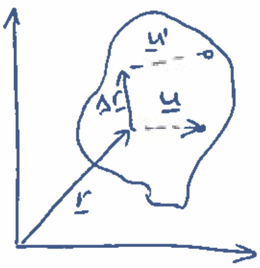
FOLYTONOS KÖZEGEK MECH. EMELT.

2013. 02. 18.

Deformáció jellemzése



$\underline{r}$: deformálatlan állapotban egy pont helye

$\underline{u}(\underline{r},t)$ – elmozdulás fv.

$$\underline{r} \to \underline{r}+\underline{u}(\underline{r})$$

$$\underline{r}+\Delta\underline{r} \to \underline{r}+\Delta\underline{r}+\underline{u}(\underline{r}+\Delta\underline{r})$$

$$\Delta\underline{r}' = \Delta\underline{r} + \underline{u}(\underline{r}+\Delta\underline{r}) - \underline{u}(\underline{r})$$

$$u_i(\underline{r}+\Delta\underline{r}) - u_i(\underline{r}) \approx \frac{\partial u_i}{\partial r_j}\Delta r_j$$

$$\Delta r_i' \approx \Delta r_i + \frac{\partial u_i}{\partial r_j}\Delta r_j = \left(\delta_{ij} + \frac{\partial u_i}{\partial r_j}\right)\Delta r_j$$

$$\beta_{ij} = \frac{\partial u_i}{\partial r_j} \quad : \text{ disztorzió}$$

$$\hat{\beta} = \frac{\partial \underline{u}}{\partial \underline{r}}$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\beta_{ij}+\beta_{ji}\right) = \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right)$$

ε deformációs tenzor

$$\boxed{\begin{aligned} &\underline{F} = -\mathrm{grad}\,\phi \\ &\phi(\underline{r}+\Delta\underline{r}) - \phi(\underline{r}) = -\underline{F}\Delta\underline{r} = \mathrm{grad}\,\phi\,\Delta \\ &\phi(\underline{r}+\Delta\underline{r}) - \phi(\underline{r}) = \frac{\partial\phi}{\partial r_i}\Delta r_i \end{aligned}}$$

Forgás: $\underline{v} = \underline{v}_0 + \underline{\omega}\times\underline{r}$

$$\underline{u}(\underline{r},t+\Delta t) - \underline{u}(\underline{r},t) = \Delta\underline{\varphi}\times\underline{r}$$

$$\Delta\left[\underline{u}(\underline{r}+\Delta\underline{r}) - \underline{u}(\underline{r})\right] = \Delta\underline{\varphi}\times\Delta\underline{r} = \Delta\hat{\Omega}\,\Delta\underline{r}$$

$\Downarrow$ ekkor tartozik $\hat{\beta}$.

$\uparrow$ antiszimmetrikus mátrix

$$\underline{\Delta x}' = \left(1+\frac{\partial u_1}{\partial r_1}, \frac{\partial u_2}{\partial r_1}, \frac{\partial u_3}{\partial r_1}\right)\Delta x$$

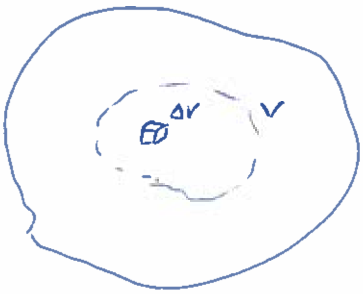
$$\underline{\Delta y}' = \left(\frac{\partial u_1}{\partial r_2}, 1+\frac{\partial u_2}{\partial r_2}, \frac{\partial u_3}{\partial r_2}\right)\Delta y$$

$$\underline{\Delta z}' = \left(\frac{\partial u_1}{\partial r_3}, \frac{\partial u_2}{\partial r_3}, 1+\frac{\partial u_3}{\partial r_3}\right)\Delta z$$

$$\Delta V' = \begin{vmatrix} 1+\frac{\partial u_1}{\partial r_1} & \frac{\partial u_2}{\partial r_1} & \frac{\partial u_3}{\partial r_1} \\ \frac{\partial u_1}{\partial r_2} & 1+\frac{\partial u_2}{\partial r_2} & \frac{\partial u_3}{\partial r_2} \\ \frac{\partial u_1}{\partial r_3} & \frac{\partial u_2}{\partial r_3} & 1+\frac{\partial u_3}{\partial r_3} \end{vmatrix} \Delta x \Delta y \Delta z \approx \left(1+\frac{\partial u_1}{\partial r_1}\right)\left(1+\frac{\partial u_2}{\partial r_2}\right)\left(1+\frac{\partial u_3}{\partial r_3}\right)\Delta x \Delta y \Delta z =$$

$$\approx \left[1+\left(\frac{\partial u_1}{\partial r_1}+\frac{\partial u_2}{\partial r_2}+\frac{\partial u_3}{\partial r_3}\right)\right]\Delta x \Delta y \Delta z = (1+\mathrm{Sp}\,\hat{\varepsilon})\,\Delta V$$

Dinamika



$$\Delta \underline{F}_k = \underline{f}(\underline{r})\,dV$$

$$\Delta \underline{F}_f \sim \Delta \underline{A}$$

$$\Delta \underline{F}_f = \hat{\sigma}(\underline{r})\,\Delta \underline{A}$$

$$\underline{F}_k = \int_V \underline{f}(\underline{r})\,dV$$

$$\underline{F}_f = \oint_F \hat{\sigma}(\underline{r})\,d\underline{A}$$

$$\boxed{\underline{F} = \int_V \underline{f}(\underline{r})\,dV + \oint_{\partial V} \hat{\sigma}(\underline{r})\,d\underline{A}}$$

A fel. integrál 1. komponense: $\oint \sigma_{11}\,dx + \sigma_{12}\,dy + \sigma_{13}\,dz = \int \left(\frac{\partial \sigma_{11}}{\partial r_1} + \frac{\partial \sigma_{12}}{\partial r_2} + \frac{\partial \sigma_{13}}{\partial r_3} \right) dV$

$$\uparrow \quad \frac{\partial \sigma_{1j}}{\partial r_j}$$

$$\oint \hat{\sigma}\, d\underline{A} = \int \operatorname{div} \hat{\sigma}\, dV$$

$$\left(\int \operatorname{div} \hat{\sigma}\, dV \right)_1 = \int \frac{\partial \sigma_{1j}}{\partial r_j}\, dV$$

$\Downarrow$

$$\underline{F} = \int_V \underbrace{\left[f(\underline{r}) + \operatorname{div} \hat{\sigma} \right]}_{=0 \text{ egyensúlyban}} dV = 0$$

