline{v} = \underline{r}$

$V_x = x$

$V_y = y$

$V_z = z$

$\operatorname{div} \underline{v} = 3$

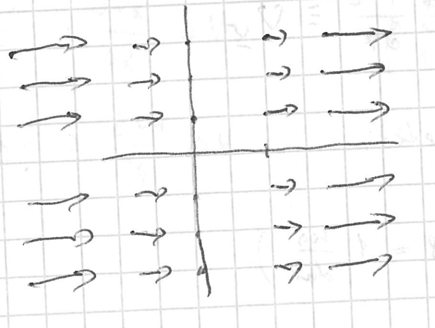
így mozognak a galaxisok az univerzumban.

kellisben lévő tinta így mozog

2. $v_x = x^2$

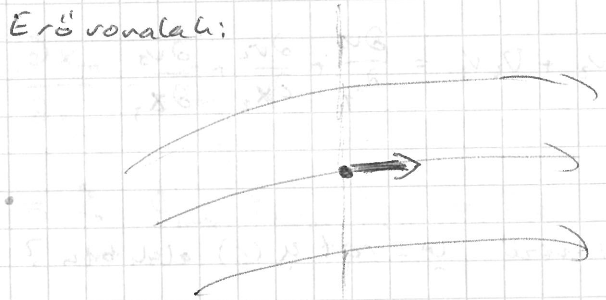
$v_y = 0$

$v_z = 0$

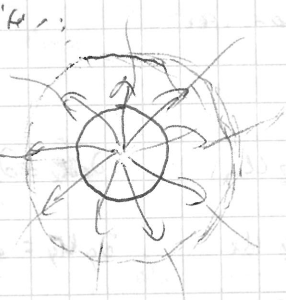


$\operatorname{div} \underline{v} = 2x$

Erővonalak:



Coulomb erőtér:



Állít: Csak azok a vektormezők szemléltethetők erővonalakkal

amiknek a ~~[illegible]~~ divergenciája ~~[illegible]~~ nulla

$$\mathrm{div}(\phi \underline{v}) = \underline{\nabla}(\phi \underline{v}) = \underline{\nabla}(\phi \underline{v}) + \underline{\nabla}(\phi \underline{v}) = \underline{v} \cdot \mathrm{grad}\,\phi + \phi\, \mathrm{div}\, \underline{v}$$

$$\mathrm{rot}(u \times v) = \nabla \times (\underline{u} \times \underline{v}) + \nabla \times (\underline{u} \times \underline{v}) =$$

$$u(\underline{\nabla} \underline{v}) - \underline{v}(\underline{\nabla} \underline{u})$$

X. 2. 4. óra

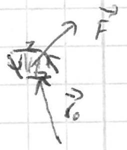
gradiens: $\Delta \phi \approx \mathrm{grad}\, \phi \cdot \Delta \vec{r} \Rightarrow \int_{G(A \to B)} \mathrm{grad}\, \phi \, d\vec{r} = \phi(B) - \phi(A)$ ha grad létezik

↙ amelyik körül járom a kis felületet

rotáció: $\oint \vec{v}\, d\vec{r} \approx \mathrm{rot}\, \vec{v}\, (\vec{r}_0)\, \Delta \vec{F} =$

kicsi hurokra

$= (\vec{e}\ \mathrm{rot} \vec{v})\, |\Delta \vec{F}|$ $\quad \Delta \vec{F} = \vec{e}\, |\Delta \vec{F}|$



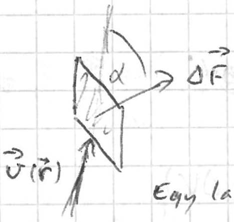
divergencia:
(forráserősség) $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} + \dfrac{\partial v_z}{\partial z} \equiv \underline{\nabla}\, \underline{v}$

Div: alkalmazása és szemléltetés:

$\vec{v}(\vec{r})$ víz sebesség $\left(\rho = 1 \dfrac{\mathrm{kg}}{\mathrm{dm}^3} \right)$
↑
adott fv.

Mennyi víz megy át rajta?

Vízhozam



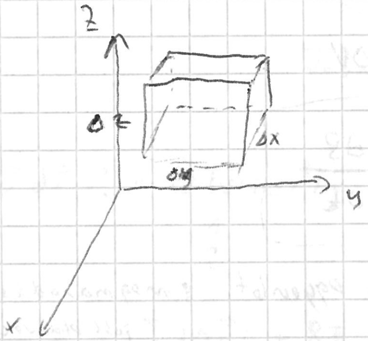
Egy lapka

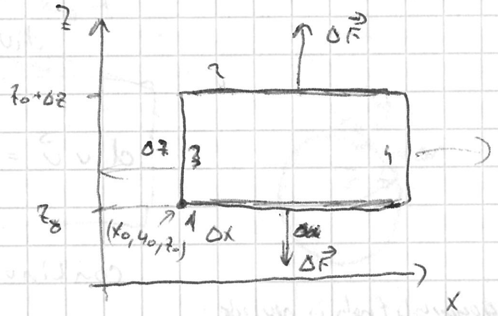
$|\vec{v}|\Delta t$ vízmolekulák mennek át ezen felületen

$$\frac{\text{átáramló víz tömege}}{\text{idő}} = \vec{v}\cdot\Delta\vec{F} = |\vec{v}||\Delta\vec{F}|\cos\alpha$$

$\uparrow$

fluxus = hozam: egységnyi idő alatt átáramló v. mi.





z komp. negatív irányú

1. felület: $\vec{v}\,\Delta\vec{F} \approx -v_z(x_0,y_0,z_0)\,\Delta x\,\Delta y$

hanem $z_0+\Delta z$

2. felület: $\vec{v}\,\Delta\vec{F} \approx +v_z(x_0,y_0,z_0)\,\Delta x\,\Delta y$

(ami az egyik felületen bejön az a másik oldalon kimegy)

Módosítva összeadva:

$$(1)+(2)\quad \frac{v_z(x_0,y_0,z_0+\Delta z)-v_z(x_0,y_0,z_0)}{\Delta z}\cdot(\Delta x\Delta y\Delta z)$$

$$(1)+(2):\ \approx \sum \vec{v}\,\Delta\vec{F} \approx \frac{\partial v_z}{\partial z}\underbrace{(\Delta x\Delta y\Delta z)}_{\Delta V\ \text{térfogat}}$$

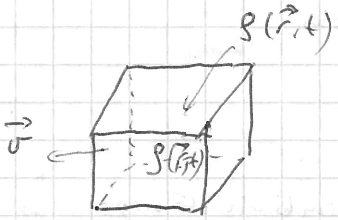
$$(3)+(4):\quad \sum \vec{v}\,\Delta\vec{F} \approx \frac{\partial v_x}{\partial x}\,\Delta V$$

$$(5)+(6):\quad \sum \vec{v}\,\Delta\vec{F} \approx \frac{\partial v_y}{\partial y}\,\Delta V$$

$$\sum_{1}^{6} \vec{v}\,\Delta\vec{F} \approx (\operatorname{div}\vec{v})\,\Delta V$$

div $\vec{v}$ mennyivel jön több ki mint amennyi jön be.

Megmaradási tétel (bármire, aminek az áramlását a $\vec{v}$ írja le.)



kiáramlik Δt idő alatt: $\sum_1^6 \vec{v}\,\Delta\vec{F}\cdot\Delta t$

~~$(\rho\Delta V)_{t+\Delta t} - \rho\Delta V$~~

$$(\rho\Delta V)_{t+\Delta t} - (\rho\Delta V)_t$$

Ha nincs „szökés": $$\sum \underbrace{\vec{v}\,\Delta\vec{F}}_{\operatorname{div}\vec{v}\cdot\Delta V}\cdot\cancel{\Delta t} = -\frac{(\rho\Delta V)_{t+\Delta t} - (\rho\Delta V)_t}{\Delta t}\cdot\cancel{\Delta t}$$

$$\boxed{\operatorname{div}\vec{v} = -\frac{\partial\rho}{\partial t}}$$

continuitási egyenlet ≠ megmaradási tv. jobb elnevezés

Mérlegegyenletnek is nevezik:

~~mérlegegyenlet~~

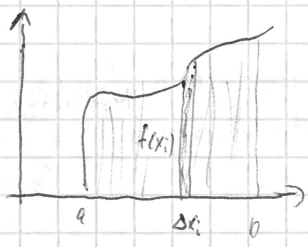
$$\operatorname{div}\vec{v} + \frac{\partial\rho}{\partial t} = f(\vec{r},t)$$

↑ forrássűrűség (utánpótlás sűrűség)

megmaradási tv. esetén $f=0$

Emlékeztető:

$$I = \int_a^b f(x)\,dx \approx \sum_{i=1}^{N} f(x_i)\,\Delta x_i$$



pl: $$\int_{t_1}^{t_2} v(t)\,dt = \text{megtett út}$$

$$\int F(x)\,dx = W \text{ munka}$$

kiszámítás: $f(x) = F'(x)$

$$I = F(b) - F(a)$$

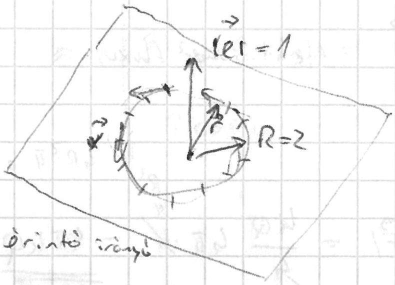
Vektorokkal (3 dimenzióban):

$\oint_G$ ~~$\vec F(\vec r)\, d\vec r \approx \Sigma F$~~ $\quad \int_G \vec F(\vec r_i)\, d\vec r \approx \Sigma \vec F(\vec r_i)\, \Delta \vec r_i \approx$

vonalintegrál

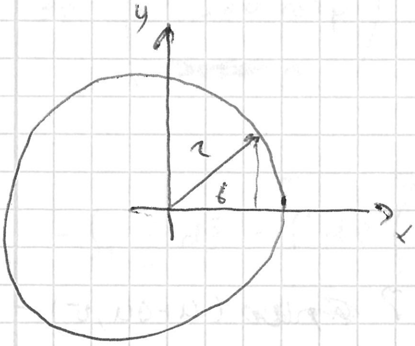
$$\approx \sum \vec F \frac{\Delta \vec r}{\Delta t} \Delta t \approx \sum \underbrace{\vec F \; \vec r\,'(t)}_{\text{skalár}} \Delta t \approx \int_{t_1}^{t_2} (\vec F \vec r\,') \, dt$$

Pl: $\vec v_{(\vec r)} = \vec e \times \vec r$



$$\oint \vec v \, d\vec r = ? \approx \Sigma \underset{\overset{\uparrow}{2}}{|\vec v|} |\Delta \vec r| =$$

$$= 2 \cdot \underbrace{\Sigma |\Delta \vec r|}_{\text{kör } 2\pi \underset{2}{\overset{\shortparallel}{R}}} = \underline{\underline{8\pi}}$$



$$\underline{r}(t) = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 0 \end{pmatrix} \qquad r'(t) = \begin{pmatrix} -2\sin t \\ 2\cos t \\ 0 \end{pmatrix}$$

$$\vec v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} = \begin{pmatrix} 2\cos t \\ 2\sin t \\ 0 \end{pmatrix}$$

$$\oint \vec v \, d\vec r = \int_{t_1}^{t_2} \underbrace{\vec v(\vec r(t)) \; \vec r\,' t}\, dt$$

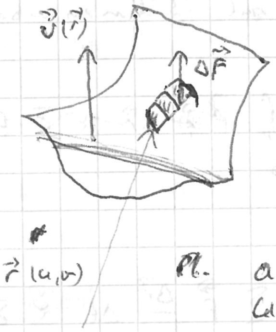
$$\int_0^{2\pi} (4\sin^2 t + 4\cos^2 t + 0)\, dt = 4 \cdot \int_0^{2\pi} dt = \underline{\underline{8\pi}}$$

Felületi int.

gyakorlatban skalár szorzat

$$\iint \vec{v}\, d\vec{F} \simeq \sum_i \vec{v}(\vec{r}_i)\, \Delta\vec{F}^{(i)}$$

$\vec{v}$ vektormező felületi integrálja



kis sík felület darabokra osztom

Pl. alagút hiszemplére felület kis darabka

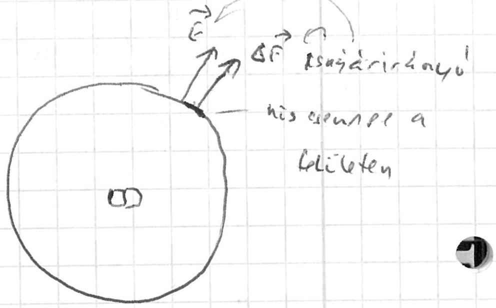
Pl: $\vec{v} = \frac{kQ}{r^3}\vec{r} \equiv \vec{E}(\vec{r})$ Coulomb tér

$$\oiint \vec{E}\, d\vec{F} = \text{elektromos fluxus} \approx$$

$|r| = R$ gömb

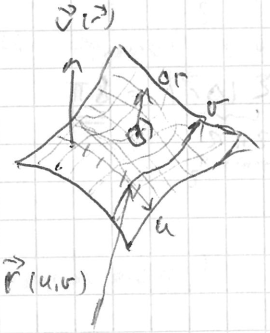
$4R^2\pi$ a gömb felszíne

$$\approx \sum \underset{\substack{\shortparallel \\ \frac{kQ}{4}}}{|\vec{E}|}|\Delta\vec{F}| = \frac{kQ}{\cancel{4}} 4\pi \cdot \overset{R^2}{\cancel{4}} = \underline{\underline{4\pi kQ}}$$



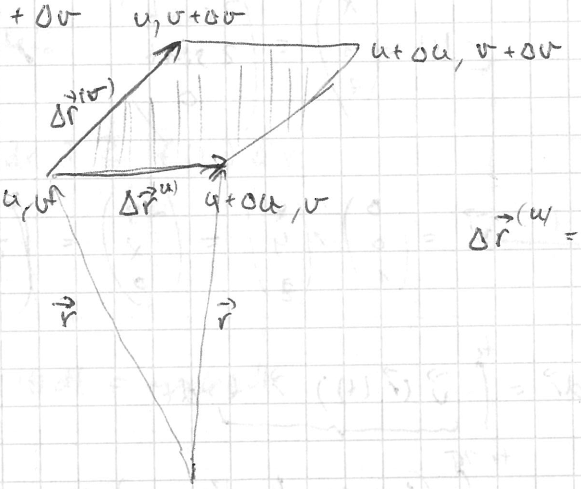
elvben (és a vizsgán)

$$\iint \vec{v}\, d\vec{F} \approx \sum_i \vec{v}(\vec{r}_i)\, \Delta\vec{r}^{(i)}$$



$u,\ u+\Delta u$

$v,\ v+\Delta v$



$$\Delta\vec{r}^{(u)} = \vec{r}\, ~~(u+\Delta u)~~ (u+\Delta u, v) - \vec{r}(u,v) \approx$$

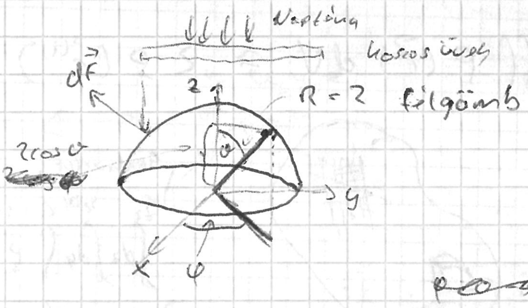
$$\approx \frac{\partial\vec{r}}{\partial u}\Delta u$$

$$\Delta\vec{r}^{(v)} \approx \frac{\partial\vec{r}}{\partial v}\Delta v$$

$$\iint \vec{v}\, d\vec{F} = \sum \vec{v}(\vec{r}(u,v))\left(\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right)\Delta u\, \Delta v \approx \iint \underbrace{\left(\vec{v}, \frac{\partial r}{\partial u}, \frac{\partial r}{\partial v}\right)}_{f(u,v)} du\, dv$$

gyakorlatban

pl: $\iint \vec{v}\, d\vec{F} = ?$ kW/m² napfény



csíkokban koszos az üveg

~~$\underline{j} = \begin{pmatrix} 0 \\ 0 \\ -1+0{,}1y^2 \end{pmatrix}$~~ $\begin{pmatrix} 0 \\ 0 \\ -1+0.1y^2 \end{pmatrix}$

~~φ 0 – 360°~~

~~φ 0 – 2π~~

~~θ: 0 → $\frac{\pi}{2}$~~

$$\frac{\text{energia}}{\text{idő}} = -\iint_{\text{félgömb}} \underline{j}\, d\underline{F}$$

$$\underline{r}(\vartheta,\varphi) = \begin{pmatrix} 2\sin\vartheta\cos\varphi \\ 2\sin\vartheta\sin\varphi \\ 2\cos\vartheta \end{pmatrix}$$

$0 \le \varphi \le 2\pi$

$0 \le \vartheta \le \frac{\pi}{2}$

$u = \vartheta$
$v = \varphi$

$$\underline{j} = \begin{pmatrix} 0 \\ 0 \\ -1+0{,}1 \cdot 2 \cdot 2 \sin\vartheta^2 \cdot \sin^2\varphi \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial \vartheta} = \begin{pmatrix} 2\cos\vartheta\ \cancel{\cos\varphi}\ \cos\varphi \\ 2\cos\vartheta\ \sin\varphi \\ -2\sin\vartheta \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -2\sin\vartheta\sin\varphi \\ +2\sin\vartheta\cos\varphi \\ \cancel{2\cos\vartheta}\ 0 \end{pmatrix}$$

$$-\iint \underline{j}\, d\underline{F} = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta \begin{vmatrix} 0 & 2\cos\vartheta\cos\varphi & -2\sin\vartheta\sin\varphi \\ 0 & 2\cos\vartheta\sin\varphi & 2\sin\vartheta\cos\varphi \\ -1+0{,}4\sin^2\vartheta\sin^2\varphi & -2\sin\vartheta & 0 \end{vmatrix} \cdot$$

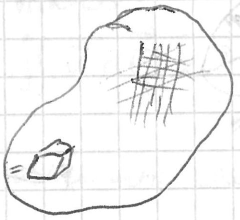
dot helyett

$$\cdot\, 2\cdot 2\,(-1+0{,}4\sin^2\vartheta\sin^2\varphi)\cos\vartheta\sin\vartheta =$$

$$= 4\pi \int_0^{\pi/2} d\vartheta\, \left(2\pi + 0{,}4\sin^2\vartheta \cdot \frac{1}{2}2\pi\right)\cos\vartheta\sin\vartheta =$$

$= \ldots$

$$\iiint \rho(\vec{r})\, dV \simeq \sum \rho(\vec{r}^{(i)})\, \Delta V^{(i)}$$ elvi



gyakorlat:

$$\int_{z_1}^{z_2} dz \int_{y_1(z)}^{y_2(z)} dy \int_{x_1(y,z)}^{x_2(y,z)} \rho(x,y,z)\, dx = \dots$$

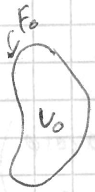
$\int \vec{v}\, d\vec{r}$ $\oint$

$\iint \vec{v}\, d\vec{F}$ $\oiint$

$\iiint \rho\, dV$

2 tétel:

$$\iiint_{V_0} \operatorname{div} \vec{v}\, dV = \oiint_{F_0} \vec{v}\, d\vec{F}$$