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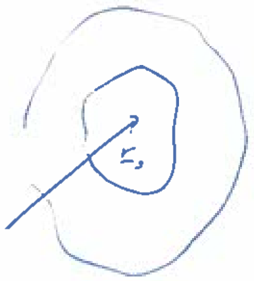


$$\underline{F}_k = \int \underline{f}\, dV + \oint \hat{\sigma}\, d\underline{A} = \int (\underline{f} + \mathrm{div}\, \hat{\sigma})\, dV$$

$$\Delta \underline{F} = \hat{\sigma} \Delta \underline{A}$$

$$\underline{f} + \mathrm{div}\,\hat{\sigma} = 0 \quad \text{egyensúlyban}$$

$$M \ddot{\underline{r}}_0 = \underline{F}_k \qquad M \ddot{\underline{r}}_0 = \sum_{i=1}^{N} m_i \ddot{\underline{r}}_i = \frac{d}{dt}\left[\sum_{i=1}^{N} m_i \underline{v}_i\right]$$

$$M \ddot{\underline{r}}_0 = \frac{d}{dt}\left[\int \dot{\underline{u}}\, \rho(\underline{r})\, dV\right] \approx \int \ddot{\underline{u}}\, \rho\, dV$$

kicsik a deformációk

$$\boxed{\rho \ddot{\underline{u}} = \underline{f} + \mathrm{div}\,\hat{\sigma}}$$

$$\underline{F}(\underline{r}, \underline{v}, t)$$

$$\Downarrow$$

$$\hat{\sigma}(\hat{\varepsilon}, \dot{\hat{\varepsilon}}, t) \leftarrow$$ Anyagra jellemző, meg kell adni...

$\sigma_{ij} = \sigma_{ji} \Rightarrow$ A forgatónyomaték 0.

↳ Teljesül a pontrendszer impulzusmomentumára vonatkozó tétel.

↳ Pl. betonban léphetnek fel forgatónyomatékok: Cosserat-tenzorok...

$$\sigma_{ij} = C_{ijkl}\, \varepsilon_{kl}$$ Általánosított Hooke-törvény

$$\hat{\sigma} = \hat{\hat{C}} : \hat{\varepsilon}$$

81 darab változó, de

$$C_{ijlk} = C_{jilk} \quad , \text{mert} \quad \sigma_{ij} = \sigma_{ji}$$

$$C_{jilk} = C_{ijkl} \quad , \text{mert} \quad \varepsilon_{ij} = \varepsilon_{ji} \text{, mert úgyis kiesik}$$

$\Downarrow$

36 független változó

↳ Rugalmas test energiája

$$\sigma = E\varepsilon$$

$$\frac{F}{A} = E\frac{\Delta l}{l} \qquad F = \frac{EA}{l}\Delta l = D\Delta l$$

$$W = \frac{D}{2}\cdot\Delta l^2 = \frac{1}{2}\frac{EA}{l}\Delta l^2 = \frac{1}{2}EAl\left(\frac{\Delta l}{l}\right)^2$$

$$w = \frac{W}{V} = \frac{1}{2}E\varepsilon^2 = \frac{1}{2}\sigma\varepsilon$$

Általános esetben:

$$\boxed{w = \frac{1}{2}\varepsilon_{ij}C_{ijlk}\varepsilon_{kl}}$$

– Azt feltételezzük, hogy az összes tulajdonság az energiából származtatható legyen.

$\Downarrow$

$$C_{ijlk} = C_{lkij}$$

↳ 21 független változó

$$\boxed{\sigma_{mn} = \frac{dw}{d\varepsilon_{mn}}} = \frac{d}{d\varepsilon_{mn}} \sum_{ijkl} \frac{1}{2} \varepsilon_{ij} C_{ijkl} \varepsilon_{kl} = C_{mnij} \varepsilon_{ij}$$

↳ ez is lehetne a $\hat{\sigma}$ definíciója

21 független változó ⇒ a kristály szimmetriája tudja ezt csökkenteni.

Izotróp anyagok

Pl. polikristályos anyagok, szilárd

∃ $Sp\,\hat{\varepsilon}$ $\quad Sp(\hat{\varepsilon}\cdot\hat{\varepsilon})$ $\quad Sp(\hat{\varepsilon}\cdot\hat{\varepsilon}\cdot\hat{\varepsilon})$ ← ez harmadrendű, ezért $w = \varepsilon_{ij} \cdot C_{ijkl} \varepsilon_{kl}$ típusú tagot nem adhat

↳ ezek minden koordinátarendszerben megegyeznek

$$w = \frac{\lambda}{2}(Sp\,\hat{\varepsilon})^2 + \mu \cdot Sp(\hat{\varepsilon}\cdot\hat{\varepsilon}) = \mu\, \varepsilon_{ij}\varepsilon_{ij} + \frac{\lambda}{2}(\varepsilon_{ll})^2$$

μ, λ: Lamé-állandók.

$$\sigma_{mn} = \frac{dw}{d\varepsilon_{mn}} = 2\mu\,\varepsilon_{mn} + \lambda\,\varepsilon_{ll}\cdot\delta_{mn}$$

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\delta_{ij}\,\varepsilon_{ll} = 2\mu\underbrace{\left(\varepsilon_{ij} - \frac{1}{3}\delta_{ij}\,\varepsilon_{ll}\right)}_{\text{tiszta nyírás esetén csak ez a tag van}} + \underbrace{\left(\frac{2\mu}{3} + \lambda\right)}_{K\text{: kompressziómodulus}}\delta_{ij}\,\varepsilon_{ll}$$

μ: nyírási modulus

Ha csak izotróp összenyomás van akkor csak ez számít.

Mi az $\varepsilon(\sigma)$ összefüggés?

$$\sigma_{ll} = 2\mu\,\varepsilon_{ll} + 3\lambda\,\varepsilon_{ll} \quad \rightarrow \quad \varepsilon_{ll} = \frac{1}{2\mu+3\lambda}\,\sigma_{ll}$$

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \frac{\lambda}{2\mu+3\lambda}\,\delta_{ij}\,\sigma_{ll}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{2\mu+3\lambda}\,\delta_{ij}\,\sigma_{ll}\right]$$

Egytengelyű nyújtás:

$$\hat{\sigma} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Downarrow$

$$\hat{\varepsilon} = \begin{pmatrix} \frac{1}{2\mu}\left(1 - \frac{\lambda}{2\mu+3\lambda}\right)\sigma & 0 & 0 \\ 0 & -\frac{1}{2\mu}\frac{\lambda}{2\mu+3\lambda}\sigma & 0 \\ 0 & 0 & -\frac{1}{2\mu}\frac{\lambda}{2\mu+3\lambda}\sigma \end{pmatrix}$$

$$\frac{1}{E} = \frac{1}{2\mu}\left(1 - \frac{\lambda}{2\mu+3\lambda}\right) = \frac{1}{2\mu}\left(\frac{2\mu+3\lambda-\lambda}{2\mu+3\lambda}\right) = \frac{1}{\mu}\,\frac{\mu+\lambda}{2\mu+3\lambda}$$ $\leftarrow$ nem lehet negatív

$$\nu = -\frac{\varepsilon_{22}}{\varepsilon_{11}}$$

$\llcorner$ teljesíti a $\nu < 0{,}5$ (!) feltételt

Ismétlés

Deformáció

$\underline{u}(x,y,z)$

$$\Downarrow$$

$$\hat{\beta}(x,y,z) = \nabla \underline{u} = \frac{\partial \underline{u}}{\partial \underline{r}} \quad ; \quad \beta_{ij} = \frac{\partial u_i}{\partial r_j} = \partial_j u_i$$

$$\Downarrow$$

$$\hat{\varepsilon} = \operatorname{sym} \hat{\beta} = \frac{1}{2}(\hat{\beta} + \hat{\beta}^T) \qquad \varepsilon_{ij} = \frac{1}{2}(\beta_{ij} + \beta_{ji}) = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$$

Feszültség

$$\Delta \underline{F} = \hat{\sigma} \Delta \underline{A}$$

$$\hat{\sigma} = \hat{\hat{C}} \hat{\varepsilon} \quad ; \quad \sigma_{ij} = C_{ijkl} \varepsilon_{lk}$$

Izotróp anyag: $\sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{ll} = 2\mu \delta_{il} \delta_{jk} \varepsilon_{lk} + \lambda \delta_{ij} \delta_{kl} \varepsilon_{lk}$

$$\hookrightarrow C_{ijkl} = 2\mu \delta_{il} \delta_{jk} + \lambda \delta_{ij} \delta_{kl}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{2\mu + 3\lambda} \delta_{ij} \sigma_{ll}\right]$$

Mozgásegyenlet

$$\rho \ddot{\underline{u}} = \underline{f} + \operatorname{div} \hat{\sigma} \quad ; \quad \rho \ddot{u}_i = f_i + \frac{\partial}{\partial r_j} \sigma_{ij} = f_i + \partial_j \sigma_{ij}$$

~~Egyensúlyi~~

$$\cancel{\partial_j \sigma_{ij} + f_i = 0}$$

Izotróp anyag: $\rho \ddot{u}_i - f_i = +\partial_j \sigma_{ij} = +\partial_j \left(2\mu \varepsilon_{ij} + \lambda \delta_{ij} \varepsilon_{ll}\right) = +\partial_j \left(\cancel{2}\mu \cancel{\tfrac{1}{2}}(\partial_j u_i + \partial_i u_j) + \lambda \delta_{ij} \partial_l u_l\right) =$

$$= \mu \partial_j^2 u_i + \mu \partial_j \partial_i u_j + \lambda \delta_{ij} \partial_j \partial_l u_l = \mu \partial_j^2 u_i + (\lambda + \mu) \partial_i \partial_j u_j$$

$$\boxed{\rho \ddot{u}_i - f_i = \mu \partial_j \partial_j u_i + (\lambda + \mu) \partial_i \partial_j u_j \quad , \quad \rho \ddot{\underline{u}} - \underline{f} = \mu \operatorname{div} \operatorname{grad} \underline{u} + (\lambda + \mu) \operatorname{grad} \operatorname{div} \underline{u}}$$

Izotróp anyag mozgásegyenlete. Δ

Egyensúlyban:

$$\boxed{-f_i = \mu\, \partial_j \partial_j u_i + (\lambda+\mu)\, \partial_i \partial_j u_j \quad ; \quad -\underline{f} = \mu \Delta \underline{u} + (\lambda+\mu)\, \mathrm{grad}\, \mathrm{div}\, \underline{u}}$$

$$\Delta \phi = -\frac{1}{\varepsilon_0} \varrho$$

<u>Saját súlya alatt megnyúló rúd esete</u>

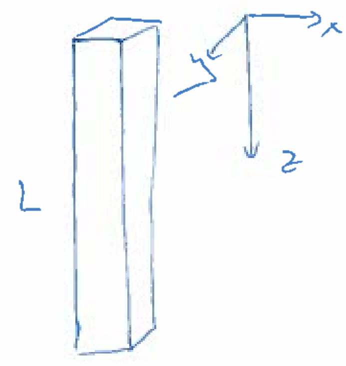
$$\underline{u}(x,y,z) \approx \begin{pmatrix} u_x(x,y,z) \\ u_y(x,y,z) \\ \boxed{u_z(z)} \end{pmatrix} \Rightarrow \quad \hat{\varepsilon}(x,y,z) = \begin{pmatrix} \partial_x u_x & \frac{1}{2}(\partial_x u_y + \partial_y u_x) & \frac{1}{2}\partial_z u_x \\ \frac{1}{2}(\partial_x u_y + \partial_y u_x) & \partial_y u_y & \frac{1}{2}\partial_z u_y \\ \frac{1}{2}\partial_z u_x & \frac{1}{2}\partial_z u_y & \partial_z u_z \end{pmatrix}$$

A feszültség mint az egytengelyű nyújtásnál:

$$\hat{\sigma} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sigma(z) \end{pmatrix}$$

Innen: $\varepsilon_{ij} = \frac{1}{2\mu}\left[\sigma_{ij} - \frac{\lambda}{2\mu+3\lambda}\delta_{ij}\sigma_{ll}\right] =$

$$\hat{\varepsilon} = \begin{pmatrix} -\frac{1}{2\mu}\frac{\lambda}{2\mu+3\lambda}\sigma(z) & 0 & 0 \\ 0 & -\frac{1}{2\mu}\frac{\lambda}{2\mu+3\lambda}\sigma(z) & 0 \\ 0 & 0 & \underbrace{\frac{1}{2\mu}\left(1-\frac{\lambda}{2\mu+3\lambda}\right)}_{\frac{1}{E}}\sigma(z) \end{pmatrix}$$



Így

$$\underline{\partial_z u_z = u_z'(z) = \frac{1}{E}\sigma(z)}$$

$$-\underline{f} = \mathrm{div}\,\hat{\sigma} = \cancel{\partial_z \sigma(z) = \sigma(z)'} \begin{pmatrix} 0 \\ 0 \\ \partial_z \sigma_z(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma'(z) \end{pmatrix}$$

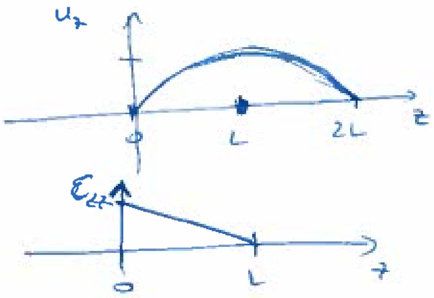
$$\begin{pmatrix} 0 \\ 0 \\ -\varrho g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sigma'(z) \end{pmatrix} \Rightarrow \quad -\varrho g = \sigma'(z) = E \cdot u_z''(z)$$