Gauss – Ostrogradszky tétel

$$\iint_{F_0} \operatorname{rot} \vec{v}\, d\vec{F} = \oint_{G_0} \vec{v}\, d\vec{r}$$

Stokes – tétel

$$\oint \vec{v}\, d\vec{r} \approx \operatorname{rot} \vec{v}\, \Delta\vec{F}$$

X. 9.

5. óra

Lenymásolat

6. óra

X. 16.

Gauss-Ostrogor. tétel:

$$\iiint \mathrm{div}\,\vec{v}\, dV = \oiint \vec{v}\, d\vec{F}$$

Legyen $\vec{v} = \phi(\vec{r})\ \mathrm{grad}\ \psi(\vec{r})$

$$\mathrm{div}\,(\phi\,\mathrm{grad}\,\psi) = \underline{\nabla}(\phi\,\underline{\nabla}\psi) = (\underline{\nabla}\phi)(\nabla\psi) + \phi(\underline{\nabla}^2\psi) =$$

$$= \mathrm{grad}\,\phi\ \mathrm{grad}\,\psi + \phi\ \underbrace{\mathrm{div\,grad}}_{\substack{\text{Laplace-operator}\\ \Delta}}\psi$$

I. $\iiint \left[\mathrm{grad}\,\phi\ \mathrm{grad}\,\psi + \phi\,\Delta\psi\right] dV = \oiint \phi\,(\mathrm{grad}\,\psi)\, d\vec{F}$ 1. Green-tétel

~~[illegible]~~, aszimmetrikus

II. $\phi \leftrightarrow \psi$ egy. csere

$$\iiint (\phi\,\Delta\psi - \psi\,\Delta\phi)\, dV = \cancel{\phi}\oiint (\phi\,\mathrm{grad}\,\psi - \psi\,\mathrm{grad}\,\phi)\, d\vec{F}$$

szimmetrikus G.-t.

pl. $\mathrm{div}\,\underline{E} = \frac{1}{\varepsilon_0}\rho$

~~$E = \mathrm{grad}\,\phi$~~ $\underline{E} = -\mathrm{grad}\,\phi$

$$\left.\right\} \Delta\phi = -\frac{1}{\varepsilon}\rho$$

$\psi = \frac{1}{|\underline{r} - \underline{r}_0|}$

$\iiint \frac{\rho(\underline{r})}{|\underline{r} - \underline{r}_0|}\, dV$ töltésfelhő potenciálja

---

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$x_i \rightarrow u_i$

$x(u, v, w) \longleftrightarrow u(x, y, z)$

$y(u, v, w) \longleftrightarrow v(\quad)$

$z(u, v, w) \longleftrightarrow w(\quad)$

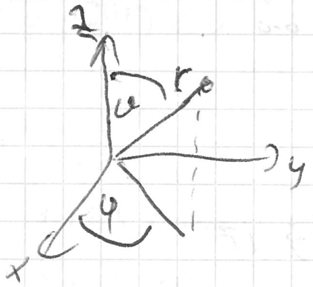
derékszögű egyenesvonalú

görbevonalú k.r.

Pl.: $u = r$

$v = \vartheta$

$w = \varphi$

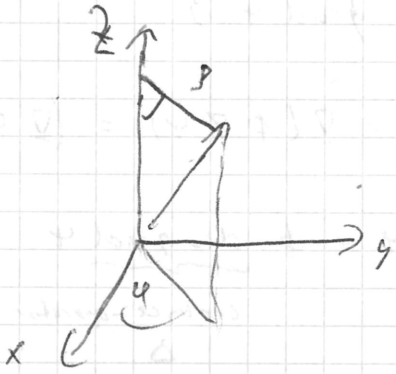


gömbi polár-koord.

$u = \rho$

$v = \varphi$

$w = z$



Henger-koord.

$u \rightarrow u + \Delta u$ -t teszek: $\}$ →

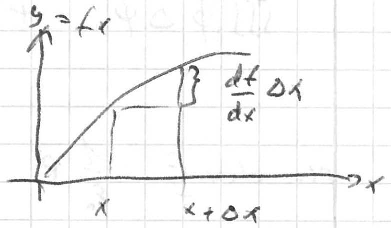
v, w állandó

ki mondanaketted

$$\Delta x = \frac{\partial x}{\partial u} \cdot \Delta u + \frac{\partial x}{\partial v} \Delta v + \frac{\partial x}{\partial w} \Delta w$$

$v \rightarrow v$ ha v változik

$w \rightarrow w$ -11-



$\phi(x)$

$\Delta \phi \approx (\text{grad}\ \phi)\ \Delta \underline{r}$

$$\Delta y = \frac{\partial y}{\partial u} \Delta u + \frac{\partial y}{\partial v} \Delta v + \frac{\partial x}{\partial w} \Delta w$$

Teljes differenciál

$\Delta z = \ldots$

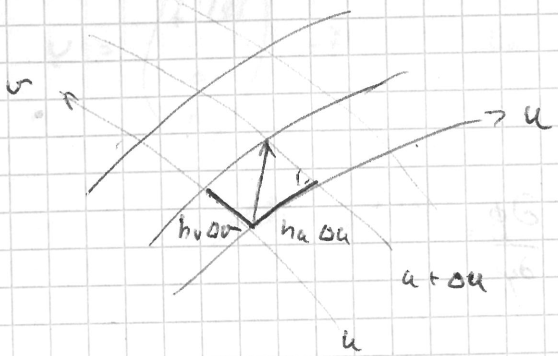
$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = (\overset{}{h_u^2}\ )\Delta u^2 + (\overset{h_v^2}{})\Delta v^2 + (\overset{h_w^2}{})\Delta w^2 +$$

$+ (\quad) \Delta u \Delta v + \ldots$

h_u geom. jelentése: Ha csak $\Delta u \neq 0$ akkor $\Delta s^2 = h_u^2 \Delta u^2$

$$\boxed{\Delta s = h_u \Delta u}$$

Mennyi a tényleges elmozdulás Pl. 1km-rel



ortogonális rendszerekben:

$$\Delta s^2 = h_u^2 \Delta u^2 + h_v^2 \Delta v^2 + h_w^2 \Delta w^2$$

pl.: $u = \rho$ | $x = \rho \cos\varphi$

$v = \varphi$ | $y = \rho \sin\varphi$

nem komplex konjugált: $w = z^*$ | $z = z^*$ ennek megváltozása

$$\Delta x = \cos\varphi \cdot \Delta\rho - \rho \sin\varphi \Delta\varphi$$

$$\Delta y = \sin\varphi \Delta\rho + \rho ~~cos~~ \cos\varphi \Delta\varphi$$

$$\Delta z = \Delta z^*$$

+

$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 = 1^2 \cdot \Delta\rho^2 + \rho^2 \Delta\varphi^2 + \Delta z^{*2} + \cancel{0 \cdot \Delta\rho \cdot \Delta\varphi}$$

$h_\rho = 1$ $h_\varphi = \rho$ $h_{z^*} = 1$ mert a rendszer ortogonális

polár koord.: $h_r = 1$

$h_\vartheta = r$ számításokat elvégezni derivál + négyzetre emel

$h_\varphi = r \cdot \sin\vartheta$ + összead.

$$\phi(\vec{r}) \to \phi(u, v, w)$$

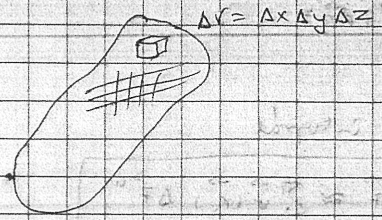
$$\Delta\phi(\vec{r}) \approx \text{grad}\,\phi \cdot \Delta\vec{r} = (\text{grad}\,\phi)_u h_u \cdot \Delta u + (\text{grad}\,\phi)_v h_v \Delta v + $$
$$+ (\text{grad}\,\phi)_w h_w \cdot \Delta w)$$

$$(\text{grad}\,\phi)_u = \frac{\Delta\phi}{h_u \Delta u}\bigg|_{v=\text{áll}} \longrightarrow \frac{1}{h_u}\frac{\partial\phi}{\partial u}$$

= térfogati integrál

$$\boxed{\iiint \rho(\vec{r})\, dV \approx \sum \rho(\vec{r}^{(i)})\, \Delta V^{(i)}}$$

elvi



- gyakorlatban:

$$\int_{z_1}^{z_2} dz \int_{y_1(z)}^{y_2(z)} dy \int_{x_1(y,z)}^{x_2(y,z)} \rho(x,y,z)\, dx$$

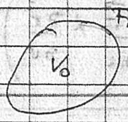
$\int \vec{v}\, d\vec{r}$ $\oint$ zárt görbe

$\iint \vec{v}\, d\vec{F}$ $\oiint$ zárt felület

$\iiint \rho\, dV$

2 tétel:

① $$\iiint_{V_0} \operatorname{div} \vec{v}\, dV = \oiint_{F_0} \vec{v}\, d\vec{F}$$



Gauss - Ostrogradskij - tétel

② $$\iint_{F_0} \operatorname{rot} \vec{v}\, d\vec{F} = \oint_{G_0} \vec{v}\, d\vec{r}$$

Stokes - tétel

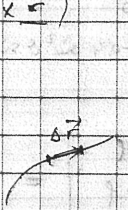
X. 9. ~~.~~ 5. óra

er: grad ϕ $(\text{grad}\, \phi)_i = \frac{\partial \phi}{\partial x_i} = \partial_i \phi$ → leggyorsabb változás iránya

ír: div $\underline{v}$ $\operatorname{div} \underline{v} = \sum_i \partial_i v_i = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \underline{\nabla}\, \underline{v}$ → forráserősség

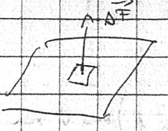
tér: rot $\underline{v}$ $(\text{rot}\, \underline{v}) = \sum_{j,k} \varepsilon_{ijk}\, \partial_j v_k = (\underline{\nabla} \times \underline{v})$ → örvényerősség

$\int_G (\quad)\, d\vec{r} \approx \sum (\quad)\, \Delta\vec{r}$



az integrált megszorozzuk egy kicsiny mennyiséggel és összeadjuk

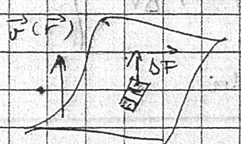
$\iint (\quad)\, d\vec{F} \approx \sum (\quad)\, \Delta\vec{F}$



$\iiint (\quad)\, dV \approx \sum (\quad)$

$$\int_G \vec{v}(\vec{r})\,d\vec{r} =$$

– felületi integrál



$$\boxed{\iint \vec{v}\,d\vec{F} \approx \sum_i \vec{v}(\vec{r}_i)\,\Delta\vec{F}^{(i)}}$$

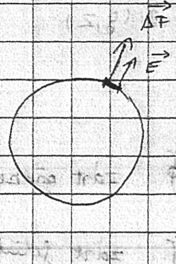
pl. a felületen átáramló víz

↑
előbb

pl: $\vec{v} = \frac{kQ}{r^3}\vec{r} = \vec{E}(\vec{r})$

$$\oint\!\!\oint \vec{E}\,d\vec{F} \approx \sum|\vec{E}||\Delta\vec{F}| = \sum\frac{kQ}{4}\Delta F = \frac{kQ}{4}\sum 4R^2\pi =$$

$(|r| = R = 2$ (gömb$)$ elektromos fluxus

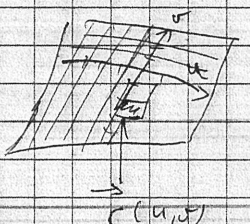


A gömb felszíne: $4R^2\pi$

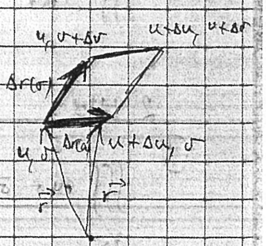
$= 4\pi kQ$

gyakorlatban:

$$\iint \vec{v}\,d\vec{F} \approx \sum_{u,v}\vec{v}(\vec{r}(u,v))\left(\frac{\partial\vec{r}}{\partial u}\times\frac{\partial\vec{r}}{\partial v}\right)\Delta u\Delta v \approx$$



$u,\ u+\Delta u$
$v,\ v+\Delta v$



$$\approx \iint \underbrace{\left(\vec{v},\frac{\partial r}{\partial u},\frac{\partial r}{\partial v}\right)}_{f(u,v)\ \text{vegyes szorzat}} du\,dv$$

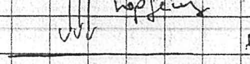
$$\Delta\vec{r}^{(u)} = \vec{r}(u+\Delta u, v) - \vec{r}(u,v) \approx$$

$$\frac{\partial\vec{r}}{\partial u}\cdot\Delta u$$

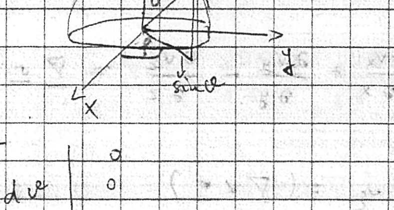
$$\Delta r^{(v)} \approx \frac{\partial\vec{r}}{\partial v}\cdot\Delta v$$

$$\iint \vec{j}\,d\vec{F} = ?$$

pl:



$$\underline{j} = \begin{pmatrix}0\\0\\-(1+0,1y^2)\end{pmatrix} \quad \left[\frac{kW}{m^2}\right]$$



$$\frac{\text{energia}}{\text{idő}} = -\iint_{\text{felgömbre}} \underline{j}\,d\underline{F} = \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta \begin{vmatrix} 0 & 2\cos\vartheta\cos\varphi & -2\sin\vartheta\sin\varphi \\ 0 & 2\cos\vartheta\sin\varphi & 2\sin\vartheta\cos\varphi \\ -1+0,4\sin^2\vartheta\sin^2\varphi & -2\sin\vartheta & 0 \end{vmatrix} =$$

$$= \int_0^{2\pi} d\varphi \int_0^{\pi/2} d\vartheta\ 4\cdot(-1+0,4\sin^2\vartheta\sin^2\varphi)\cos\vartheta\sin\vartheta =$$

$\pi/$

$$\left\{\begin{array}{l} 0 \leq \varphi \leq 2\pi \\ 0 \leq \vartheta \leq \frac{\pi}{2} \\ r(\vartheta,\varphi) = \begin{pmatrix}2\sin\vartheta\cos\varphi\\2\sin\vartheta\sin\varphi\\2\cos\vartheta\end{pmatrix} \\ \underline{j} = \begin{pmatrix}0\\0\\-1+0,1\cdot 4\sin^2\vartheta\sin^2\varphi\end{pmatrix}, \\ \frac{\partial\vec{r}}{\partial\vartheta} = \begin{pmatrix}2\cos\vartheta\cos\varphi\\2\cos\vartheta\sin\varphi\\-2\sin\vartheta\end{pmatrix} \\ \frac{\partial\vec{r}}{\partial\varphi} = \begin{pmatrix}-2\sin\vartheta\sin\varphi\\2\sin\vartheta\cos\varphi\end{pmatrix} \end{array}\right.$$

Kiszámítás:

$$\int (\ \ )\ \frac{d\vec{r}\,t}{dt}\ dt$$

$$\iint (\ \ )\left(\frac{\partial \vec{r}}{du} \times \frac{\partial \vec{r}}{dv}\right) du\,dv$$

$$\iiint (\ \ \ \ )\,dx\,dy\,dz$$

Gauss – Osztrogradszkij – tétel:

$$\oint \vec{v}\,d\vec{F} = \iiint \operatorname{div} \vec{v}\,dV$$

bal oldal jobb oldal

$$\approx \operatorname{div} \vec{v}\Big|_{\vec{r}=\vec{r_0}} \Delta V = \sum_1^6 \vec{v}\,\vec{dF}$$

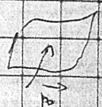
nagyon kis térrészre integrálunk

Stokes – tétel:

$$\oint \vec{v}\,d\vec{r} = \iint \operatorname{rot} \vec{v}\ d\vec{F}$$

bal oldal jobb oldal

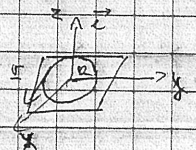
kicsiba $\operatorname{rot}\vec{v}\Big|_{\vec{r_0}} \Delta\vec{F} = \sum_1^4 \vec{v}\,\Delta\vec{r}$



~~Pl~~ Felt: folytonosan diff.ható: a derivált is folytonos

Pl: $\vec{v} = \vec{e} \times \vec{r}$ $|\vec{e}| = 1$

G:



$|\underline{v}| = R$

$\oint \vec{v}\,d\vec{r} = ?$

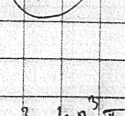
$\iint \operatorname{rot} \vec{v}\ d\vec{F} = ?$ $\operatorname{rot} \vec{v} = 2\vec{e}$

$$\sum(2\vec{e})\,\vec{\Delta F} = 2\vec{e} \sum \vec{\Delta F} = 2\vec{e}\,(R^2\pi\,\vec{e}) = 2R^2\pi \quad (\underbrace{\vec{e}\,\vec{e}}_{1})$$

$$\oint \vec{v}\,d\vec{r} = \sum(\vec{e} \times \vec{r})\,\Delta\vec{r} = \sum \underbrace{|\vec{e} \times \vec{r}|}_{R}\,|\Delta \vec{r}| = R\,(2R\pi) = 2R^2\pi$$

azonos irányúak

Pl: $\vec{v} = \vec{r}$



gömb

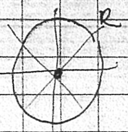
$\operatorname{div} \underline{v} = 3$

Gauss – O – tétel:

$$\iiint \operatorname{div} \vec{v}\ dV = \sum 3\,\Delta V = 3\,\frac{4R^3\pi}{3} = 4R^3\pi$$

$$\oint \vec{v}\,d\vec{F} \approx \sum |\vec{v}|\,|\vec{\Delta F}| = R\,4R^2\pi = 4R^3\pi$$

pl: $\vec{v} = kQ \frac{\vec{r}}{|\vec{r}|^3} = E$



G-O:

$\mathrm{div}\,\vec{E} = 0 \quad |\vec{r}| \neq 0$

$\iiint \mathrm{div}\,\vec{v}\, dV \overset{?}{=} 0$

$\oint\!\!\oint \vec{E}\, d\vec{F} \approx \sum |\vec{E}|\,|\Delta\vec{F}| = \frac{kQ}{R^2} \cdot 4R^2\pi = 4\pi k Q$

⚠ nem alkalmazható a G-O?