$u_z(z) = A + Bz - \frac{\rho g}{2E} z^2$

Határfeltételek:

$$\begin{cases} u_z(0) = 0 \Rightarrow A = 0 \\ \sigma(L) = 0 \Rightarrow \cancel{B - \frac{\rho g}{2E} L} \end{cases} \qquad B - \frac{\rho g}{E} L = 0 \Rightarrow B = \frac{\rho g}{E} L$$

$$\boxed{u_z(z) = \frac{\rho g}{E}\left(Lz - \frac{z^2}{2}\right) = -\frac{\rho g}{2E} z (z - 2L)}$$



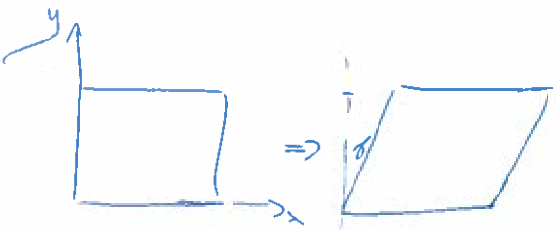
$\varepsilon_{zz}(z) = u_z'(z) = \frac{\rho g}{E}(L - z)$

$\sigma(z) = E\,\varepsilon_{zz}(z) = \rho g (L - z) \Rightarrow \sigma(0) = \rho g L$

$L' = L + u(L) = L + \frac{\rho g}{2E} \cdot L^2$ $\qquad F = \sigma(0) \cdot A = \rho A L \cdot g = mg$

Csavarás

— Nyírási deformáció:

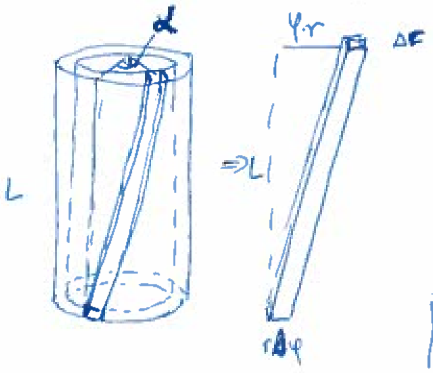


$u_x(y) = y \cdot \tan\gamma \approx y \cdot \gamma$

$$\underline{u}(x,y,z) = \begin{pmatrix} u_x(y) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} y \cdot \gamma \\ 0 \\ 0 \end{pmatrix} \Rightarrow \hat{\varepsilon} = \begin{pmatrix} 0 & \frac{1}{2}\gamma & 0 \\ \frac{1}{2}\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\sigma_{ij} = 2\mu\, \varepsilon_{ij} + \lambda\, \delta_{ij}\, \varepsilon_{ll}$

$$\hat{\sigma} = \begin{pmatrix} 0 & \overset{\tau}{\mu\gamma} & 0 \\ \mu\gamma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \tau = \mu\gamma, \quad \gamma = \frac{\tau}{\mu}$$

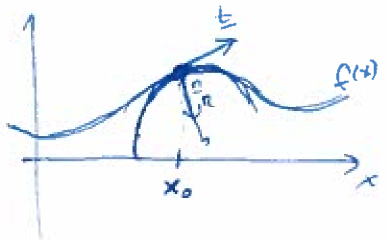


$\Rightarrow \gamma = \frac{\alpha r}{L} \rightarrow \tau = \mu\gamma = \mu \frac{r}{L} \alpha \Rightarrow \Delta F = \tau \Delta A = \mu \frac{r}{L} \alpha\, r \Delta\varphi \Delta r$

$\Delta M = \Delta F \cdot r = \frac{\mu}{L} r^3 \alpha \Delta\varphi \Delta r$

$$\boxed{M = \int_0^R \int_0^{2\pi} \frac{\mu}{L} \alpha \cdot r^3 \, d\varphi \, dr = 2\pi \frac{\mu}{L} \alpha \frac{R^4}{4} = \frac{\pi \mu}{2} \cdot R^4 \cdot \frac{\alpha}{L}}$$

Görbületi sugár



$$\begin{cases} x(t) = t \\ y(t) = f(t) \end{cases} \qquad v_x(t) = 1 \qquad a_x(t) = 0$$

$$v_y(t) = f'(t) \qquad a_y(t) = f''(t)$$

$$v(t) = \sqrt{1 + f'(t)^2}$$

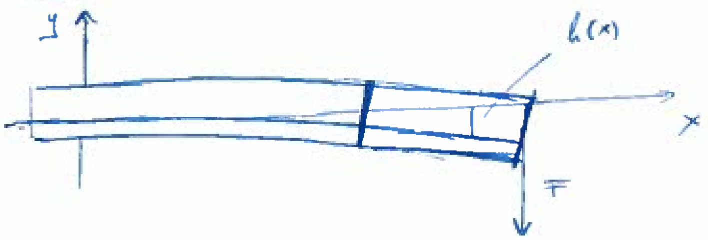
$$\underline{t} = \frac{\underline{v}}{v} = \frac{1}{\sqrt{1+f'(t)^2}} \begin{pmatrix} 1 \\ f'(t) \end{pmatrix} \qquad \underline{a} = \dot{v}\,\underline{t} + \frac{v^2}{R}\,\underline{n} \quad / \cdot \underline{n}$$

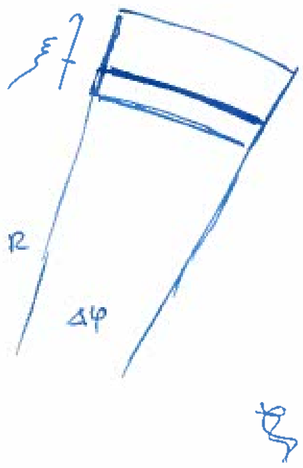
$$|\underline{a}\cdot\underline{n}| = \frac{v^2}{R}$$

$$\underline{n} = \frac{1}{\sqrt{1+f'(t)^2}} \begin{pmatrix} -f'(t) \\ 1 \end{pmatrix}$$

$$\underline{a}\cdot\underline{n} = \frac{1}{\sqrt{1+f'(t)^2}}\, f''(t) \implies \frac{v^2}{R} = \frac{1}{\sqrt{1+f'(t)^2}} \cdot |f''(t)| \implies \boxed{\frac{1}{R} = \frac{|f''(t)|}{(1+f'(t)^2)^{3/2}}}$$

Lehajlás





$$\varepsilon_{xx}(\xi) = \frac{(R+\xi)\Delta\varphi - R\Delta\varphi}{R\Delta\varphi} = \frac{\xi}{R} = -\xi\{h''(x)\}$$

$$\sigma_{xx}(\xi) = E \cdot \varepsilon_{xx}(\xi) = -E \cdot \xi\{h''(x)\}$$

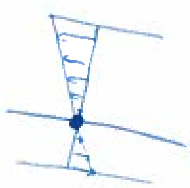
A lehajló része: $F_e = 0$, $M = 0$

$\Downarrow$

neutrális vonal meghatározása

Innentől köv. óra

$M=0$:



$$M = \int_A \sigma_{xx}(\xi)\cdot\xi\, d\zeta\, d\xi - F(L-x) = -E\{h''(x)\} \underbrace{\int_A \xi^2\, d\zeta\, d\xi}_{I} - F(L-x) =$$

I ← másodrendű felületi nyomaték

$$= -EI\{h''(x)\} - F(L-x) = 0$$

$$\{h''(x)\} = -\frac{F}{EI}(L-x) \implies h''(x) = \frac{F}{EI}(x-L)$$

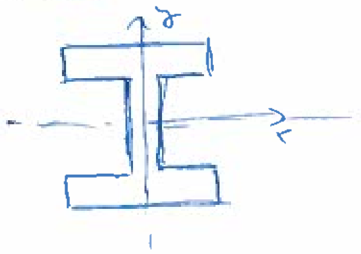
$$h''(x) = \frac{F}{EI}(x-L) \Rightarrow h'(x) = \frac{F}{EI}\left(\frac{x^2}{2} - Lx\right) + B \Rightarrow h(x) = \frac{F}{EI}\left(\frac{x^3}{6} - L\frac{x^2}{2}\right) + Bx + A$$

Határfeltételek: $\begin{cases} h(0) = 0 \Rightarrow A = 0 \\ h'(0) = 0 \Rightarrow B = 0 \end{cases}$

$$\boxed{h(x) = \frac{F}{EI}\left(\frac{x^3}{6} - L\frac{x^2}{2}\right)}$$

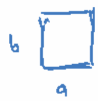
A lehajlás: $\boxed{s = |h(L)| = \left|\frac{F}{EI}\left(\frac{L^3}{6} - \frac{L^3}{2}\right)\right| = \frac{F}{EI}\cdot\frac{L^3}{3}}$

A másodrendű felületi nyomaték:



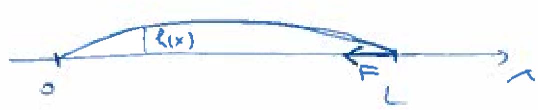
$$I = \int_A y^2\,dx\,dy$$

téglalapra: $\displaystyle\int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{b}{2}}^{\frac{b}{2}} dy\, y^2 = a\left[\frac{y^3}{3}\right]_{-\frac{b}{2}}^{\frac{b}{2}} = \frac{ab^3}{12}$



Behajlás: $s = \frac{1}{EI}\cdot\frac{F}{2}\cdot\frac{1}{3}\left(\frac{L}{2}\right)^3 = \frac{F}{EI}\cdot\frac{L^3}{48}$

Kihajlás



$$M = F\,h(x) + EI\,h''(x) = 0$$
$$\Downarrow$$
$$h''(x) + \frac{F}{EI}h(x) = 0$$

Megoldás: $h(x) = 0$

$$h(x) = A\sin(kx + x_0), \quad k = \sqrt{\frac{F}{EI}}$$

Határfeltételek: $h(0) = 0 \Rightarrow x_0 = 0$

$h(L) = 0 \Rightarrow k\cdot L = n\pi$

$n=0$: $h(x) = 0$

$n=1$: $k = \frac{\pi}{L} = \sqrt{\frac{F}{EI}} \Rightarrow \boxed{F = EI\cdot\frac{\pi^2}{L^2} = F_1}$ Euler-erő

$n=2$: $k = 2\frac{\pi}{L} = \sqrt{\frac{F}{EI}} \Rightarrow F = 4EI\frac{\pi^2}{L^2} = 4F_1$

~~2013. 03. 11.~~

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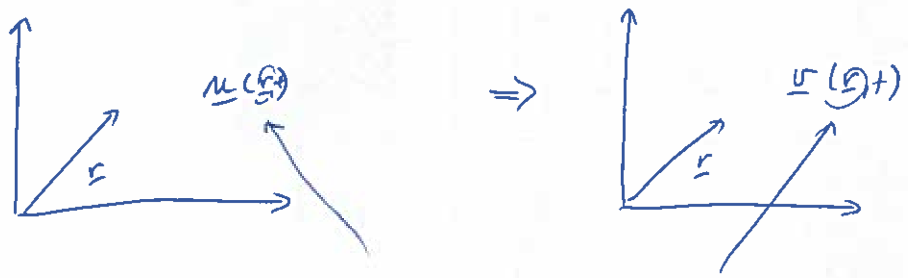
Folyadékok

$$\underline{\Phi} = (\underline{r}, \dot{\underline{r}}, t) \Rightarrow \hat{\sigma}(\hat{\varepsilon}, \dot{\hat{\varepsilon}}, t)$$

csak $\hat{\varepsilon}$: szilárdtest

csak $\dot{\hat{\varepsilon}}$: folyadék

mindkettő: viszkoplasztikus, pl. kátrány



A két $\underline{r}$ nem ugyanaz

referenciaállapot $\longleftrightarrow$ megfigyelési pont

Áttranszformálhatóak: ~~$\underline{v}(\underline{r},t)$~~

$$\frac{\partial \underline{u}(\underline{r},t)}{\partial t} = \underline{v}\left(\underline{r} + \underline{u}(\underline{r},t), t\right)$$

folyadék: $\rho(\underline{r},t) \approx$ áll.

gáz $\rho(\underline{r},t)$ nem állandó.

$\varrho(\underline{r},t)$, $\underline{v}(\underline{r},t)$, $\hat{\sigma}(\underline{r},t)$

ϱ és $\underline{v}$ nem független

$\varrho \underline{v}\,\underline{n} \cdot \Delta t \cdot \Delta A$

$$\Delta m = -\oint \varrho\, \underline{v}\, \underline{n}\, dA\, \Delta t = \oint \varrho \underline{v}\, d\underline{A} \cdot \Delta t =$$

$$= \left(\frac{d}{dt} \int \varrho\, dV\right) \Delta t$$

$$\frac{d}{dt} \int \varrho\, dV = -\oint \varrho\, \underline{v}\, d\underline{A}$$

$$\int \frac{\partial \varrho}{\partial t}\, dV = -\int \operatorname{div}(\varrho \underline{v})\, dV$$

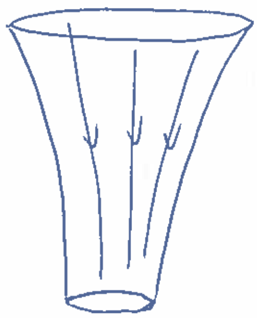
$$\boxed{\frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \underline{v}) = 0}$$ Kontinuitási egyenlet

---

Ha $\varrho =$ áll. $\Rightarrow$ $\operatorname{div} \underline{v} = 0$

Stacionárius áramlás

$\underline{v}(\underline{r})$, $\varrho(\underline{r})$



$$\oint \varrho \underline{v}\, d\underline{A} = 0$$

$$\varrho_1 v_1 A_1 = \varrho_2 v_2 A_2$$

$\hat{\sigma}(\dots) = ?$

$\Delta \underline{F} \sim \Delta \underline{A} \Rightarrow \Delta \underline{F} = -p \Delta \underline{A} \Rightarrow \quad \hat{\sigma} = \begin{pmatrix} -p(r) & 0 & 0 \\ 0 & -p(r) & 0 \\ 0 & 0 & -p(r) \end{pmatrix}$