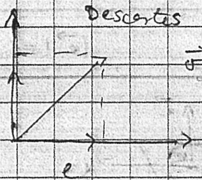
G-O felt: div $\vec{v}$ létezzen és folytonos legyen

Alkalmazzuk a G-O-t erre,

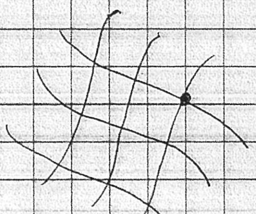
A kis gömbön ugyanannyi fluxus megy át, mint a nagyon, csak ellentétes irányba

$\rightarrow \quad 4\pi kQ - 4\pi kQ = 0$ ✓

Vektoranalízis görbevonalú koordinátarendszerben



$\vec{v} = (\;)\,\vec{e}^{(1)} + (\;)\,\vec{e}^{(2)}$



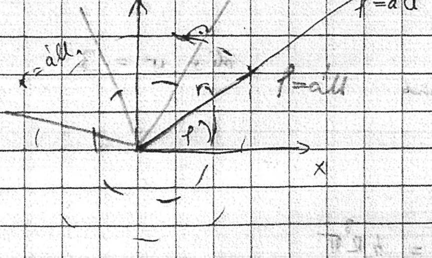
pl: $\left.\begin{aligned} u &= r \\ v &= \varphi \end{aligned}\right\}$ síkban

$x = r\cos\varphi$

$y = r\sin\varphi$

$r = \sqrt{x^2 + y^2}$

$\varphi = \mathrm{arctg}\,\frac{y}{x}$

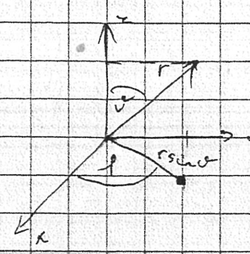


– ortogonális görbe vonalú koordináta-rsz.

Coulomb-potenciál:

$$\Phi = \frac{kQ}{\sqrt{x^2 + y^2 + z^2}} = \frac{kQ}{r}$$

Térbeli polár KR

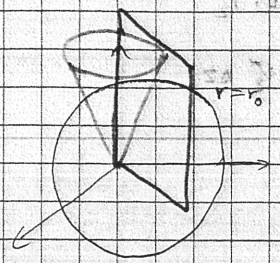


$$r = \sqrt{x^2+y^2+z^2}$$

$$\vartheta = \arccos\frac{z}{r} = \arccos\frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\varphi = \operatorname{arctg}\frac{y}{x}$$

(r, ϑ, φ)

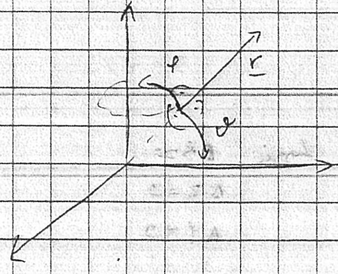


koordináta-felületek:

$r =$ áll → „hagymahéjak" $0 < r < \infty$ („tengerszint feletti mag.)

$\vartheta = \vartheta_0$ → „kúp" $(0 < \vartheta < \pi)$ „szélességi koordináta)

$\varphi = \varphi_0$ → „z-n átmenő sík" → „hosszúsági koordináta" $(0 < \varphi < 2\pi)$ $(-\pi, \pi)$



→ ahogy arrébb megyünk elfordul, mert a koordináta vonalak görbék

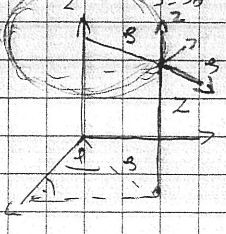
$r(x, y, z)$ $\quad x(r, \vartheta, \varphi)$ $\quad x = r\sin\vartheta\cos\varphi$

$\vartheta(x, y, z)$ $\quad y(r, \vartheta, \varphi)$ $\quad y = r\sin\vartheta\sin\varphi$

$\varphi(x, y, z)$ $\quad z(r, \vartheta, \varphi)$ $\quad z = r\cos\vartheta$

Az origóban φ és ϑ határozatlanok /r=0/ → a polár KR-ben az origó szinguláris /É-i és D-i sark is/

Henger KR → síkbeli polárkoordináták + z



(ρ, φ, z) → ortogonális, görbevonalú

$\rho = \rho_0$ → „henger"

görbevonalú rendszer 1 adott pontban Descartes-rendszerrel közelíthető.

$$\begin{tikzcd}\phi(\vec{r}) \arrow[r, "K^{(b)}"] & \phi(r, \vartheta, \varphi) \arrow[d, phantom, "|||"] \\ \arrow[u] & \phi(u, v, w)\end{tikzcd}$$

görbevonalú KR-ben

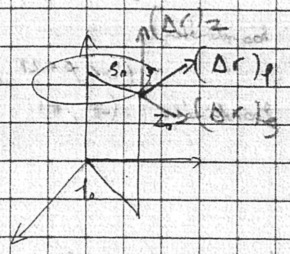
$$\vec{v}(\vec{r}) \xrightarrow{K^{(b)}} v_r(r, \vartheta, \varphi)$$ $\quad \operatorname{div}\vec{v}$ míg a konstans vektorok is len

Térbeli ~~polár~~ $\overset{\text{henger}}{}$ KR: (ρ, φ, z)

$\vec{v} = \text{grad}\ \phi\ (\rho, \varphi, z) = ?$

$\Delta\phi \approx (\overrightarrow{\text{grad}\ \phi})\ \Delta\vec{r} = (\text{grad}\ \phi)_\rho (\Delta r)_\rho + (\text{grad}\ \phi)_\varphi (\Delta r)_\varphi + (\text{grad}\ \phi)_z (\Delta r)_z$

$\phi(\rho_0 + \Delta\rho, \varphi_0 + \Delta\varphi, z_0 + \Delta z) - \phi(\rho_0, \varphi_0, z_0) \approx v_\rho\ \Delta\rho + v_\varphi\ \rho\ \Delta\varphi + v_z\ \Delta z$



$(\Delta r)_z = \Delta z$

$(\Delta r)_\rho = \Delta\rho$

$(\Delta r)_\varphi = \rho\ \Delta\varphi$

Legyen $\Delta\varphi = 0$
$\Delta z = 0$

$v_\rho \approx \left.\dfrac{\Delta\phi}{\Delta\rho}\right|_{\substack{\varphi = \text{áll}\\ z = \text{áll}}} \rightarrow \dfrac{\partial\phi}{\partial\rho}$

Legyen $\Delta\rho = 0$
$\Delta\varphi = 0$

$v_z = \dfrac{\partial\phi}{\partial z}$

Legyen $\Delta\rho = 0$
$\Delta z = 0$
$\Delta\varphi \neq 0$

$v_\varphi = \dfrac{\Delta\phi}{\rho\ \Delta\varphi} \rightarrow \dfrac{1}{\rho}\dfrac{\partial\phi}{\partial\varphi}$

Ha $(u, v, w) \rightarrow (u + \Delta u, v + \Delta v, w + \Delta w)$

s = Tényleges elmozdulás $= \sqrt{h_u^2\ \Delta u^2 + h_v^2\ \Delta v^2 + h_w^2\ \Delta w^2}$ → elmozdulások egymásra ⊥ → Pit.-tétel

$\Delta s = h_u\ \Delta u$, ha csak u változik, $h_u\ (u, v, w)$

h_u → mennyi az $\overset{\text{tényleges}}{\text{elmozdulás}}$, ha $\Delta u = 1$

$(\text{grad}\ \phi)_u = \dfrac{1}{h_u}\dfrac{\partial\phi}{\partial u}$

$(\text{grad}\ \phi)_v = \dfrac{1}{h_v}\dfrac{\partial\phi}{\partial v}$

$(\text{grad}\ \phi)_w = \dfrac{1}{h_w}\dfrac{\partial\phi}{\partial w}$

Polár KR

$h_r = 1$

$h_\vartheta = r$ $\qquad \Delta s = r\ \Delta\vartheta$

$h_\varphi = r \sin\vartheta$ $\qquad \Delta s = r \sin\vartheta\ \Delta\varphi$

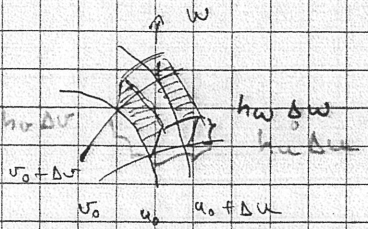
$$(\text{grad}\,\phi)_r = \frac{\partial\phi}{\partial r}$$

$$(\text{grad}\,\phi)_\vartheta = \frac{1}{r}\,\frac{\partial\phi}{\partial\vartheta}$$

$$(\text{grad}\,\phi)_\varphi = \frac{1}{r\sin\vartheta}\,\frac{\partial\phi}{\partial\varphi}$$

Divergencia:

$$\text{div}\,\vec{v} \simeq \frac{1}{\Delta V}\sum_1^6 \vec{v}\,\overrightarrow{\Delta F}$$



$$\Delta V = h_u\,h_v\,h_w\;\Delta u\,\Delta v\,\Delta w$$

$$/\!/(1+2)\qquad v_u(u_0+\Delta u, \ldots, \ldots)\,h_v\,\Delta v\,\overset{h}{\blacktriangle}_w\,\Delta w - v_u(u_0, \ldots, \ldots)\,h_v\,\Delta v\,h_w\,\Delta w$$

$$(\text{div}\,\vec{v})_{(u)} = \frac{\partial}{\partial u}\left(v_u\,h_v\,h_w\right)\cdot\frac{1}{h_u\,h_v\,h_w} + (\quad) + (\quad)$$

X. 16 — 6. óra

Gauss - Ostr. - tétel:

$$\iiint \text{div}\,\vec{v}\;dV = \oint\!\!\!\oint \vec{v}\;\overrightarrow{dF}$$

$$\underline{\nabla}(\phi\,\underline{\nabla}\,\psi)$$
$$\underline{\nabla}(\phi\,\underline{\nabla}\,\psi)$$

Legyen $\vec{v} = \phi(\vec{r})\;\text{grad}\,\psi(\vec{r})$

$$\text{div}\,(\phi\,\text{grad}\,\psi) = \underline{\nabla}(\phi\,\underline{\nabla}\,\psi) = (\underline{\nabla}\,\phi)(\nabla\psi) + \phi(\underline{\nabla}^2\psi) =$$

$$= (\text{grad}\,\phi)(\text{grad}\,\psi) + \phi(\underbrace{\text{div}\,\text{grad}}\,\psi)$$

Laplace - operátor: $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$$\iiint\left[(\text{grad}\,\phi)(\text{grad}\,\psi) + \phi\,\Delta\psi\right]dV = \oint\!\!\!\oint \phi\,(\text{grad}\,\psi)\,\overrightarrow{dF}$$

$-(\phi \to \psi$ produktív felírva$)$ 1. (asszimetrikus) Green - tétel

$$\iiint \phi\,\Delta\psi - \psi\,\Delta\,\phi)\,dV = \oint\!\!\!\oint\left(\phi\,\text{grad}\,\psi - \psi\,\text{grad}\,\phi\right)\overrightarrow{dF}$$

(szimmetrikus) Green - tétel

$\Downarrow$

Gauss - Ostr. speciális esete

pl: Maxwell: $\text{div}\,\underline{E} = \frac{1}{\varepsilon_0}\,\varrho$, $\underline{E} = -\,\text{grad}\,\phi$ $\Big\}$ $\Delta\phi = -\frac{1}{\varepsilon_0}\,\varrho$

$\psi = \frac{1}{\ }$ $\rightarrow \iiint \frac{\varrho(r)}{\ }\,dV$ → Coulomb - potenciál

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

x_i $\qquad$ u_i

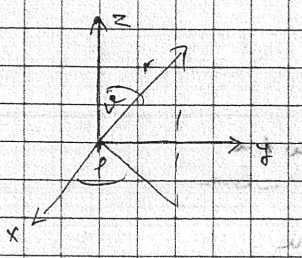
$x(u, v, w)$ $\qquad$ $u(x, y, z)$

$y(\;-''-\;)$ $\longleftrightarrow$ $v(\;-''-\;)$

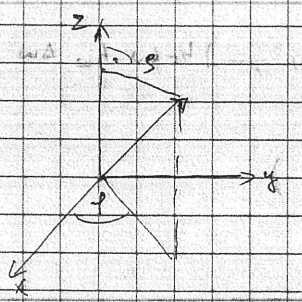
$z(\;-''-\;)$ $\qquad$ $w(\;-''-\;)$

erékszögű → görbevonalú
yenes vonalú $\quad$ ortogonális kr.

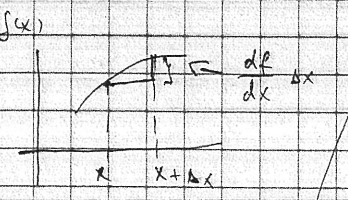
Pl: $u = r$ $\qquad$ térbeli polárkoordináta -rsz
)
$v = \vartheta$
$w = \varphi$



)
$u = \rho$ $\qquad$ hengerkoordinátál
$v = \varphi$
$w = z$



$$\left.\begin{matrix} u \rightarrow u + \Delta u \\ v, w \text{ állandó} \end{matrix}\right\} \rightarrow \Delta x = \frac{\partial x}{\partial u}\Delta u \qquad v \rightarrow v + \Delta v \; : \; \frac{\partial x}{\partial v}\Delta v$$



$$\Delta x = \frac{\partial x}{\partial u}\Delta u + \frac{\partial x}{\partial v}\Delta v + \frac{\partial x}{\partial w}\Delta w$$

$$\Delta y = \frac{\partial y}{\partial u}\Delta u + \frac{\partial y}{\partial v}\Delta v + \frac{\partial y}{\partial w}\Delta w$$

$$\Delta z = \text{-----}$$

$\phi(\underline{x})$

$\Delta\phi \approx (\text{grad}\,\phi)\,\underline{\Delta r}$

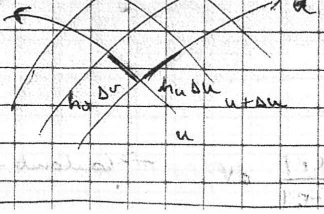
$$\Delta s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \qquad \leftarrow \text{elmozdulás}$$

$$\Delta s^2 = (h_u^2)\,\Delta u^2 + (h_v^2)\,\Delta v^2 + (h_w^2)\,\Delta w^2 + (\quad)\,\Delta u \Delta v + \ldots$$

$\equiv$ 0, ha ⊥ a koordináta-rsz.

Ha csak $\Delta u \neq 0$ $\rightarrow \Delta s^2 = h_u^2 \Delta u^2 \rightarrow \boxed{\Delta s = h_u \Delta u}$

h_u → megadja, mennyi a tényleges elmozdulás, ha „1 lépéssel" továbbmegyünk



$\overset{0}{(\;)}\Delta u \Delta v$

pl. $(\mathrm{grad}\,\phi)_r = \frac{\partial \phi}{\partial r}$

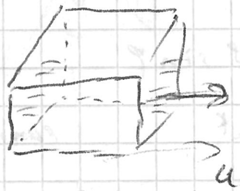
$(\quad)_\vartheta = \frac{1}{r}\frac{\partial \phi}{\partial \vartheta}$

$(\quad)_\varphi = \cancel{\frac{1}{r\sin\varphi}} \; \frac{1}{r\sin\vartheta} \frac{\partial \phi}{\partial \varphi}$

$(\mathrm{grad}\,\phi)_i = \frac{1}{h_i}\frac{\partial \phi}{\partial u_i}$

$\mathrm{div}\,\vec{a} = ?$

$\mathrm{div}\,\vec{a}\,\cancel{\Delta V} = \sum_1^6 \vec{a}\,d\vec{F} \cdot \frac{1}{\Delta V}$



$$\mathrm{div}\,\vec{a} = \frac{1}{h_u h_v h_w}\left[\frac{\partial}{\partial u}(a_u h_v h_w) + \frac{\partial}{\partial v}(a_v h_u h_w) + \frac{\partial}{\partial w}(a_w h_u h_v)\right]$$

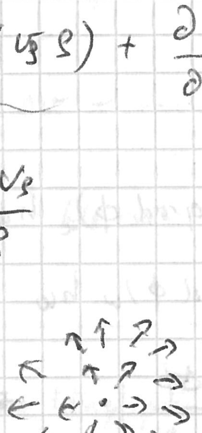
$$\mathrm{div}\,\vec{a} = \cancel{\frac{1}{\prod_1^3 h_k}} \; \frac{1}{\prod_{k=1}^{3} h_k} \sum_{i=1}^{3} \frac{\partial}{\partial u_i}\left(a_i \frac{h_1 h_2 h_3}{h_i}\right)$$

pl. henger koord. $h_\rho = 1$, $h_\varphi = \rho$, $h_z = 1$

$$\mathrm{div}\,\vec{v} = \frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(v_\rho \rho) + \frac{\partial}{\partial \varphi}(v_\varphi) + \frac{\partial}{\partial z}(v_z \rho)\right) =$$

$$= \underbrace{\frac{1}{\rho}\left(\frac{\partial}{\partial \rho}(v_\rho \rho)\right.}_{\frac{\partial v_\rho}{\partial \rho} + \frac{v_\rho}{\rho}} \left. + \frac{\partial v_\varphi}{\partial \varphi}\right) + \frac{\partial v_z}{\partial z}$$

pl. $v_\rho =$ áll.



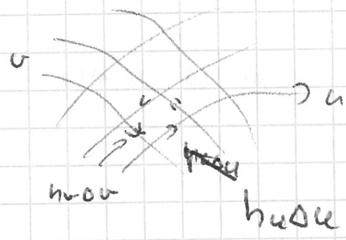
forrása van a mezőnek

$$v = \begin{pmatrix} f(\varrho) \\ 0 \\ 0 \end{pmatrix} \overset{?}{=} \quad \text{ha } \operatorname{div} \vec{v} \equiv 0$$

$$\cancel{\varrho^{f(\varrho)}} \cdot \varrho^{f(\varrho)} = k \Rightarrow \boxed{f(\varrho) = \frac{k}{\varrho}}$$

k = állandó

$$\operatorname{rot} \vec{a} \, \Delta \vec{F} \approx \sum_{1}^{4} \vec{a} \, \Delta \vec{r}$$



Stokes t. kicsiben:

$$(\operatorname{rot} \vec{a})_w = h_u \Delta u \cdot h_v \Delta v = a_v h_u \Delta v \Big|_{u+\Delta u} - a_v h_v \Delta v \Big|_{u} + \dots$$

$$(\operatorname{rot} \vec{a})_w = \frac{1}{h_u h_v} \left[\frac{\partial}{\partial u} (a_v h_v) - \frac{\partial}{\partial v} (a_u h_u) \right]$$

$$(\operatorname{rot} \vec{a})_3 = \frac{1}{h_1 h_2} \left[\frac{\partial}{\partial u_1} (a_2 h_2) - \frac{\partial}{\partial u_2} (a_1 h_1) \right]$$

$$\boxed{(\operatorname{rot} a)_i = \frac{h_i}{h_1 h_2 h_3} \sum_{k,l} \varepsilon_{ikl} \frac{\partial}{\partial u_k} (v_l h_l)}$$

pl. hengerben $\quad \varrho, \varphi, z$

$$(\operatorname{rot} \vec{v})_z = \frac{1}{h_\varrho h_\varphi} \left[\frac{\partial}{\partial \varrho} v_\varphi h_\varphi - \frac{\partial}{\partial \varphi} v_\varrho h_\varrho \right] =$$

$$= \frac{1}{\varrho} \left\{ \frac{\partial}{\partial \varrho} (v_\varrho \varrho) - \frac{1}{\varrho} \frac{\partial v_\varrho}{\partial \varphi} \right.$$

$$\operatorname{rot} \underbrace{v^{\hat{z}}}_{\text{[illegible]}} = \frac{1}{\varrho} \frac{d}{d\varrho} \left(\varrho \cdot f(\varrho) \right)$$

pl.