Pascal - törvény

$0 = \operatorname{div} \hat{\sigma} + \underline{f}$

$(\operatorname{div} \hat{\sigma})_x = -\frac{\partial p}{\partial x} \Rightarrow \operatorname{div} \hat{\sigma} = -\operatorname{grad} p$

$-\operatorname{grad} p + \underline{f} = 0$

gravitációs erőtérben: $\underline{f} = \rho \cdot \underline{g} = -\rho \cdot \operatorname{grad} \phi$ ← grav. potenciál

$$\boxed{\operatorname{grad} p + \rho \cdot \operatorname{grad} \phi = 0}$$

! p függhet a ρ-tól: $pV = \frac{m}{M} RT \Rightarrow \frac{p}{\rho} = \frac{R}{M} T$

$\rho = $ áll. összenyomhatatlan folyadék.

⇓

$\operatorname{grad}(p + \rho\phi) = 0$

$p + \rho\phi = c = $ áll.

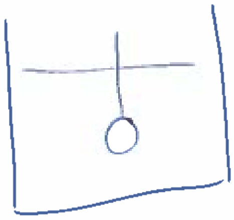
(azonos nyomású és ekvipotenciális felületek megegyeznek.)

Föld felszínén:

$p - \rho g h = p_0$

$p = p_0 + \rho g h$

Felhajtóerő



$$\underline{F}_f = \oint \hat{G}\, d\underline{A}$$

Egyensúlyban: $\mathrm{div}\,\hat{G} + \underline{f} = 0$

Tfh víz van a test helyén: $\underline{F}_f = \oint \hat{G}\, d\underline{A} = \int \mathrm{div}\,\hat{G}\, dV = -\int \underline{f}\, dV =$

$$= -\rho_f \underline{g} V$$

barometrikus magasságformula

$$-\mathrm{grad}\, p + \rho \underline{g} = 0$$

$$-\frac{dp}{dz} \cancel{+} - \rho g = 0$$

$$\frac{p}{\rho} = \frac{R}{M} T = \text{áll} \quad \Rightarrow \quad \rho = \frac{p}{\frac{R}{M} \cdot T}$$

$$\frac{dp}{dz} = -\frac{g}{\frac{R}{M} \cdot T} \cdot p$$

$$\boxed{p(z) = p_0 \cdot e^{-\frac{g \cdot z}{\frac{R}{M} T}}} \quad \Rightarrow \quad \rho(z) = \rho_0 \cdot e^{-\frac{g \cdot z}{\frac{R}{M} T}} = \rho_0 e^{-\frac{mgz}{k_B T}} = \rho_0 e^{-\frac{E_p}{kT}}$$

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Barometrikus magasságformula - kísérlet

Forgó folyadék felszíne

← összenyomhatatlan folyadék

$$-\text{grad}\, p + f = 0 \qquad f = -\rho\, \text{grad}\, \phi$$

forgó koordinátarendszerben:

$$f = \rho g + \rho\omega^2 \cdot \underline{s} = -\rho\, \text{grad}\left(gz - \frac{\omega^2}{2}(x^2+y^2)\right)$$

$$\Downarrow$$

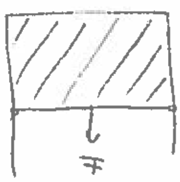
$$\text{grad}\left(-\rho g z + \rho\frac{\omega^2}{2}(x^2+y^2) - p\right) = 0$$

így $-\rho g z + \rho \frac{\omega^2}{2}(x^2+y^2) - p = -p_0$ ← bárhol

$$\boxed{\rho g z = \rho\frac{\omega^2}{2}(x^2+y^2)}$$ ← a felületen $(p = p_0)$

→ forgási paraboloid

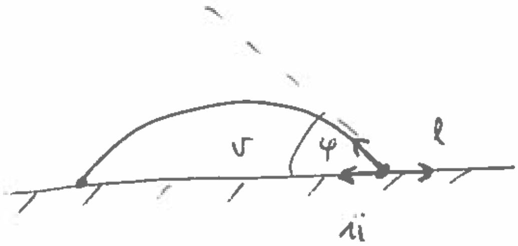
Felületi feszültség (kísérletek)



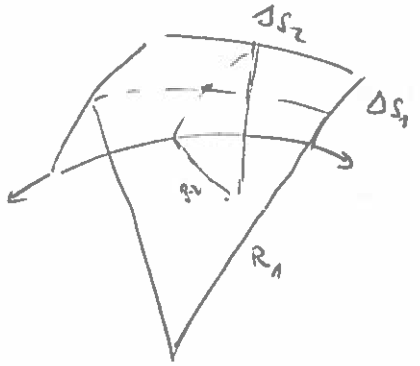
$$F = 2\alpha \cdot l$$ ← 2 oldala van a folyadéknak

$$\Delta W = F \cdot \Delta x = \alpha \cdot 2\, l \cdot \Delta x = \alpha \cdot \Delta A$$

$$\boxed{\Delta W = \alpha \cdot \Delta A}$$



$$\cos\varphi = \frac{\alpha_{ül} - \alpha_{üv}}{\alpha_{vl}}$$



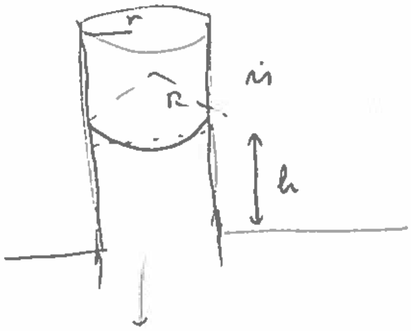
$$\Delta F = \alpha \Delta s_1$$

$$\Delta F_f = 2 \cdot \alpha \cdot \Delta s_1 \cdot \cos\varphi = \cancel{2}\alpha \Delta s_1 \frac{\Delta s_2}{\cancel{2} R_1}$$

$$\Delta F_f^t = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\Delta s_1 \Delta s_2$$

$$p = \alpha\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$$ ([illegible])

Kapillaritás



$r = R \cdot \cos\varphi$

$$E(h) = \rho \cdot r^2\pi \cdot h \cdot g \frac{h}{2} - \alpha_{lü} 2\pi r h + \alpha_{üv} 2\pi r h$$

$$\frac{dE}{dh} = \rho r^2\pi \cdot g h - (\alpha_{lü} - \alpha_{üv}) 2\pi r = 0$$

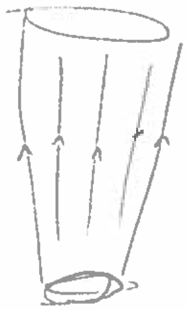
$$\boxed{h = \frac{2(\alpha_{lü} - \alpha_{üv})}{\rho \cdot r \cdot g}} = \frac{2}{\rho r g}\alpha_{vl} \cdot \cos\varphi =$$

$$= \frac{2\alpha_{vl}}{\rho g R}$$

Áramlások

Pohl-féle kísérlet

$\underline{v}(\underline{r},t)$, $\rho(\underline{r},t)$, $p(\underline{r},t)$



$\rho v A = c$

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0$$

$$\hat{\sigma} = \begin{pmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{pmatrix} \leftarrow \text{ideális folyadék}$$

Energiamegmaradás:

$$\frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 = p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t + \Delta m U_1 - \Delta m U_2 + \Delta(\Delta W_b)$$

összenyomhatatlan folyadékra $=0$, mert ~~csak~~ kényszererők vannak.

$$\frac{1}{2}v_2^2 - \frac{1}{2}v_1^2 = \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} + U_1 - U_2$$

$$(*)\quad \boxed{\frac{1}{2}v^2 + \frac{p}{\rho} + U = \text{áll.}} \quad - \text{Bernoulli tv.}$$

Kissé áramlás állandó keresztmetszetű csőben

Gázok esetén $\rho \neq áll.$:

$$\delta Q = + dE_b + \delta W_b = \oint dE_b - \delta W_k$$

Adiabatikus folyamatra $\delta Q = 0$:

$$\frac{1}{2}\Delta m v_2^2 - \frac{1}{2}\Delta m v_1^2 = p_1 A_1 v_1 \Delta t - p_2 A_2 v_2 \Delta t + \Delta m U_1 - \Delta m_2 U_2 + \Delta(-\Delta E_b)$$

$$E_b = m C_v T$$

$$pV = \frac{m}{M} R T \Rightarrow \frac{p}{\rho} = \frac{R}{M} T \qquad E_b = m C_v \frac{M}{R} \frac{p}{\rho}$$

$$C_p - C_v = \frac{R}{M} \quad \text{(R-M egyenlet)} \longrightarrow$$

$$C_v \frac{M}{R} = \frac{C_v}{C_p - C_v} = \frac{1}{\frac{C_p}{C_v} - 1} = \frac{1}{\kappa - 1}$$

$$E_b = m \frac{1}{\kappa - 1} \frac{p}{\rho}$$

$$\Downarrow$$

$$\frac{1}{2} v_2^2 - \frac{1}{2} v_1^2 = \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} + U_1 - U_2 + \frac{1}{\kappa-1}\frac{p_1}{\rho_1} - \frac{1}{\kappa-1}\frac{p_2}{\rho_2}$$

$$\boxed{\frac{1}{2}v^2 + \frac{\kappa}{\kappa - 1}\frac{p}{\rho} + U = áll}$$

Bernoulli tv. ideális gázra adiabatikus közlítés esetén

Kísérletek: lebegő labda, rezgő papír, Pitot-Prandtl cső, szárnyprofil

Áramlás változó keresztmetszetű csőben

Magnus-effektus

$\rho v A = \text{áll.}$

$\frac{1}{2}v^2 + \frac{\kappa}{\kappa+1}\frac{p}{\rho} + U = \text{áll.}$

$$d(\rho v A) = 0$$

$$vA\,d\rho + \rho A\,dv + \rho v\,dA = 0$$

$$\boxed{\frac{d\rho}{\rho} + \frac{dv}{v} + \frac{dA}{A} = 0}$$

$U = \text{áll}$

$\Downarrow$

$$v\,dv + \frac{\kappa}{\kappa-1}\frac{dp}{\rho} - \frac{\kappa}{\kappa-1}\frac{p}{\rho^2}\,d\rho \cancel{\varepsilon} = 0$$

$$dp = \frac{dp}{d\rho}\,d\rho =: c^2\,d\rho$$

$\downarrow$

$$\frac{\frac{N}{m^2}}{\frac{kg}{m^3}} = \frac{m^2}{s^2}$$

$$v \cdot dv + \frac{\kappa}{\kappa-1}\left(c^2\frac{d\rho}{\rho} - \frac{p}{\rho^2}\,d\rho\right) = 0$$

$pV^{\kappa} = \text{áll}$

$\frac{p}{\rho^{\kappa}} = \text{áll} \quad \Rightarrow \quad p = \alpha \cdot \rho^{\kappa}$

$$\frac{dp}{d\rho} = \alpha \cdot \kappa\,\rho^{\kappa-1} = \alpha \cdot \kappa \cdot \frac{\rho^{\kappa}}{\rho} = \kappa \cdot \frac{p}{\rho}$$

$$v\,dv + \frac{\kappa}{\kappa-1}\frac{d\rho}{\rho}\left(c^2 - \frac{c^2}{\kappa}\right) = 0$$

$$v\,dv + c^2 \cdot \frac{d\rho}{\rho} = 0$$

$$\Rightarrow \quad \frac{d\rho}{\rho} = -\frac{v}{c^2}\,dv \quad \longrightarrow \quad -\frac{v}{c^2}\,dv + \frac{1}{v}\,dv + \frac{dA}{A} = 0$$

$$\boxed{\frac{dv}{v}\left(1 - \frac{v^2}{c^2}\right) + \frac{dA}{A} = 0}$$

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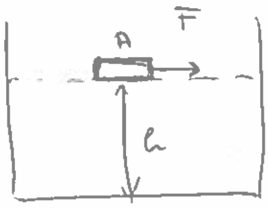
$$c^2 = \frac{dp}{d\rho}$$

Ideális közegben: $\hat{\sigma} = \begin{pmatrix} -p & & \\ & -p & \\ & & -p \end{pmatrix}$

Valódi közegben: $\hat{\sigma} = \begin{pmatrix} -p & & \\ & -p & \\ & & -p \end{pmatrix} + \hat{\sigma}'$



$F = \eta \frac{A \cdot v}{h} \Rightarrow \boxed{\tau = \eta \cdot \frac{dv_x}{dy}}$ $\dot{\varepsilon}_{12}$

Newton-féle viszkozitás

$$F(r, v, t)$$
$$\downarrow$$
$$\hat{\sigma}(\hat{\varepsilon}, \dot{\hat{\varepsilon}}, t)$$

Folyadék: $\hat{\sigma}'(\dot{\hat{\varepsilon}})$

$$\dot{\varepsilon}_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) \qquad \sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\delta_{ij}\varepsilon_{ll}$$

Hasonlóan (izotrópia): $\sigma'_{ij} = 2\eta\dot{\varepsilon}_{ij} + \eta'\delta_{ij}\dot{\varepsilon}_{ll}$

$$\frac{\partial}{\partial r_i}\sigma'_{ij} = \eta \frac{\partial}{\partial r_i}\left(\frac{\partial v_i}{\partial r_j} + \frac{\partial v_j}{\partial r_i}\right) + \eta' \frac{\partial}{\partial r_j}\frac{\partial v_l}{\partial r_l} =$$

$$= \eta \frac{\partial^2}{\partial r_i^2} v_j + (\eta + \eta')\frac{\partial}{\partial r_j}\frac{\partial}{\partial r_i} v_i$$