$v_\varrho \equiv 0$

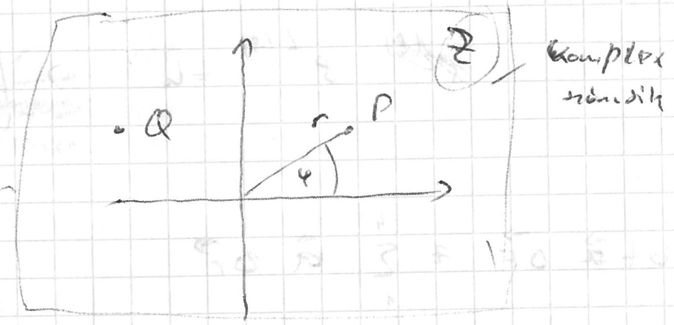
$v_\varphi \equiv f(\varrho) \qquad$ ha $f(\varrho) = \frac{k}{\varrho}$

Tornádó sodorja az [illegible]

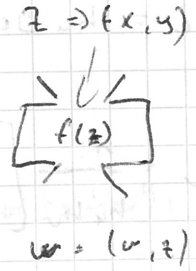
Komplex változós függvények (+ alkalmazásuk a fizikában)

~~adiabatikus változás:~~

$z = x + iy = re^{i\varphi}$



$z = x + iy \rightarrow f(z) \rightarrow w = u + iv$



$x = \mathrm{Re}\, z \quad y = \mathrm{Im}\, z$

$w = u + iv$

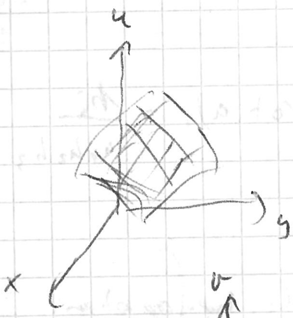
$u\,(x, y)$

$v\,(x, y)$

pl: $w = z^2 = (x+iy)^2 = \underbrace{x^2 - y^2}_{} + i(2xy)$

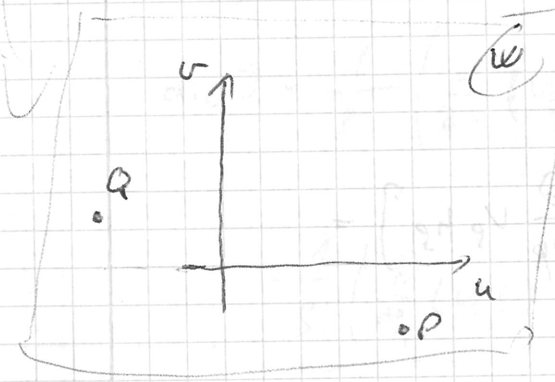
$u = x^2 - y^2$

$v = 2xy$



komplex számsík leképezés

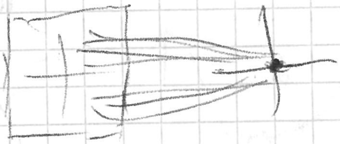
leképezzük a különböző pontokat



nem érdemes
szét választani
ez így nem
megy!!

pl:

$w = 0$ egész komplex síkot egy pontba viszi



$w = z$ saját magát rendeli hozzá   ~~azonosság~~ azonosság

$w = z + z_0$ $u = x + x_0$ eltolás z_0-lal.

↑ adott kompl. szám

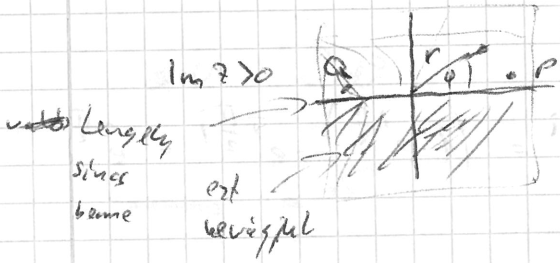
$v = y + y_0$

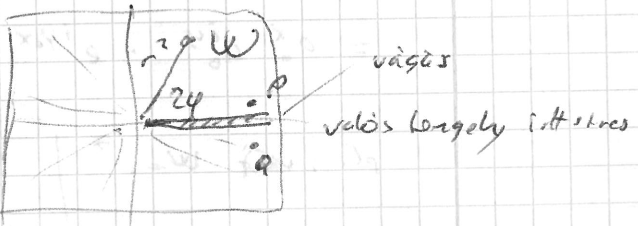
$w = 2z$ nagyítás   ált.: λz

$w = z_0 \cdot z = r_0 e^{i\varphi} \cdot r e^{i\varphi} = (r \cdot r_0)\, e^{i(\varphi+\varphi_0)}$

nagyítás és forgatás = hasonlósági transzf.

$w = z^2$ $(r e^{i\varphi})^2 = r^2 e^{i2\varphi}$ Pl. ~~kínai legyező teljesen kinyitja~~



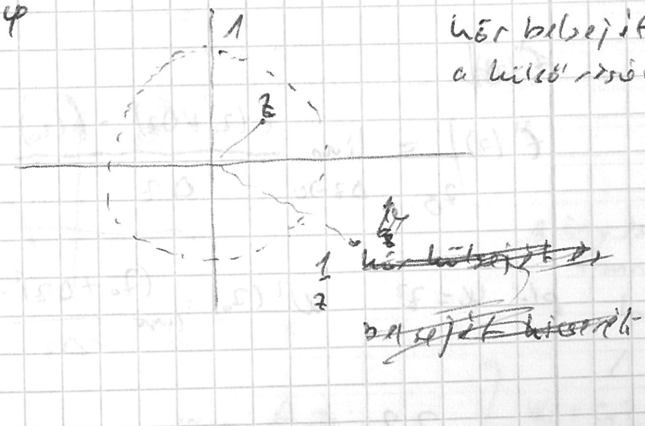


Pl. legyező teljesen kinyitva

$w = \frac{1}{z}$

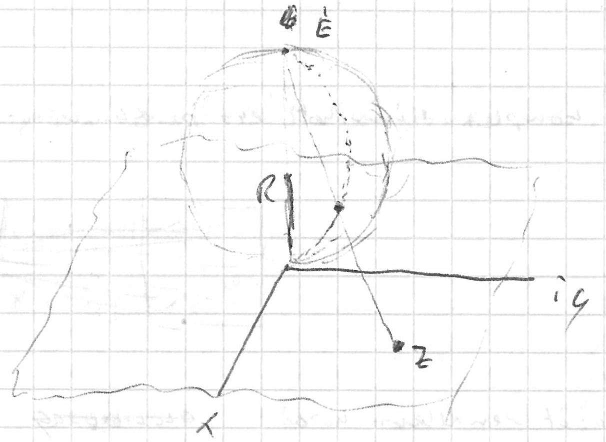
$z = r e^{i\varphi} \rightarrow \frac{1}{r} e^{-i\varphi}$

kör belsejét kívülre
a külső részét belsőre



0-nak ∞ távoli pont

∞-nek 0 pont a képe



sztereografikus

leképezés

$$w = \ln z = \ln r + i\varphi$$

$$\uparrow$$

$$z = r e^{i\varphi}$$

pl. $\ln i = \ln\left(1 \cdot e^{i\frac{\pi}{2}}\right) = 0 + i\frac{\pi}{2}$

$w = a^z \qquad a \in \mathbb{C}$

$$a^z = \left(a_0 e^{i\varphi_0}\right)^{x+iy} = a_0^{x+iy} \cdot e^{i\varphi_0 (x+iy)} =$$

$$= a_0^x \, a_0^{iy} \cdot e^{i\varphi_0 x} \, e^{-\varphi_0 y}$$

pl.: $w = i^z \Big|_{z=i} = i^i = \left(1 \cdot e^{i\frac{\pi}{2}}\right) = e^{i\frac{\pi}{2}i} = e^{-\frac{\pi}{2}}$

deriválás $\qquad\qquad$ integrálás

(?) $\qquad\qquad$ (?)

~~$f(z)$~~

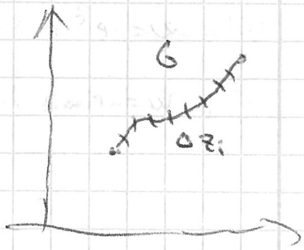
$$f'(z)\Big|_{z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

pl. $w = z^2 \qquad w'(z_0) = \lim \dfrac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} =$

$$= \lim \frac{2 z_0 \Delta z + \Delta z^2}{\Delta z} = 2 z_0 = 2z$$

integrálás:

$$\int f(z)\,dz = \lim \sum f(z_i)\,dz_i$$

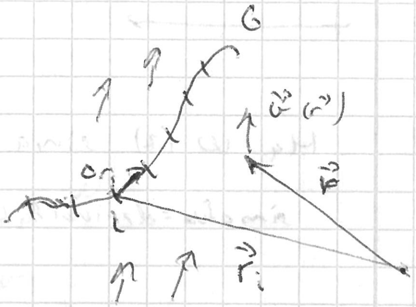


vektormező integrálja nagyon hasonló

fizika

XI. 6.

$$\int_G \vec{v}(\vec{r})\,d\vec{r} \approx \sum_{i=1}^{n} \vec{v}(\vec{r}_i)\,\Delta\vec{r}_i$$



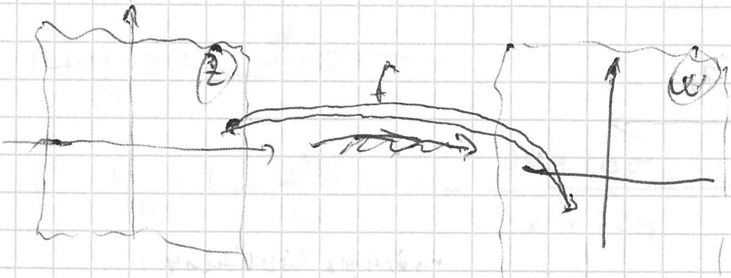
pl:
$$W = \int \vec{F}(\vec{r})\,d\vec{r}$$

$$U = \int \vec{E}\,d\vec{r}$$

$$U_{mágneses} = \int \vec{B}\,d\vec{r}$$

Komplex fv.

$w = f(z)$



arány és szögtartó transzformáció

komplex

$w = z + z_0$;

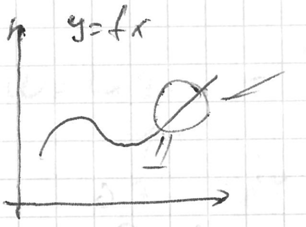
$w = e^{i\alpha} \cdot z = r e^{i(\varphi + \alpha)}$.

$w = A z$;

komplex $|A| \cdot e^{i\alpha}$

$w = c_1 z + c_2$

pl. $\left.\begin{array}{l} w = z^2 \\ w = \frac{1}{z} \\ w = e^z \\ w = \sin z \end{array}\right\}$ Nem aránytartó ⇒ nem szögtartó leképezés



kis környezetben ~~[illegible]~~ lineárissal közelíthető

$$y\,(x \approx x_0) = y'(x_0)(x - x_0) = C_1 x + C_2$$

Ha $w(z)$ sima $\quad w\,(z \approx z_0) \approx C_1 z + C_2$

simafv = deriválható fv.

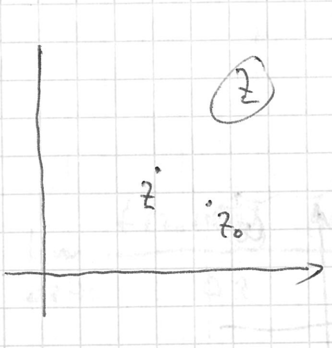
Def: $$\left. w'(z)\right|_{z_0} = \lim_{z \to z_0} \underbrace{\frac{w(z) - w(z_0)}{z - z_0}}_{\frac{\Delta w}{\Delta z}}$$

pl.

$w = z^2$

$w'(z_0) = 2z_0$

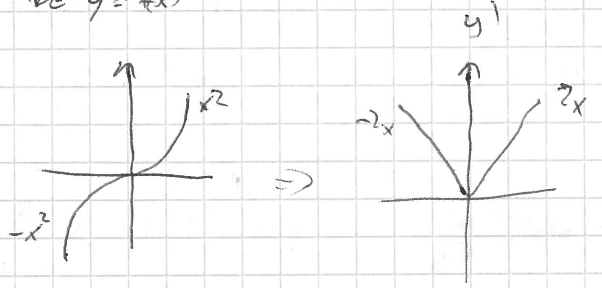
$$\frac{w(z) - w(z_0)}{z - z_0} = \frac{\overset{(z-z_0)(z+z_0)}{z^2 - z_0^2}}{\cancel{z - z_0}} = z + z_0 = 2z$$



$f(x,y) = \frac{y}{x}$

másképp közelítünk

$y = f(x)$

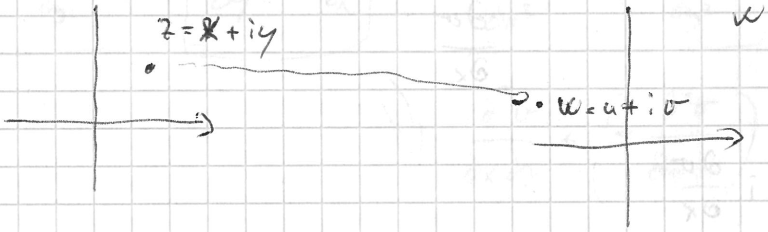


valós fv. nem mindig deriválható $2x$, $3x$.

a komplex fv. ∞-sokszor deriválható :

Deriválás a komplex számsíkon (Cauchy - Riemann relációk)

$w = f(z)$



$w(x,y)$ $\quad u(x,y)$

$\quad v(x,y)$

pl.

$$w = z^2 = (x+iy)^2 = \underset{u(x,y)}{(x^2 - y^2)} + i\,\underset{v(x,y)}{(2xy)}$$

ilyet gyakorolni kell vizsgán !!!

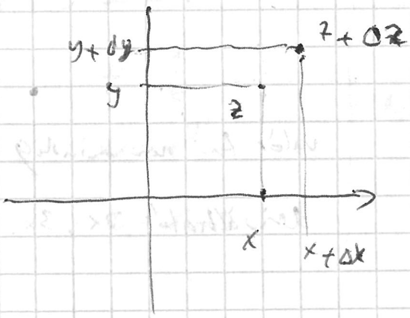
$$w = e^z = e^{(x+iy)} = e^x \cdot \underset{\cos y + i \sin y}{e^{iy}} = (\underbrace{e^x \cos y}_{u}) + i(\underbrace{e^x \sin y}_{v})$$

~~$f'(z)\big|_{z_0} = \lim$~~

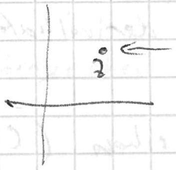
$$f'(z)\Big|_{z_0} = \lim \frac{\Delta w}{\Delta z} = \lim \frac{\Delta u + i \Delta v}{\Delta x + i \Delta y}$$

ahol $\Delta u = u(x + \Delta x, y + \Delta y) - u(x,y)$

$\Delta v = \ldots$



Legyen $\Delta y = 0$



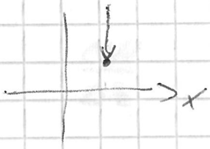
$$f'(z) = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\bigg|_{\Delta y=0} + i \lim \frac{\Delta v}{\Delta x}\bigg|_{\Delta y=0}$$

$$\qquad \parallel \qquad\qquad\qquad \parallel$$

$$\qquad \frac{\partial u}{\partial x} \qquad\qquad\quad \frac{\partial v}{\partial x}$$

$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

Legyen $\Delta x = 0$



$$\frac{1}{i} = -i$$

$$f'(z) = \lim \frac{\Delta u + i \Delta v}{i \Delta y} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\parallel$$

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \text{és} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$

C-R reláció

ha ez teljesül akkor akármilyen irányból is mehetek hiszen

e kettő lin kombinációjából kijön a bármilyen irány pl.

$w = z^2$ ⇒ $u = x^2 - y^2$

$v = 2xy$

C-R reláció teljesül:

$\underline{\underline{\frac{\partial u}{\partial x}}} = \underline{\underline{2x}} = \frac{\partial v}{\partial y} = \underline{\underline{2x}}$; $\frac{\partial u}{\partial y} = \underline{\underline{-2y}} = -\frac{\partial v}{\partial x} = \underline{\underline{-2y}}$

$e^z \Rightarrow u = e^x \cos y$

$v = e^x \sin y$

$\underline{\underline{\frac{\partial u}{\partial x} = e^x \cos y}}$ $\frac{\partial v}{\partial y} = e^x \cos y$

$\underline{\underline{\frac{\partial u}{\partial y} = -e^x \sin y}}$ $\underline{\underline{-\frac{\partial v}{\partial x} = -e^x \sin y}}$

(Laplace operator) $u = 0$ $\Delta u = 0$ — div grad u = 0

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ és } \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \Rightarrow \boxed{\frac{\partial u^2}{\partial x^2} + \frac{\partial u^2}{\partial y^2} = 0} \text{ és } \boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

$$\Downarrow \quad \frac{\partial^2 v}{\partial x \partial y} + \left(-\frac{\partial^2 v}{\partial y \partial x}\right) = 0$$

$\Delta \phi (x, y) = 0$

$\vec{v} = \text{grad}\, \phi$

$\text{div}\, \vec{v} = 0$ $\text{rot}\, \vec{v} = 0$

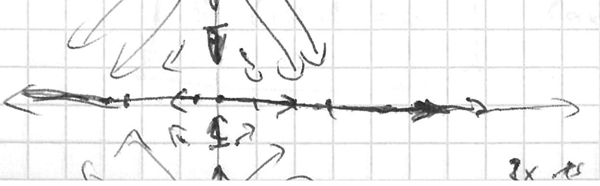
forrásmentes rotációmentes

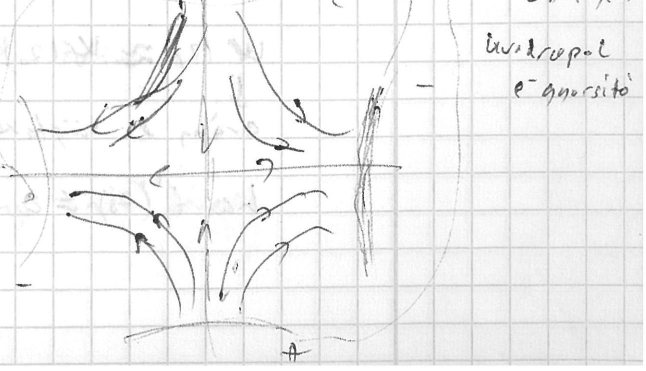
pl:

$w = z^2 = (x + iy)^2 = (x^2 - y^2) + i(2xy)$

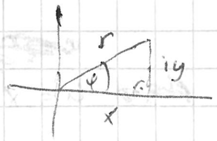
$u = \phi(x,y)$ $v = \psi(x,y)$

$\vec{v} = \text{grad}\,(x^2 - y^2) = \begin{pmatrix} 2x \\ -2y \end{pmatrix}$





pl: $w = \underset{\substack{\uparrow \\ \text{valós}}}{} A \ln \underset{\substack{\uparrow \\ r\cdot e^{i\varphi}}}{z} = A(\ln r + i\varphi)$



$$u(x,y) = \underbrace{A \ln\sqrt{x^2+y^2}}_{\phi} + iA \operatorname{arctg}\frac{y}{x}$$

kielégíti a Laplace egyenletet. ennek is

$$\phi(x,y) = A \ln\sqrt{x^2+y^2} = A \ln r$$

$$(\operatorname{grad}\phi)_r = \frac{\partial\phi}{\partial r} = \frac{A}{r}$$

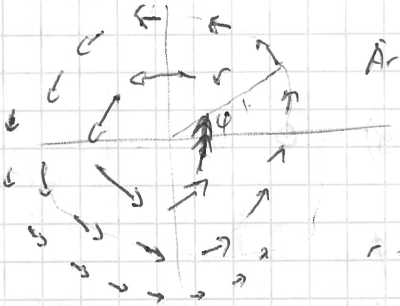
$$(\operatorname{grad}\phi)_\varphi = \frac{1}{r}\frac{\partial\phi}{\partial\varphi} = 0$$

Ha ez E(r) akkor
ez végtelen vezető

$$\Psi = A \operatorname{arc\,tg}\frac{y}{x} = A\varphi$$

$$(\operatorname{grad}\Psi)_r = \frac{\partial\Psi}{\partial r} = 0$$

$$(\operatorname{grad}\Psi)_\varphi = \frac{1}{r}\frac{\partial\Psi}{\partial\varphi} = \frac{A}{r}$$



Áramjárta vezető
mágneses
tere, körkörös alakú

r függő belül nagyobb

kint kisebb

~~körkörös alakú~~

Ha $w = f(z)$ deriválható

$$\frac{w(z) - w(z_0)}{z - z_0} \approx f'(z_0)$$

$$\Downarrow$$

$$w(z) \approx w(z_0) + f'(z_0)(z - z_0) = c_1 z + c_0$$

$\Downarrow$
arány és szögtartás kicsiben vagyis lokálisan (helyileg)

ha $f'(z_0) \neq 0$

Pl: $w = z^2$ ugyanez !!! nem szögtartó de aránytartó

$f' = 2z$

Komplex vonalintegrálok

$w = f(z)$

$$I_1 + i I_2 = \int_G f(z)\,dz = \lim_{|\Delta z_i| \to 0} \sum_{i=1}^{N} \overset{u+iv}{\downarrow}f(z_i)\, \overset{\Delta x + i\Delta y}{\nearrow}\Delta z_i =$$

$$= \underbrace{\lim \sum (u \cdot \Delta x - v \Delta y)}_{I_1} + i \underbrace{\lim \sum (v \Delta x + u \cdot \Delta y)}_{I_2} \; *$$

$$\int_G \vec{v}\, d\vec{r} = \lim \sum (v_x \Delta x + v_y \Delta y)$$

$$I_1 = \int_G \vec{a}(\vec{r})\, d\vec{r} \qquad I_2 = \int_G \vec{b}(\vec{r})\, d\vec{r}$$

*

$a_x = u\,(x,y)$ $\qquad$ $b_x = v\,(x,y)$

$a_y = -v\,(x,y)$ $\qquad$ $b_y = u\,(x,y)$

$a_z = 0$ $\qquad$ $b_z = 0$

$$(\mathrm{rot}\,\vec{a})_x = \frac{\partial}{\partial y} a_z - \frac{\partial}{\partial z} a_y = 0$$

$$(\mathrm{rot}\,\vec{a})_y = \frac{\partial}{\partial z} a_x - \frac{\partial}{\partial x} a_z = 0$$

$$(\quad)_z = \frac{\partial}{\partial x} a_y - \frac{\partial}{\partial y} a_x = -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

u.ez. a képzetes részre is.

$\mathrm{rot}\,\vec{a} = 0$

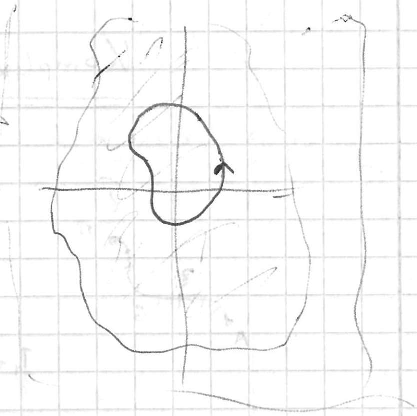
$\mathrm{rot}\,\vec{b} = 0$ Cauchy - Riemann rel. szerint.

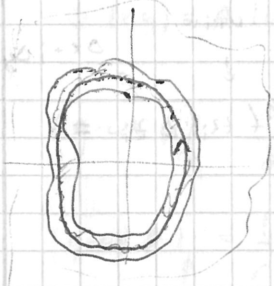
$$\boxed{\oint f(z)\,dz = 0}$$ Cauchy int. tétel.

$\mathrm{rot}\,\vec{a} = 0$

$\mathrm{rot}\,\vec{b} = 0$ $\xrightarrow[\text{Stokes}]{}$ $\oint \vec{a}\,d\vec{r} = \oint \vec{b}\,d\vec{r} = 0$

egyszeresen
összefüggő
tartomány

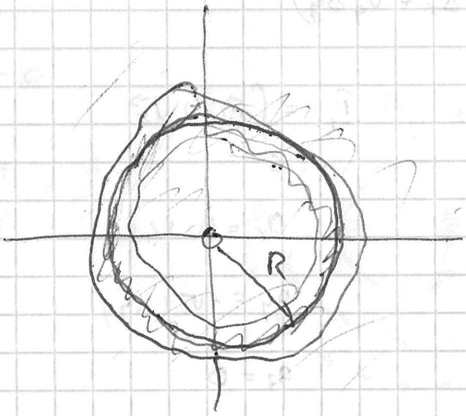




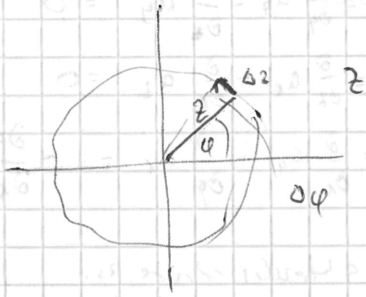
$\oint \vec{a}\,d\vec{r} \neq 0 \Rightarrow \oint f(z)\,dz \neq 0$

Pl.: $f(z) = \frac{1}{z}$

mindenhol
érvényes
kivéve a 0
helyen



$\oint \frac{1}{z}\,dz =$



$z = Re^{i\varphi}$

$\Delta z = \underbrace{\frac{\Delta z}{\Delta\varphi}}_{iRe^{i\varphi}} \cdot \Delta\varphi$

$|z| = R$

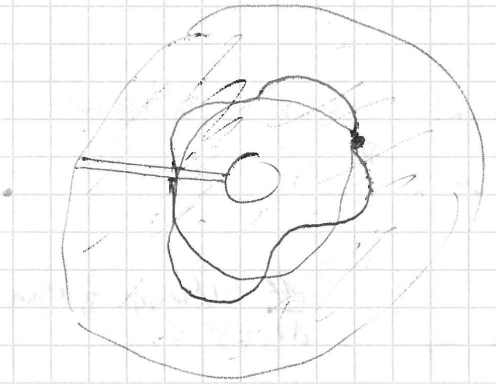
$\oint \frac{1}{z}\,dz \approx \sum \frac{1}{z}\,\Delta z =$

$= \sum \frac{1}{R}\,e^{-i\varphi} \cdot iRe^{i\varphi}$

$= i$

$\lim i(\sum \Delta\varphi) = 2\pi i$

$\oint \frac{1}{z}\,dz = 2\pi i$ Cauchy - rel.



kivágom ezt a csíkot 1 összefüggő lesz

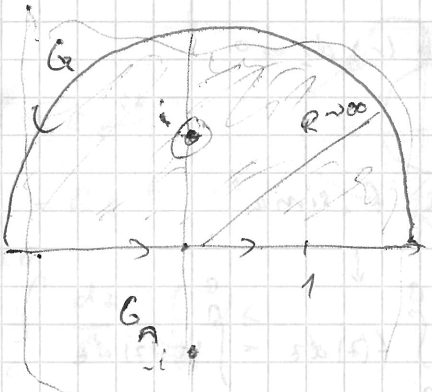
Bármely görbére ami 1x járja körbe az origót ha ~~tsz~~ n-szer megyünk körbe ⇒ $n \cdot 2\pi i$

$$\oint \frac{1}{z} dz = 2\pi i$$

de $\oint \frac{1}{z^2} dz = 0$

$\oint \frac{1}{z^n} dz = 0$, ha $n \neq 1$

Alkalmazás



csak x-tengely mentén mozgok.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx =$$

$$= \int_{G_1} \frac{1}{1+z^2} dz$$

innentől zárt görbe ⇒ $\oint$

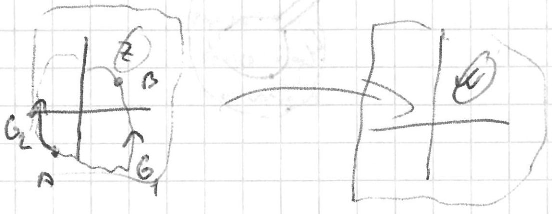
$$\sim \frac{1}{R^2} R\pi \rightsquigarrow 0$$

$\oint \frac{1}{1+z^2} dz$ i és -i nél szinguláris pont

$$\oint \frac{1}{1+z^2} dz = \dots = \pi$$

XI. 13. 8. óra

$$z = x + iy \xrightarrow{f \text{ „sima”}} w = u + iv$$



$\frac{df}{dz}$ létezik „sima”

$\downarrow$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{(C-R-ek)}$$

$\downarrow$

div grad $u \equiv 0$

div grad $v \equiv 0$

$\downarrow$

Fizika (síkban)

Stokes

$$\sum f(z_i)\,\Delta z_i \approx \oint f(z)\,dz = 0$$

$\Downarrow$ $\Downarrow$

$$\oint \underline{u}(\underline{r})\,d\underline{r} = 0 \qquad \oint_{G_1} f(z)\,dz = \int_{G_2} f(z)\,dz$$

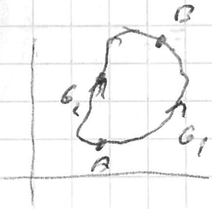
$$\oint \underline{b}(\underline{r})\,d\underline{r} = 0$$

Ha f „sima”

$\downarrow$

$F(B) - F(A) =$

$$= \sum \frac{\Delta F}{\Delta z}\Delta z \approx \int_{A\,G_1}^{B} f(z)\,dz = \int_{A\,G_2}^{B} f(z)\,dz$$



mit jelent ez? Szemléltetés

$$\text{Ha } f(z) = F'(z) \approx \frac{\Delta F}{\Delta z}$$

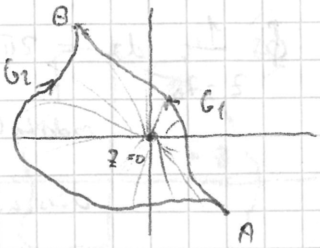
de ha a ~~f(z)~~ $f(x) = \frac{1}{z} \quad z \neq 0$

$$F(z) = \ln z \quad F'(z) = f(z)$$

$$\ln z = \ln(re^{i\varphi}) = \ln r + i\varphi = \ln\sqrt{x^2+y^2} + i\,\text{arctg}\,\frac{y}{x}$$

$$(\ln z)' = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$$

$$\int_A^B (\ln z)' \, dz = \ln z \Big|_{z_B} - \ln z \Big|_{z_A}$$



$$\oint \frac{1}{z} \, dz \neq 0!$$

$= 2\pi i \cdot \underset{4}{k}$ ⇐ topológia szám

hányszor mentem körbe az origót

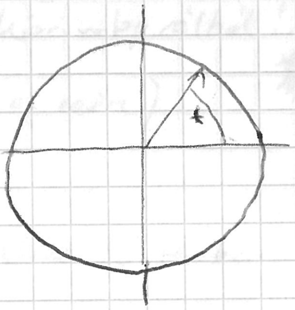
$0, \pm 1, \pm 2, \pm 3 \ \ldots$

Cauchy-tétel:

$\oint \frac{1}{z} \, dz = 2\pi i$ $\quad k = 1$ tehát 1 szer jártam körbe a origót

Biz:

$z = R e^{i\varphi}$ $\quad R = $ áll $\quad \varphi \, [0, 2\pi]$



~~$z = R e^{it} = R e^{it}$~~

$z = R e^{i\varphi} = R e^{it}$

$$\oint \frac{1}{z} \, dz = \int_0^{2\pi} \frac{dz}{dt} \frac{1}{z(t)} \, dt$$

$$i \cancel{R e^{it}} \frac{1}{\cancel{R e^{it}}} \, dt$$

$$\oint \frac{1}{z^n} \, dz = \int_0^{2\pi} i R e^{it} \frac{1}{(R e^{it})^n} \, dt = \int_0^{2\pi} i \, R^{1-n} \underset{\substack{\| \\ \cos(1-n)t + i \sin(1-n)t \\ \| \\ \cos(n-1)t - \sin(n-1)t}}{e^{i(1-n)t}} \, dt = 0$$

$z = R e^{it}$

$$\frac{dz}{dt} = i R e^{it}$$

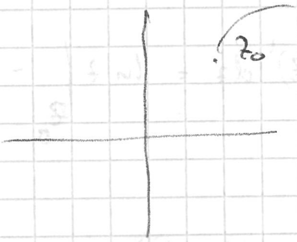
$\oint \frac{1}{z^n} \, dz = 0$, ha $n \neq 1$

$= 2\pi i$ ha $n = 1$

$$\oint \frac{1}{z-z_0}\, dz = 2\pi i$$

odatolom a koord.-t is

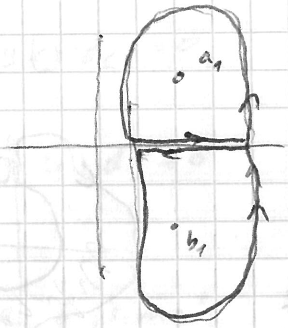
visszavezetem az előzőre



$\frac{1}{z^n} \Rightarrow$ n-edrendű pólus

$$\frac{\overset{\swarrow \text{ póluserősség}}{a_1}}{z-\underset{\uparrow\ \text{pólus helye}}{z_0}} = f(z)$$

$$\oint f(z)\, dz = 2\pi i\, a_1$$



$$\frac{a_1}{z-z_a} + \frac{b_1}{z-z_b}$$

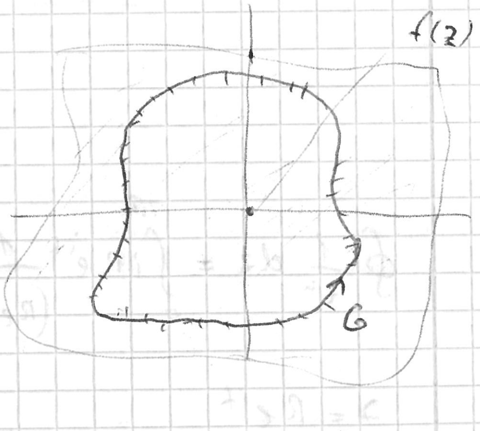
Cauchy tétel: $\oint f(z)\, dz = \sum\limits_{\text{pólusokra}} 2\pi i\ (\text{póluserősség})$

$$\boxed{\oint \frac{\overset{\swarrow \text{ sima}}{f(z)}}{z-z_0}\, dz = 2\pi i\, f(z_0)}$$

pl.

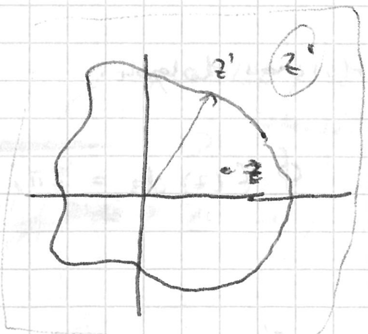
$$\frac{\cos z}{z-i}$$

analitikus fv.



a külső „körből" meghatározható a belső pont.

$$\frac{1}{2\pi i}\oint\frac{f(z')}{z'-z}\,dz' = f(z) \Rightarrow f'(z) =$$



$$= \frac{1}{2\pi i}\,1\oint\frac{f(z')}{(z'-z)^2}\,dz' \Rightarrow f''(z) =$$

$$= \frac{1}{2\pi i}\,2\oint\frac{f(z')}{(z'-z)^3}\,dz' \rightarrow$$

$$f^{(n)}(z) = \left(\frac{d}{dz}\right)^n f(z) = \frac{1}{2\pi i}\,n!\oint\frac{f(z')}{(z'-z)^{n+1}}\,dz'$$

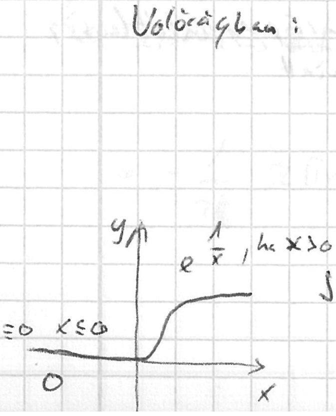
Ha f(z) sima (deriválható) egy tartományon akkor akárhányszor deriválható és ~~előtt~~ előállítható a tartomány szélén futó körintegrálból! (Visszafele is igaz a pontból meghatározható az egész)

Valós fv.-tanban NEM igaz!!

J(t) | jövőbeli jellemző pl. tőzsdeárfolyam

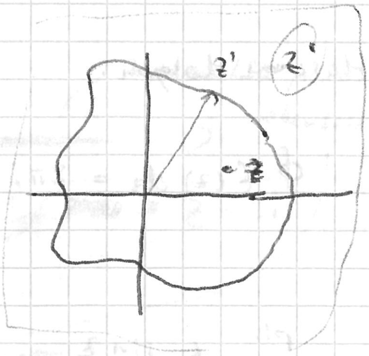
$$J(t) \approx a - bt + ct^2 + \underset{\substack{\shortparallel\\0}}{(?)}\,t^3 + \underset{\substack{\shortparallel\\0}}{(!)}\,t^4$$

Valóságban: $$\cancel{J(t) = \begin{cases} a - bt + ct^2, & \text{ha } t \leq t_0 \\ a - bt + ct^2 - d\,e^{-\frac{\tau}{t-t_0}}, & \text{ha } t > t_0 \end{cases}}$$



$$J(t) = \begin{cases} a - bt + ct^2, & \text{ha } t \leq t_0 \\ a - bt + ct^2 - d\,e^{-\frac{\tau}{t-t_0}}, & \text{ha } t > t_0 \end{cases}$$

$$\frac{1}{2\pi i}\oint \frac{f(z')}{z'-z}\,dz' = f(z) \Rightarrow f'(z) =$$



$$= \frac{1}{2\pi i}\, 1 \oint \frac{f(z')}{(z'-z)^2}\,dz' \Rightarrow f''(z) =$$

$$= \frac{1}{2\pi i}\, 2 \oint \frac{f(z')}{(z'-z)^3}\,dz' \rightarrow$$

$$f^{(n)}(z) = \left(\frac{d}{dz}\right)^n f(z) = \frac{1}{2\pi i}\, n! \oint \frac{f(z')}{(z'-z)^{n+1}}\,dz'$$

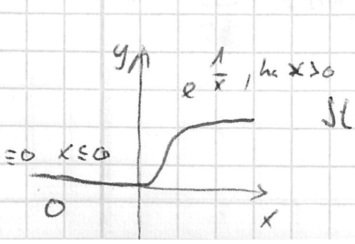
Ha $f(z)$ sima (deriválható) egy tartományon akkor akárhányszor deriválható és ~~előáll~~ előállítható a tartomány szélén futó körintegrálból! (Visszafelé is igaz a pontból meghatározható az egész)

Valós fv.-tanban NEM igaz!!

$S(t)$ jövőbeli idősor pl. tőzsdeárfolyam

$$S(t) \approx a - bt + ct^2 + (?)t^3 + (?)t^4$$
$$\overset{\shortparallel}{0} \qquad \overset{\shortparallel}{0}$$

Valóságban: ~~$S(t) = \begin{cases} a - bt + ct^2 \,\, , \text{ ha } t \leq t_0 \\ a - bt + ct^2 - d\, e^{-\frac{\tau}{t-t_0}} \, , \text{ ha } t > t_0 \end{cases}$~~



$$S(t) = \begin{cases} a - bt + ct^2 \, , \text{ ha } t \leq t_0 \\ a - bt + ct^2 - d\, e^{-\frac{\tau}{t-t_0}} \, , \text{ ha } t > t_0 \end{cases}$$

Hasznos dolgok:

$$\oint f(z)\,dz = 2\pi i \sum \text{póluserősség}$$

Cauchy-féle reziduum tétel

pl. $$\oint \frac{\sin z}{z}\,dz = 0, \text{ mert } \sin z\Big|_{z=0} = 0$$

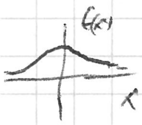
$$\oint \frac{\cos z}{z}\,dz = 2\pi i \cdot 1 \qquad \cos z\Big|_{z=0} = 1$$

de $$\oint \frac{e^{z}}{z^{2}}\,dz = 1 \cdot 2\pi i \quad \text{vizsgán!!!}$$

$$e^{z} \underset{z\to 0}{\approx} 1 + z + \frac{z^2}{2!}$$

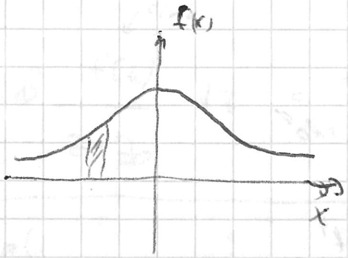
$$\oint \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i\, f'(z_0)$$

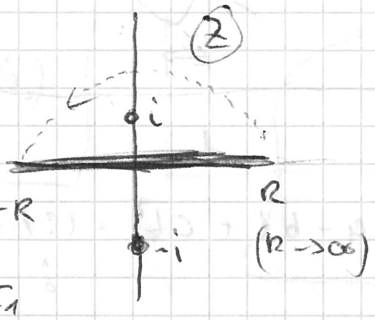
$$* \oint \frac{dz}{(z+i)(z-i)}$$



$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \underset{\text{valós}}{\int_{-\infty}^{\infty}} \frac{dz}{1+z^2} = \oint \frac{dz}{1+z^2} = \sum 2\pi i \text{ pólus erősség} = 2\pi i \left(\frac{1}{z+i}\right)_{z=i} =$$

$$= 2\pi i \frac{1}{2i} = \underline{\underline{\pi}}$$





$1+z^2=0$

$z^2=-1$

$z=\pm i$

$$\Bigg|\underset{\text{félkörre}}{\int} \frac{1}{1+z^2}\,dz\Bigg| \le \int \overset{\le \frac{1}{R^2-1}}{\left|\frac{1}{1+z^2}\right|}\,|dz| \le$$

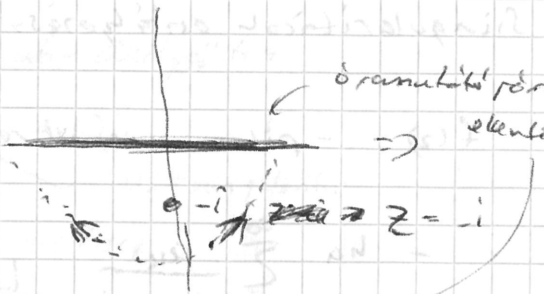
háromszög egyenlőtlenség miatt

$$\approx \frac{1}{R^2}$$

$$\left|\sum f(z_i)\,\Delta z_i\right|$$

$$\le \frac{1}{R^2-1} \underbrace{\int |dz|}_{R\pi} \approx \frac{1}{R} \xrightarrow[R\to\infty]{} 0$$

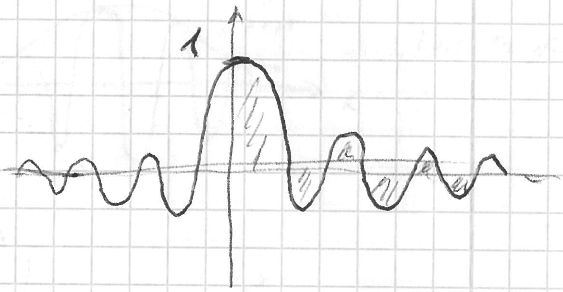
$$\cancel{* = \oint \frac{dz}{(z+i)(z-i)}}$$



óramutató járással
ellentétes irány

$$= \oint \frac{dz}{(z+i)(z-i)} = 2\pi i \frac{1}{(-2i)} \cdot (-1) = \pi$$

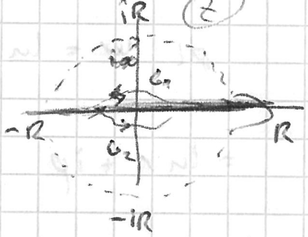
$$\int_{-\infty}^{\infty} (\mathrm{arctg}\, x)' \, dz = \pi = \int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$



$$\int_{-\infty}^{\infty} \frac{\sin x}{x} d\cancel{x} =$$

$$= \int_{-\infty}^{\infty} \frac{\sin z}{z} dz = \oint_{-\infty}^{\infty} \left(\frac{e^{iz}}{\cancel{2iz}\, 2iz} - \frac{e^{-iz}}{\cancel{2iz}\, 2iz} \right) dz =$$

$$= \int_{G_1}^{i\infty} \cancel{\frac{e^{iz}}{2iz}} dz - \int_{G_1}^{-i\infty} \frac{e^{-iz}}{2iz} dz =$$



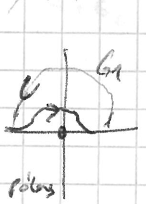
azért nem jó külön külön
mert az egyik
tag elszáll
két görbe
u.a.

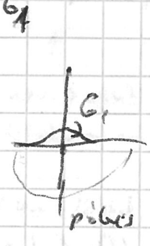
$$\sin z = \mathrm{Im}\left(e^{iz}\right) = \frac{e^{iz} - e^{-iz}}{2i}$$

def: $e^{iz} = \cos iz + i \sin z$

pl. $e^{iz} = e^{-1} = \frac{1}{e} =$

$= \cos i + i \sin i$



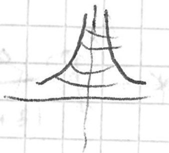


nincs körben
pólus amit
körbejárok ⇒ integrál = 0

$$= -\frac{1}{2\cancel{i}} \cancel{2}\pi \cancel{i} \cdot 1 \cdot (-1) = \pi$$

↓
körüljárás

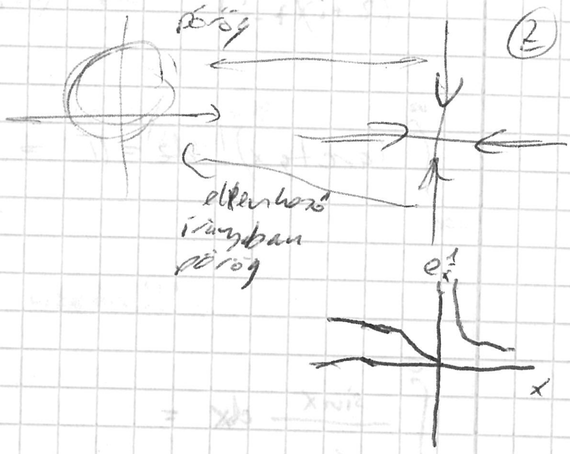
Szingularitások osztályozása



f(z) – pólus n-edrendű, ha $\sim \frac{1}{(z-z_0)^n}$

– ha $\sum_{k=1}^{\infty} \frac{a_k}{(z-z_0)^k}$ lényeges szingularitás pl. $f(z) = e^{\frac{1}{z}}$

$(z=0)$

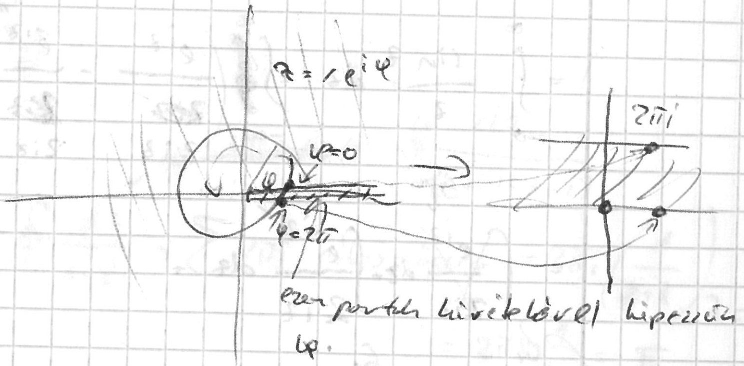


– vágás szingularitás

pl. $w = \ln z =$

$= \ln r + i\varphi$

$w = \sqrt{z}$



XI. 20.

9. óra

Analitikus (komplex változós deriválható) fv-ek

C–R:

$w = u + iv$

$z = x + iy$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Downarrow$$

$$\operatorname{div}\operatorname{grad} u = \Delta u(x,y) = 0$$

lokálisan szög- és aránytartó

$$\Delta v(x,y) = 0$$

ha $w(z) \not\equiv 0$

$\oint w(z)\,dz = 0$ egyszerűen ~~[illegible]~~ összefüggő tart.

Ha $w(z)$ - analitikus (pólusok)

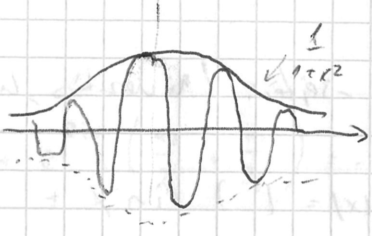
izolált szingularitás

$\oint w(z)\,dz = 2\pi i \sum$ elsőrendű pólus erősség

$\oint \frac{f(z)}{z-z_0}\,dz = 2\pi i\, f(z_0)$ pl.: $\frac{f(z)}{z-z_0}$

pl. $\oint$ -

$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\,dx =$



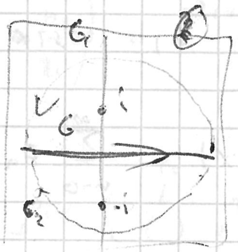
Lorentz görbe

$= \int_{-\infty}^{\infty} \frac{\cos z}{1+z^2}\,dz =$

$\cos z \to \frac{e^{iz}+e^{-iz}}{2}$

$1+z^2 \to (z+i)(z-i)$

(valós tengely G)



$$= \oint_{G+G_1} \frac{e^{iz}}{2(z+i)(z-i)}\,dz + \oint_{G+G_2} \frac{e^{-iz}}{2(z+i)(z-i)}\,dz = 2\pi i \left.\frac{e^{iz}}{2(z+i)}\right|_{z=i} + 2\pi i \left.\frac{e^{-iz}}{2(z-i)}\right|_{z=-i} \cdot(-1)$$

körüljárás miatt (-1)

nev. 0 válik

letakarom

veszem a többit ~~ta tartom~~

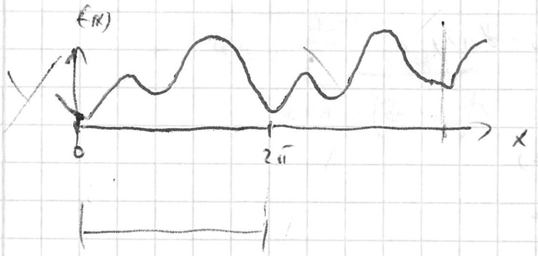
ahol a pólus van

$$= 2\pi i\left(\frac{e^{ii}}{2\cdot 2i} - \frac{e^{i(i)}}{2(-2i)}\right) = \frac{\pi}{e}$$

ilyen lesz a vizsgán!! ↑

Fourier-analízis (-sor -integrál)

Sor:



csak ezt vizsgáljuk

$f(x) = f(x - 2\pi)$

átírni a fv. 2π szerint periodikusra

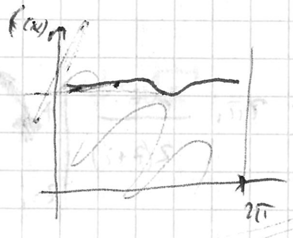
ha van 2π szerint periodikus

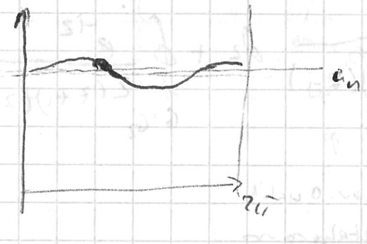
Fourier-sejtés: (közelítés hullámokkal)

$$f(x) = (\ )\sin x + (\ )\sin 2x + (\ )\sin 3x + \dots +$$

$$+ (\ )\cos x + (\ )\cos 2x + (\ )\cos 3x + \dots +$$

$$+ \cancel{c}(\ )\cos(0\cdot x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$





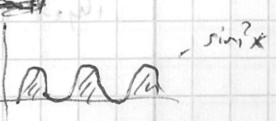
hozzáadva egy sin-t akkor jobban közelítem az eredeti fv-t.

=> b_n szorzó   majd egyre jobban közelítem sin & cos szuperpozíciójával.

$\to a_0 + b_1 \sin x$

$$\boxed{\frac{1}{\pi}\int_0^{2\pi} f(x)\sin mx\,dx = b_m} \underbrace{\int_0^{2\pi} \sin^2 mx\,dx}_{\frac{1}{2}2\pi}$$

m = egész (1, 2, 3, ...)



$$\sin\alpha\cos\beta = (\tfrac{1}{2})\sin(\alpha+\beta) + (\tfrac{1}{2})\sin(\alpha-\beta)$$

$$\frac{1}{\pi}\int_0^{2\pi} f(x)\cos mx\,dx = \begin{cases} a_m & \text{, ha } m \neq 0 \\ 2a_0 & \text{, ha } m = 0 \end{cases}$$

$$\frac{e^{ix}+e^{-ix}}{2} \qquad \frac{e^{2ix}+e^{-2ix}}{2}$$

$$f(x) = a_0 + a_1\cos x + a_2\cos 2x + a_3\cos 3x +$$

$$+ \qquad b_1\sin x + b_2\sin 2x + \ldots =$$

$$\frac{e^{ix}-e^{-ix}}{2i} \qquad \frac{e^{2ix}-e^{-2ix}}{2i}$$

$$= a_0 \quad + e^{ix}\left(\frac{a_1}{2}+\frac{b_1}{2i}\right) + e^{2ix}\left(\frac{a_2}{2}+\frac{b_2}{2i}\right) + \ldots$$

$$+ e^{-ix}\left(\frac{a_1}{2}-\frac{b_1}{2i}\right) + e^{-2ix}\left(\ldots\right) + \ldots$$

$$f(x) = \overset{c_1}{(\;)}\,e^{ix} + \overset{c_2}{(\;)}\,e^{2ix} + \overset{c_3}{(\;)}\,e^{3ix} +$$

$$+ \overset{c_0}{(\;)}\,e^{0ix} + \overset{c_{-1}}{(\;)}\,e^{-ix} + \overset{c_{-2}}{(\;)}\,e^{-2ix} + \ldots =$$

$$= \sum_{n=-\infty}^{\infty} c_n\, e^{inx}$$

komplex számok

$C_{-n} = C_n^*$, ha ~~[illegible]~~ $f(x)$ valós

$$f(x) = f^*(x)$$

Formálisan:

$$\int\sum = \sum\int$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)\,e^{-imx}\,dx \overset{(\text{ha} \ldots \text{felcserél})}{=} \cancel{\sum_{n=-\infty}^{\infty}} \sum_{n=-\infty}^{\infty} c_n \underbrace{\frac{1}{2\pi}\int_0^{2\pi} \underbrace{e^{inx}\,e^{-imx}}_{e^{i(n-m)x}}\,dx}_{\delta_{nm}} = c_m$$

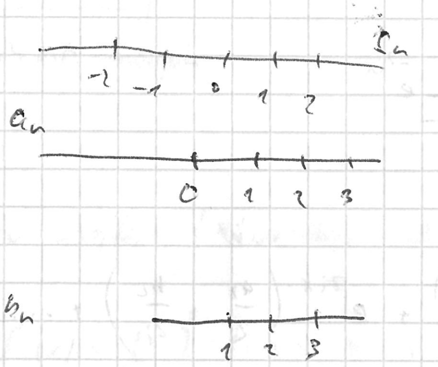
ha $n \neq m$ az integrál $= 0$

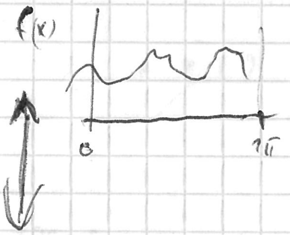
ha $n = m$ " " $= 1$

$$\boxed{\begin{aligned} f(x) &= \sum_{n=-\infty}^{\infty} c_n e^{inx} \\ c_n &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\, e^{-inx}\, dx \end{aligned}}$$

Fourier sorfejtés és integrál

kiszámitás





$$\underline{c} = \begin{pmatrix} c_0 \\ c_1 \\ c_{-1} \\ c_2 \\ c_{-2} \\ \vdots \end{pmatrix}$$

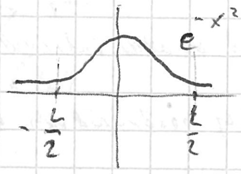
$f(x) \longleftrightarrow c_n \quad (n = -\infty, \infty)$

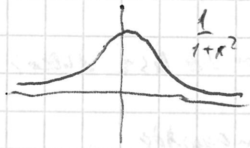
$f'(x) \longrightarrow \quad d_n = in\, c_n$

$\sum_n d_n e^{inx}$

Fourier-transzformáció (azaz nem periodikus fv-ek „sorfejtése")

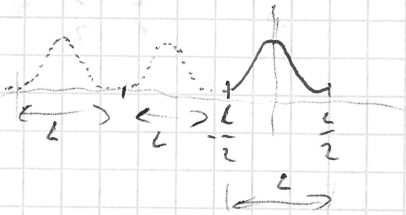
pl:





Legyen f(x) közel periodikus L hosszon

(és $L \to \infty$)



periodikus határfeltétel

$f(x) = f(x+L)$ (L nagyon nagy, $L \to \infty$)

$x = \left(\frac{L}{2\pi}\right) u \to dx = \frac{L}{2\pi} du$ u legyen 2π szerint periodikus

$x,\ x+L \to u,\ u+2\pi$

$f\left(x = \frac{L}{2\pi} u\right) = g(u) \nearrow$

$g(u) = g(u+2\pi)$

$$g(u) = \sum_{n=-\infty}^{\infty} c_n e^{inu}$$

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} g(u) e^{-inu} du$$

majd $L \to \infty$

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(u) e^{-inu} du = \frac{1}{2\pi} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-in\frac{2\pi}{L}x} \frac{2\pi}{L} dx =$$

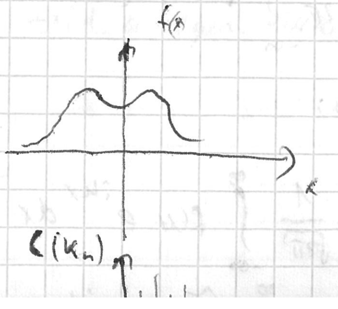
$$= \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-in\frac{2\pi}{L}x} dx$$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\frac{2\pi}{L}x}$$

jelölés: $n\frac{2\pi}{L} = k_n$

$$c_n = \frac{1}{L} \int f(x) e^{-ik_n x} dx$$

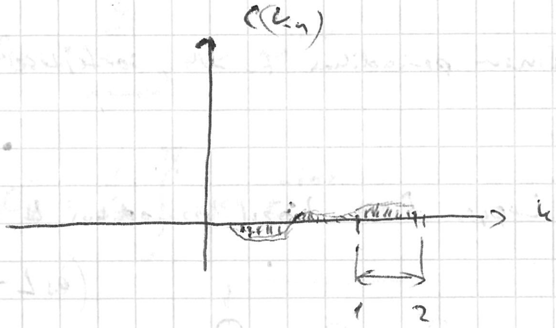
$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}$$



$\Delta k = \frac{2\pi}{L}$

$L \to \infty$ esetén ~~k besűrűsödik~~

k besűrűsödik



L nő az adott intervallumban

lévők besűrűsödnek ~~magasság~~

magasságuk csökken

L ~~→~~ 2L lesz akkor a fele

a magasság

$$\tilde{f}(k) = L \cdot c_n \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x)\, e^{ik_n x}\, dx$$

$L \to \infty$:

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

~~[illegible]~~

~~$f(x) = \sum_n c_n e^{ik_n x} \Delta k$~~

$$f(x) = \underbrace{\frac{1}{2\pi}}_{\frac{1}{\Delta k}} \sum_n c_n e^{ik_n x} \Delta k \to \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, e^{ikx}\, dk$$

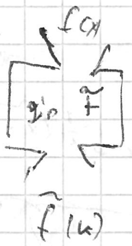
$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

~~$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk$~~

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk$$

Lehet így is:

$$\left.\begin{aligned} \tilde{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx \\ f(x) &= \frac{1}{} \int^{\infty} \tilde{f}(k)\, e^{-ikx}\, dk \end{aligned}\right\}$$ szimmetrikus alakja



$$\widetilde{\lambda F} = \lambda \tilde{F}$$

$$\widetilde{\lambda f} = \lambda \tilde{f}$$

$$\widetilde{f+g} = \tilde{f} + \tilde{g}$$

lineáris transzformáció

függvények terében

$\underline{a}\,\underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

N dimenzióban: $\sum_{i=1}^{N} a_i b_i$

komplex-értékű: $\sum$ ~~$a_1^* b_1$~~ $a_i^* b_i$

fv-eknél: $f \cdot g = \int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx$

$$f \cdot g = \tilde{f} \cdot \tilde{g}$$

$$f(x) \xrightarrow{F} \tilde{f}(k)$$

$$\boxed{f'(x)} \longrightarrow \frac{1}{\sqrt{2\pi}} \int dk\ \tilde{f}(k)\, \underbrace{\frac{d}{dx} e^{ikx}}_{(-ik)\, e^{ikx}} \qquad \boxed{= -ik\ \tilde{f}(k)}$$

Diff egy.

$$f'(x) + 3 f(x) = e^{-x^2}$$

$$\tilde{f'}(x) + 3 \tilde{f}(x) = \widetilde{e^{-x^2}}$$

$$-ik\ \tilde{f}(x)$$

Fourier transzf.

$$\tilde{f}(k) \left[3 - ik \right] = \widetilde{e^{-x^2}}$$

majd a végén vissza transzformáljuk

h(k) ismert fv.

$$\tilde{f}(k) = \frac{h(k)}{3 - ik} \Longrightarrow f(x) = \ldots$$

$\widetilde{\text{Gauss}} = \text{Gauss}$

$$\widetilde{e^{-\frac{x^2}{2}}} = \frac{1}{\sqrt{2\pi}} \int dx\, e^{-\frac{x^2}{2} + ikx} = e^{-\frac{k^2}{2}} \frac{1}{\sqrt{2\pi}} \underbrace{\int dx\, e^{-\frac{1}{2}(\overset{x'}{\overbrace{x-ik}})^2}} = \underline{\underline{e^{-\frac{k^2}{2}}}}$$

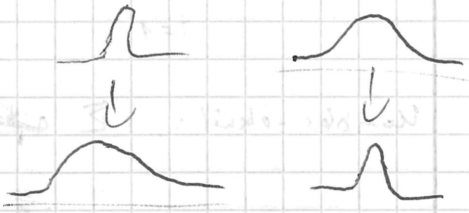
$$-\frac{x^2}{2} + ikx = -\frac{1}{2}(x-ik)^2 - \frac{k^2}{2}$$

$$\int dx'\, e^{-\frac{1}{2}x'^2} = \sqrt{2\pi}$$

Fourier transform.

$$\widetilde{e^{-x^2}} = \text{áll.}\ e^{-\frac{k^2}{4}}$$

$$\widetilde{e^{-\frac{a^2x^2}{2}}} \Longrightarrow e^{-\frac{k^2}{2a^2}}$$



HF:

$$\widetilde{e^{-a^2x^2}} = ?$$

$$\widetilde{e^{-(x-x_0)^2}}$$

XI. 27.

10. óra

$$f(x) \Rightarrow \tilde{f}(k)$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{ikx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{-ikx}\, dk$$

$f(x) \in L^2$ négyzetes integrálható

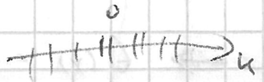
$$\int_{-\infty}^{\infty} |f(x)|^2\, dx < \infty$$

$\tilde{f} \in L$

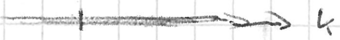
Szemléletesen: periodikus fv. $\rightarrow \sum_k \tilde{f}_k\, e^{ikx}$

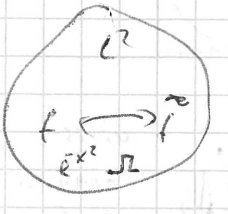
diszkrét

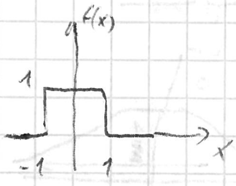
$\cos kx + i \sin kx$



nem periodikus fv. k folytonos







eddig megy

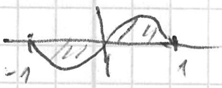
mert 1 mégis

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{\cancel{-\infty}\ -1}^{\cancel{\infty}\ 1} 1 \cdot \underbrace{e^{ikx}}\, dx =$$

~~ett~~ettől megy

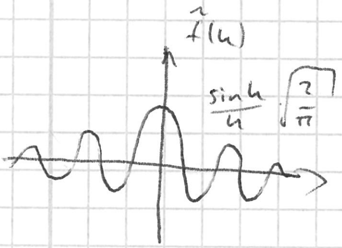
$$\cos kx + \underbrace{i \sin kx}_{} = 0$$

páros páratlan



$$= \frac{1}{\sqrt{2\pi}} \underbrace{\frac{\sin kx}{k} \Big|_{x=-1}^{x=1}}_{\frac{2\sin k}{k}} =$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{\sin k}{k}$$



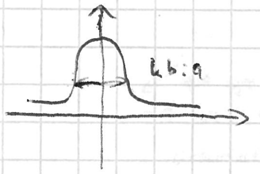
pl:

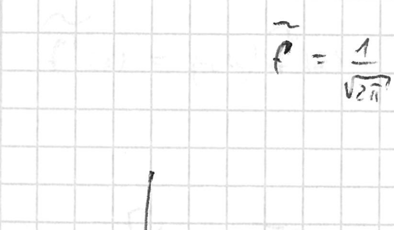
$$f(x) = e^{-\frac{x^2}{a^2}} \longrightarrow \tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int \underbrace{e^{-\frac{x^2}{a^2} + ikx}}\, dx$$

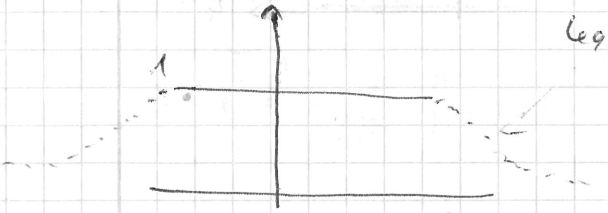
$$e^{-\frac{\left(x - i\frac{ka^2}{2}\right)^2 - \frac{k^2a^4}{4}}{a^2}}$$

$$e^{-\frac{\overbrace{\left(k - i\frac{ka^2}{2}\right)}^{\xi}{}^2}{a^2}} e^{-\frac{k^2a^2}{4}}$$

$$\tilde{f} = \frac{1}{\sqrt{2\pi}} e^{-\frac{k^2a^2}{4}} \underbrace{\int e^{-\frac{\xi^2}{a^2}} d\xi}_{\sqrt{\pi a^2}} = a\sqrt{\frac{\pi}{2}}\, e^{-\frac{k^2a^2}{4}}$$







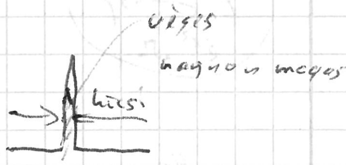
közelítünk ezzel

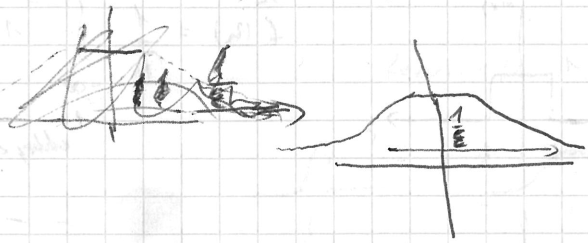
legyen $e^{-\varepsilon x^2}$ ahol $\varepsilon \ll 1$

olyan Gauss görbe $\varepsilon = \frac{1}{a^2}$

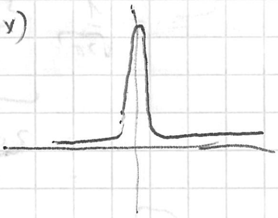
aminek a széle nagyon nagy

$\tilde{1}(x)$ értelmetlen de $\tilde{1}_\varepsilon(x) =$





$\delta_\varepsilon(x)$



$\int_{-\infty}^{\infty} \delta_\varepsilon(x)\,dx = 1$ de $\delta_\varepsilon(x) \approx 0$, kivéve $x \approx 0$

~~Ezen~~ Dirac delta

$\tilde{1} = \frac{1}{\sqrt{2\pi}}\,\delta(k)$

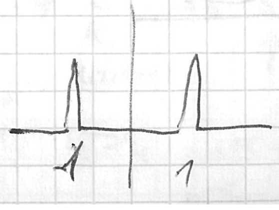
$k=0$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int \hat{f}(k)\, \overset{1}{e^{ikx}}\, dk = \frac{1}{\sqrt{2\pi}} \int \frac{\delta_\varepsilon(k)}{1}\, dk$$

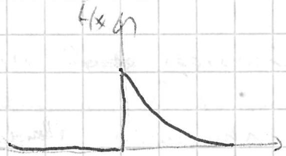
$\widetilde{\cos x} \rightarrow$ ~~áll [$\delta(k-1)$~~ áll. $[\delta(k-1)+\delta(k+1)]$

$\frac{1}{2}(e^{ix}+e^{-ix})$

$\frac{1}{2}e^{ix}+\frac{1}{2}e^{-ix}$



További példák F transzformációk



$$f(x) = \begin{cases} 0, & \text{ha } x < 0 \\ e^{-x}, & \text{ha } x > 0 \end{cases}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \underbrace{e^{-x} \cdot e^{ikx}}_{e^{-x(1-ik)}} dx = \frac{1}{\sqrt{2\pi}} \frac{1 \; i}{(1-ik)\, i} = \frac{i}{\sqrt{2\pi}} \frac{1}{k+i}$$

$$\int_0^\infty e^{-x(1-ik)} dx$$

$$\left(e^{\alpha x}\right)' = \alpha e^{\alpha x}$$

$$\frac{e^{-x(1-ik)}}{-(1-ik)}$$

$$\tilde{f}(k) = \frac{i}{\sqrt{2\pi}} \frac{1}{k+i} \longrightarrow \quad f(x) =?\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{k+i} e^{-ikx} dk = \begin{cases} 0, \text{ ha } x<0 \\ x>0 \ (2\pi i) \frac{1}{\sqrt{2\pi}} \frac{i}{\sqrt{2\pi}} \end{cases}$$

ha k = „i ∞"

$$e^{-i i \infty x} \rightarrow 0, \text{ ha } x<0$$

-1 -es a körüljárás miatt
óramutatóval egyező

x>0

$$(2\pi i) \frac{1}{\sqrt{2\pi}} \frac{i}{\sqrt{2\pi}} \cdot 1 \cdot (-1)$$

$$e^{-i(-i)x} = e^{-x}$$

$$\widetilde{f'}(x) = (-ik)\tilde{f}(k)$$

$$f'(x) = \frac{1}{\sqrt{2\pi}} \int \tilde{f}(k) \overset{\overset{(-ik)\, e^{-ikx}}{\shortparallel}}{\left(e^{-ikx}\right)'} dk$$

Diff egy.: $y'(x) + y(x) = h(x)$ h ismert fv.

$\Downarrow$

közönséges ~~[illegible]~~ elsőrendű

$\delta(x) \Rightarrow \tilde{\delta} = \frac{1}{\sqrt{2\pi}}$ inhomogén állandó együtthatós

$\widetilde{y' + y} = \tilde{h}$

$\tilde{y'} + \tilde{y}$

$(-ik)\tilde{y} + \tilde{y} = \tilde{h}$

$(1-ik)\tilde{y} = \tilde{h}$

$\tilde{y}(k) = \frac{\tilde{h}(k)}{1-ik} \longrightarrow \frac{1}{\sqrt{\pi}} \cdot \frac{1}{1-ik}$

$\Downarrow$

innen inverz F tr.

F(t) erő hat a kerékre göröngy miatt.

$$ma = F(t) - \alpha v(t)$$

$$mv'(t) + \alpha v(t) = F(t)$$

$$v'(t) + \underset{1}{\underset{\shortparallel}{\frac{\alpha}{m}}} v(t) = \underset{h}{\underset{\shortparallel}{\frac{F}{m}}}(t)$$

$$v'(t) + v(t) = h(t)$$

Legyen $F(t) = \delta(t)$

erő lökés

diff. egy speciális m.o-ja Green fv.

Green fv. fogalma: ∀ lin. dif. egy. van Green fv. ami a δ-hoz tartozik
csak lin. egyenletnek van G-fv.-e.

ami a δ-s inhomogén taghoz tartozó megoldás

$$G'(x) + G(x) = \delta(x)$$

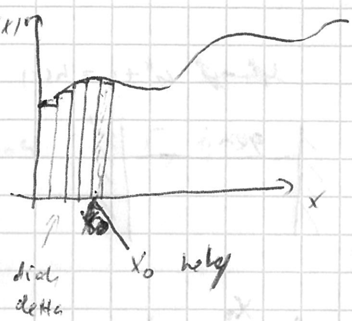
ahol a „döccenő" van

$$G'(x, x_0) + G(x, x_0) = \delta(x - x_0)$$

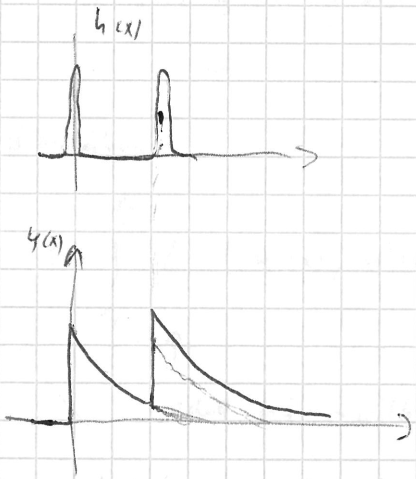
$$\Downarrow$$

$G(x, x_0) = G(x - x_0)$ ha nincs határfeltétel akkor igaz.

$$h(x) = \int h(x_0)\,\delta(x - x_0)\,dx_0$$



$$y(x) = \int G(x, x_0)\,h(x_0)\,dx_0$$



Diff Fotu.vali

$x\,y'(x) + 3y = h(x)$ döccenő, inhomogén lin. egy ~~de nem állandó~~ ~~együtthatós~~ függvény együtthatós

más: $\hat{y'}(x) + \widetilde{x\,y(x)} = \hat{h}(x) =$ 

$$\swarrow \qquad \downarrow \qquad \downarrow$$

$(-ik)\hat{y}(k) - i\,\hat{y}' = \dfrac{\sin k}{k}$ ~~[illegible]~~ (áll)

~~$\tilde{f}$~~ ~~$\tilde{f}$~~

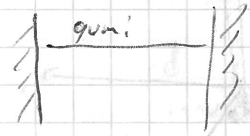
$$\frac{d}{dk}\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int dx\, e^{ikx} f(x)$$

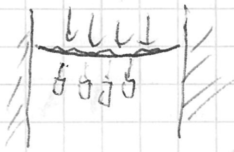
$$\tilde{f}(k)' = \frac{d\tilde{f}}{dk} = \frac{1i}{\sqrt{2\pi}}\int dx\, e^{ikx}\cdot x\cdot f(x) = -i\tilde{f} = \widetilde{xf}(x)$$

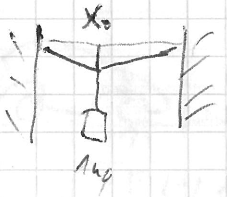
$$y''''(x) + x\,y(x) = h(x)$$

$$(-ik)^4\,\tilde{y}(k) - i\,\tilde{y}(k)' = \tilde{h}(k)$$

pl.: ~~$y''+y$~~ $y''+y=h(x)$

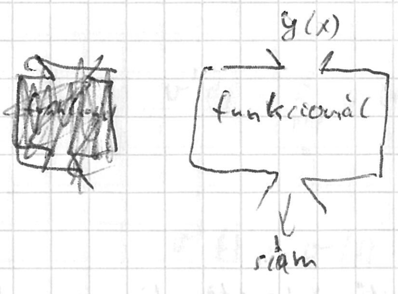






Variációszámítás

Dec 18. Elővizsga írásbeli 9^{00}-től



~~Kap~~ függvényhez számot rendel

= funkcionál

pl: 1, $I = \int_{a=1}^{b=3} y(x)\,dx$ ← ez funkcionál

lineáris funk.

2, $I = y(0)$

3, $I = \int_1^2 [y(x)]^2\,dx$ ← ez nem lin. funkcionál

4, $I = \int_a^b \left[y^2(x) + y'(x)^2 \right] dx$ $L = a^2 + b^2$

5, $I = \int_{-\infty}^{\infty} y(x)\, y(x - x_0)\, dx$ korrelációs integrál

itt milyen a fv.

lokális funkcionál (nem egy pontban) helyi viszonyok

(1, 3, 4,)

2, Dirac delta nem foglalkozunk : 2, 5, 1, 3

Variációszámítás célja: 4. típusú fv.

$I = \int y(x)\, y'(x) \cdot x\, dx$

stb $L(a,b,c) = a \cdot b \cdot c$

általában:

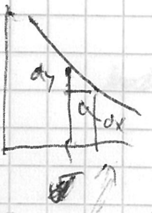
$$I = \int_{x_1}^{x_2} L\left(y(x), y'(x), x\right) dx$$

↑

adott háromváltozós fv.

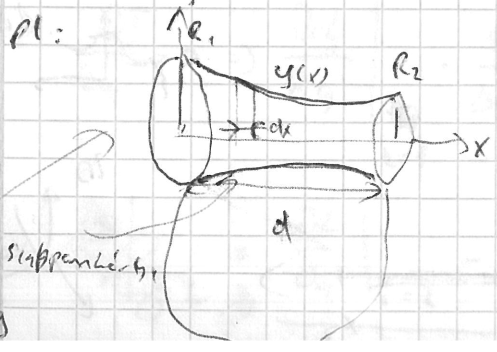
alapfeladat: ilyen funkcionál

melyik $y(x)$-nél lesz I maximális v. minimális (szélsőérték)



$\sqrt{1 + \frac{dy^2}{dx}} \cdot 2\pi y$

Pl:



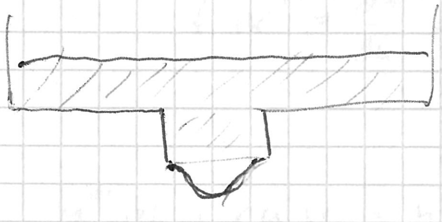
$$E_{felületi} = \sigma \overset{\text{felület nagysága}}{A} = \sigma 2\pi \int_0^d \underbrace{\sqrt{1 + y'^2(x)}\; y(x)}_{L(y,y',x)}\, dx =$$

$= \text{minimum}$

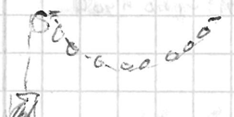
pl 2,

vízcsap: E-t minimalizálni

felületi és helyzeti E-t figyelembe véve

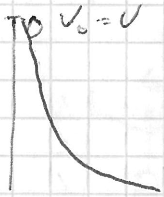


pl 3,



lánc helyzeti E kell min.-nak lenni

+ súly + fonal helyzeti E -"-

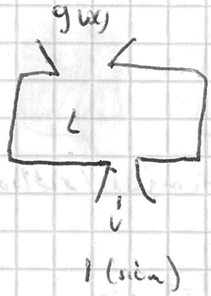


legrövidebb idő alatt érjen le a golyó

m.o. cikloisz

Euler-Lagrange : ~~$\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$~~ $\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$

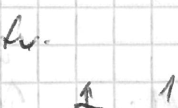
XII. 4 11. óra

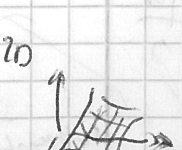


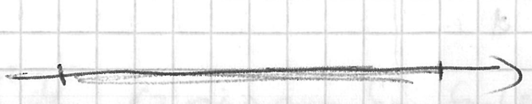
$$I = \int_a^b L\,(y(x), y'(x), x)\,dx \overset{?}{=} \text{extremális}$$

(szélsőérték)

L: alap fv. adott 3 változós fv.





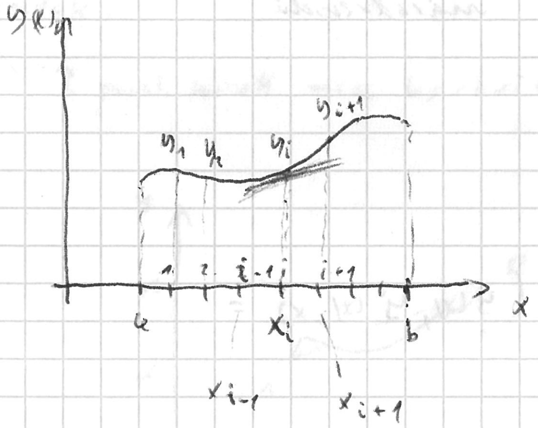


extremális: első közelítésben I nem változik

$$y(x) \rightarrow y(x) + \underset{\substack{\uparrow \\ \text{kicsi}}}{\delta y}(x)$$

ilyenkor nem változik a fv.

Euler-módszer:



$$\int (L)\,dx \rightarrow \sum_i \underbrace{(L)_{x_i}}\,\Delta x$$

$$\sum_i L\left(y(x_i), \frac{y(x_{i+1}) - y(x_i)}{\Delta x}, x_i\right)\Delta x = I(y_1, y_2, \dots y_i, \dots y_n)$$

$$\Downarrow$$

$$\frac{\partial I}{\partial y_i} = 0 \quad i = 1, 2 \dots n$$

$$I = L\left(y_1, \frac{y_2 - y_1}{\Delta x}, x_1\right)\Delta x + L\left(y_4, \frac{\overset{y_3 - y_2}{\cancel{y_3 - y_2}}}{\Delta x}, x_2\right)\Delta x + \dots$$

~~ind.~~ pl. $\frac{\partial I}{\partial y_2} = 0 \Rightarrow \left.\frac{\partial L}{\partial y'}\right|_{i=1} \cdot \frac{1}{\Delta x} + \left.\frac{\partial L}{\partial y}\right|_{i=2} + \left.\frac{\partial L}{\partial y'}\right|_{i=2} \cdot \left(\frac{-1}{\Delta x}\right) = 0$

$$\left.\frac{\partial L}{\partial y}\right|_{i=2} = \frac{\left.\frac{\partial L}{\partial y'}\right|_{i=2} - \left.\frac{\partial L}{\partial y'}\right|_{i=1}}{} = \frac{\partial L}{\partial} \Big|$$

ált.:

$$\cancel{\left.\frac{\partial L}{\partial y}\right|_i = \frac{\partial L}{\partial y'}}$$

$$\left.\frac{\partial L}{\partial y}\right|_i = \frac{\left.\frac{\partial L}{\partial y'}\right|_i - \left.\frac{\partial L}{\partial y'}\right|_{i-1}}{\Delta x} \approx \left.\frac{d}{dx}\frac{\partial L}{\partial y'}\right|_i$$

$\Delta x \to 0$
$n \to \infty$

$$\boxed{\frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}} = G$$

differenciál egyenlet $y(x)$-re

közönséges ált. nem lineáris

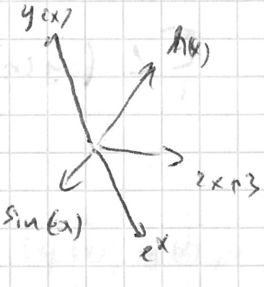
másodrendű

~~valami (y',y',x)~~
valami (y, y', x)
ismert
alak

$$\frac{d}{dx} G(y, y', x) \equiv \frac{d}{dx} G(y(x), y'(x), x) =$$

$$= \frac{\partial G}{\partial y} y' + \frac{\partial G}{\partial y'} y'' + \frac{\partial G}{\partial x} \cdot 1$$

Lagrange módszere:



$$I[y(x)]$$

$$y(x) = y_0(x) + \underbrace{\varepsilon h(x)}_{}^{\delta y(x)}$$

$y_0(x)$ ↑ keresett megoldás

$h(x)$ ↑ tetszőleges pl. $\sin x, e^x \ldots$, de $h(a) = h(b) = 0$

$$I[y(x)] = I[y_0(x) + \varepsilon h(x)] = I(\varepsilon)$$

$$\frac{dI}{d\varepsilon} = 0 \qquad I = \int L(y_0 + \varepsilon h, y_0' + \varepsilon h', x)\,dx$$

$$\frac{dI}{d\varepsilon} = \int \left[\frac{\partial L}{\partial y} \cdot h(x) + \frac{\partial L}{\partial y'} \cdot h'(x)\right] dx =$$

$$= \int \underbrace{\left[\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} \right]}_{\equiv 0} \underset{\substack{\Uparrow \\ \text{tetszőleges}}}{h(x)} \, dx$$

$$\int f(x) g'(x) dx =$$

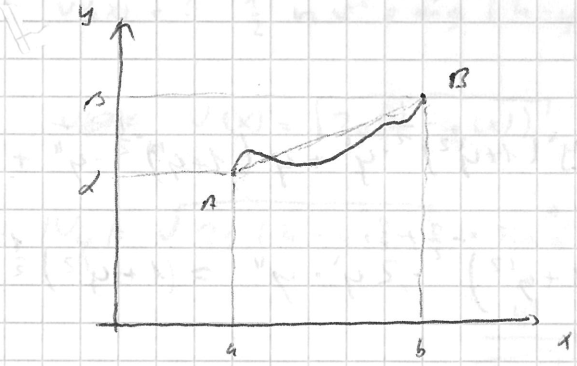
$$= \int f'(x) g(x) dx + [fg]_a^b$$

Euler – Lagrange egy.

$$\boxed{\frac{\partial L}{\partial y} = \frac{d}{dx} \frac{\partial L}{\partial y'}}$$

Példák:

2 pont között mi a legrövidebb út? (síkban)



$y(x) = ?$ de tudjuk

[illegible] $y(a) = \alpha$

$y(b) = \beta$

$$\text{ív hossz} = \underline{I} = \int_a^b \sqrt{1 + y'^2} \, dx$$

Bronstein ívhossz ~~[illegible]~~ rész

$$L(y, y', x) \equiv \sqrt{1 + y'^2}$$

$$\frac{\partial L}{\partial y} = 0 \qquad \frac{\partial L}{\partial y'} = \frac{1 \cdot 2y'}{2\sqrt{1 + y'^2}}$$

$$0 = \frac{d}{dx} \frac{y'(x)}{\sqrt{1 + y'^2_{(x)}}} \Rightarrow$$

$$\underset{\substack{\text{ez nem} \\ \text{nulla}}}{\begin{pmatrix} \text{valami} \\ y' \end{pmatrix}} \cdot y'' = 0$$

$$\downarrow$$

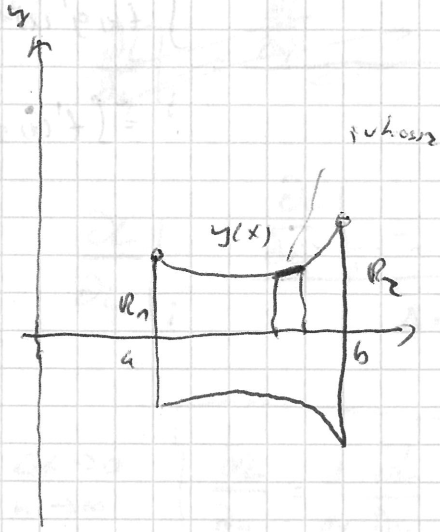
$$y'' = 0$$

$$\frac{y'}{\sqrt{1 + y'^2}} = c$$

$$y' = \text{konstans}$$

$$y = mx + k$$

a és b ponton átmenő egyenes ; mint meredk pl. 2,



forgás felület minimálfelület

Felület = $\overline{I}$ =

$$= 2\pi\int_c^b \underbrace{y(x)\sqrt{1+y'^2}}_{L}\,dx = \min$$

$$L = y\sqrt{1+y'^2}$$

x szerint deriváltja

$$\frac{\partial L}{\partial y} = \sqrt{1+y'^2} \qquad \frac{\partial L}{\partial y'} = y\cdot\frac{y'}{\sqrt{1+y'^2}} = yy'(1+y'^2)^{-\frac{1}{2}}$$

$$\frac{d}{dx}\left(yy'(1+y'^2)^{-\frac{1}{2}}\right) = y'(1+y'^2)^{-\frac{1}{2}}\cdot y' + y(1+y'^2)^{-\frac{1}{2}}\cdot y'' +$$

$$+ y\cdot y'\left(-\frac{1}{\cancel{2}}\right)\cdot(1+y'^2)^{-\frac{3}{2}+\frac{1}{2}}\cdot \cancel{2}y'\cdot y'' = (1+y'^2)^{\frac{1}{2}+\frac{1}{2}}$$

$$\cancel{y'^2\cdot y'' + y'^2 y y''}$$

$$y'^2 + \underbrace{yy'' - \frac{y'^2 y\cdot y''}{1+y'^2}}_{} = 1+y'^2$$

$$yy''\ \frac{1+y'^2-y'^2}{1+y'^2}$$

$$y'^2 + \frac{yy''}{1+y'^2} = 1+y'^2$$

pl $\operatorname{ch} x = y$

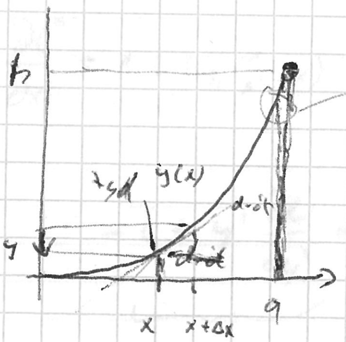
$$y' = \operatorname{sh} x \qquad \operatorname{sh}^2 x + \frac{\operatorname{ch}^2 x}{\operatorname{ch}^2 x} = 1 + \operatorname{sh}^2 x$$

$$1+y'^2 = \operatorname{ch}^2 x \qquad \operatorname{sh}^2 + 1 = 1+\operatorname{sh}^2 x$$

$$y(x) = k \cdot \mathrm{ch}\,\frac{x - x_0}{k}$$

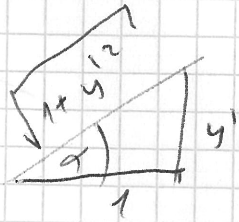
$$\boxed{\begin{matrix} R_1, & R_2 \\ a, & b \end{matrix}} \rightarrow k,\ x_0 \text{ meghatározható}$$

pl. 3/:



gyöngyszem, csúszik

legrövidebb idő



$v(x) = ?$ $\frac{1}{2} m v^2 = mg(h-y)$ energiatétellel

$$\cancel{v(x)=}\ v(x) = \sqrt{2g(h-y(x))}$$

$$\left|\frac{dy}{dt}\right| = |v_y| = \sqrt{2g(h-y(x))} \cdot \sin\alpha$$

$$\sin\alpha = \frac{y'}{\sqrt{1+y'^2}}$$

$$dt = \frac{dy}{\sqrt{2g}\sqrt{h-y(x)}} \cdot \frac{\sqrt{1+y'^2}}{y'} \qquad y' = \frac{dy}{dx}$$

$$\sqrt{2g}\,T = \min = \int_0^a dx\, \underbrace{\frac{\sqrt{1+y'^2}}{\sqrt{h-y}}}_{L(y,y',x)}$$

$$L(y,y',x) \rightarrow \frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'} \rightarrow \text{cikloís}$$

biciklisztop t. 6.

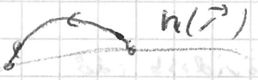
mechanika : $L = \frac{1}{2} m\, y'^2(t) - V(y)$

$$I = \int L\, dt = \min$$ $F = - V'(y)$

Legkisebb hatás elve $\vec{F} = - \mathrm{grad}\, \phi(\vec{r})$

$$\frac{d}{dt}(m y') = \boxed{m y''(t) = F}$$

Fermat elv:



Milyen L függvények:

I. $L(y, y', x)$ ha nem függ az y-tól

$$\frac{\partial L}{\partial y} = 0 \quad \frac{d}{dx}\frac{\partial L}{\partial y'} = 0 \Rightarrow \frac{\partial L}{\partial y'} = \text{állandó}$$

mech:
ciklikus koordináták

$$\frac{\partial L}{\partial y_i} = \frac{d}{dx}\frac{\partial L}{\partial y_i'}$$

i tömegpont 3 koord.

~~mech. általános koord.~~

II. $L(y, y', x)$ y'-től nem függ ?

$$\cancel{\frac{\partial L}{\partial y_i}} \quad \frac{\partial L}{\partial y'} = 0 \rightarrow \frac{\partial L}{\partial y} = 0$$

ilyen eset nincs!!

III. $L(y, y', x)$ x-től nem függ.

$$\xrightarrow{E.L.} \frac{\partial L}{\partial y} = \frac{d}{dx}\frac{\partial L}{\partial y'}$$

nem tűnik egyszerűnek, de trükk:

$$I = \int L(y, y', x)\, dx = \int \underbrace{\frac{dx}{dy} L(\quad)}\, dy$$

$$\rightarrow \quad \underset{\substack{\| \\ 0}}{\frac{\partial L^*}{\partial x}} = \frac{d}{dy}\frac{\partial L^*}{\partial x'} = \text{állandó}$$

mivel nem függ x-től

$$L^* = x' L\left(\cancel{y}, \overset{y'}{\frac{1}{x'}}, x\right)$$

$$\frac{\partial L^*}{\partial x'} = 1 \cdot L\left(y, \frac{1}{x'}, x\right) + x' \frac{\partial L}{\partial y'} \cdot \frac{-1}{x'^2} = \text{áll.}$$

$$\underbrace{-L + y' \frac{\partial L}{\partial y'}}_{\text{Hamilton fv. a mech.-ban}} = \text{áll.}$$

Mech:

$$\int L(y(t), y'(t), \cancel{t})\, dt \qquad L = \frac{1}{2} m y'^2 - V(y)$$

$$y' \frac{\partial L}{\partial y'} - L = \text{áll.} = y'(my') - \frac{1}{2} m y'^2 + V = \frac{1}{2} m y'^2 + V$$

pl:

$$L = y\sqrt{1+y'^2} \quad \text{szappanhártya}$$

$$\frac{\partial L}{\partial x} = 0 \Rightarrow y' \frac{\partial L}{\partial y'} - L = \text{áll.}$$

$$\cancel{\frac{yy'^2}{\sqrt{1+y'^2}}} - y\frac{(1+y'^2)}{\sqrt{1+y'^2}} = \cancel{\frac{y_2}{\sqrt{1+y'^2}}} \quad \frac{y}{\sqrt{1+y'^2}} = k$$

$$y^2 = k^2 + k^2 y'^2$$

$$\frac{y^2}{k^2} = 1 + y'^2$$

$$\sqrt{-1 + \frac{y^2}{k^2}} = \sqrt{y'^2} \Rightarrow \frac{dy}{dx} = y' = \sqrt{\frac{y^2}{k^2} - 1}$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{h^2}-1}} = \int dx$$

$$\int \frac{dy}{\sqrt{\frac{y^2}{h^2}-1}} = x - x_0$$

$$\frac{1}{k} \operatorname{arch}(ky) = x - x_0$$

$$ky = \operatorname{ch}(x - x_0)$$

$$\boxed{y = \frac{1}{k} \operatorname{ch} k(x - x_0)}$$

Ha több ismeretlen fv. van

pl.: $I = \int L(q_1, q_2, \ldots \dot{q}_1, \dot{q}_2 \ldots, t)\, dt$

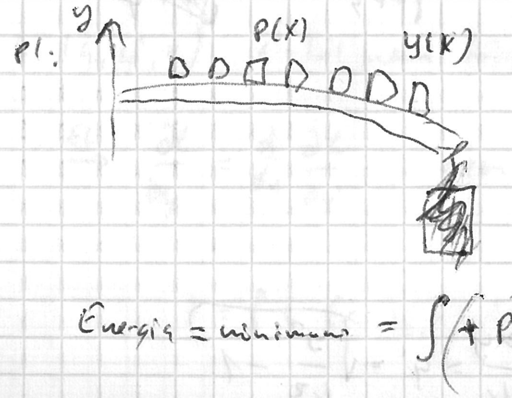
$$\frac{\partial L}{\partial q_i} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}$$

Ha $I = \int L(y, y', y'', x)\, dx$

E-L:

$$\frac{\partial L}{\partial y} - \frac{d}{dx} \frac{\partial L}{\partial y'} + \frac{d^2}{dx^2} \frac{\partial L}{\partial y''} = 0$$

pl.:



$$\text{Energia} = \text{minimum} = \int \left(+ \overbrace{P(x)\, y(x)}^{\text{súlyok helyzeti E.}} + k y''^2 \right) dx$$