$\mathrm{div}\,\hat{\sigma}' = \eta \Delta \underline{v} + \cancel{(\eta+\eta')\,\mathrm{grad}(\mathrm{div}\,\underline{v})}$ ← Összenyomhatatlan folyadékra a 2. null

Eddig: $\varrho \ddot{\underline{u}} = \operatorname{div} \hat{G} + \underline{f}$

Hogy néz ez ki a sebességtérben? $\underline{v}(\underline{r}', t)$

$\underline{v}(\underline{r}(t), t)$

részecske koordinátája

részecske gyorsulása:

$$\frac{d}{dt} v_i(\underline{r}(t), t) = \frac{\partial v_i}{\partial t} + \frac{\partial v_i}{\partial r_j} \cdot v_j$$

$$\frac{d}{dt} \underline{v}(\underline{r}(t), t) = \frac{\partial \underline{v}}{\partial t} + (\underline{v}\,\underline{\nabla})\,\underline{v}$$

$\Downarrow$

$$\begin{cases} \varrho \dfrac{\partial \underline{v}}{\partial t} + \varrho(\underline{v}\,\underline{\nabla})\underline{v} = -\operatorname{grad} p + \eta \Delta \underline{v} + (\eta + \eta')\operatorname{grad}(\operatorname{div} \underline{v}) \\ \dfrac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho \underline{v}) = 0 \qquad \cancel{p(\varrho)} \\ p(\varrho) \quad \leftarrow \text{ a gáz viselkedése} \end{cases}$$

Navier – Stokes

(+ hőmérsékletet leíró egyenlet)

Két $=$ tag aránya: v_0 – karakterisztikus sebesség
l – karakterisztikus méret

$$\varrho \cdot \frac{v_0^2}{l} \Big/ \eta \frac{v_0}{l^2} = \frac{\varrho}{\eta} \cdot v_0 \cdot l =: R \quad \text{dimenziótlan szám.}$$

Reynolds – szám

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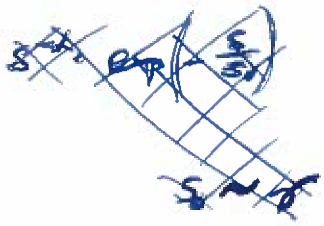
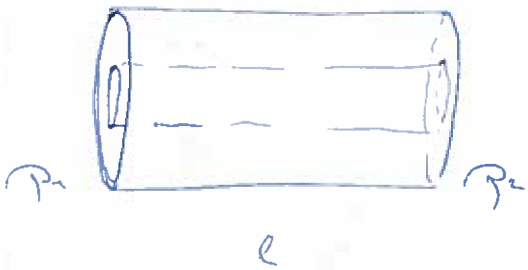


Ha a Reynolds-szám kicsi, akkor a nemlinearitás nem számít.

↳ a N-S egyenlet skála invarianciája.

Áramlás csőben



$$\eta \Delta \underline{v} + (\eta + \eta')\,\mathrm{grad}\,(\mathrm{div}\,\underline{v})$$

$$\oint \hat{\sigma}\, dA = 0 \qquad \tau = \eta \frac{dv}{dr}$$

$$2\cancel{\pi}\cancel{r}\, l\, \eta \frac{dv}{dr} - (p_2 - p_1)\, r^{\cancel{2}}\cancel{\pi} = 0$$

$$2 l \eta \frac{dv}{dr} = +(p_2 - p_1)\cdot r$$

$$\frac{dv}{dr} = -\frac{p_1 - p_2}{2 l \eta}\cdot r \quad \Rightarrow \quad v(r) = -\frac{p_1 - p_2}{4 l \eta}\cdot r^2 + C$$

↳ A határfeltétel nem adott (rézcső, vascső)

↳ $v(R) = 0$

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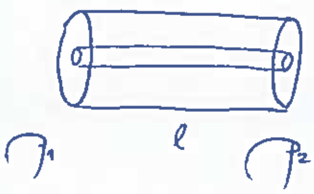


$$\oint \hat{\sigma}\, d\underline{A} = 0$$

$$v(r) = -\frac{p_1 - p_2}{4\eta l} r^2 + C \qquad v(R) = 0$$

$$\Downarrow$$

$$v(r) = \frac{p_1 - p_2}{4\eta l}\left(R^2 - r^2\right)$$

Mennyi megy át időegységenként?

$$\dot{\phi} - \rho v$$

$$\dot{\phi} = \int_{O_R} \rho v(r)\, dA = \int_0^R \rho \frac{p_1 - p_2}{4\eta l} \cdot (R^2 - r^2)\, 2r\pi \cdot dr = \rho \frac{p_1 - p_2}{2\eta l} \pi \int_0^R (R^2 - r^2)\, r\, dr =$$

$$= \rho \frac{p_1 - p_2}{2\eta l} \pi \left[\frac{R^4}{2} - \frac{R^4}{4}\right] = \boxed{\rho \frac{p_1 - p_2}{8\eta l} \pi R^4} \qquad H-P \text{ tv.}$$

Turbulens áramlás

↳ videó

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0 \qquad \rho\left[\frac{\partial v}{\partial t} + (\underline{v}\,\underline{\nabla})\underline{v}\right] = -\operatorname{grad} p + \cancel{\operatorname{div}\hat{\sigma}'} + \cancel{f}$$

$p(\rho)$

↳ Van egy megoldás: $\rho = \rho_0 = \text{áll} \implies p(\rho_0) = \text{áll}$

$\underline{v} = 0$

ρ és v pici:

$$\rho = \rho_0 + \delta\rho$$

$\underline{v}$ (pici) (és a deriváltjai is)

$$\boxed{\begin{aligned} &\frac{\partial\,\delta\rho}{\partial t} + \rho_0 \cdot \operatorname{div}\underline{v} = 0 \\ &\rho_0 \frac{\partial v}{\partial t} = -c^2 \operatorname{grad}\delta\rho \end{aligned}}$$

$$\rho_0\left[\frac{\partial v}{\partial t} + 0\right] = -\left.\frac{dp}{d\rho}\right|_{\rho_0} \operatorname{grad}\delta\rho \qquad c^2 := \left.\frac{dp}{d\rho}\right|_{\rho_0}$$

$\Downarrow$

$$\left.\begin{aligned} &\frac{\partial^2\delta\rho}{\partial t^2} + \rho_0 \cdot \operatorname{div}\frac{\partial \underline{v}}{\partial t} = 0 \\ &\rho_0 \operatorname{div}\frac{\partial \underline{v}}{\partial t} = -c^2\Delta\,\delta\rho \end{aligned}\right\} \quad \boxed{\frac{\partial^2\delta\rho}{\partial t^2} - c^2\Delta\,\delta\rho = 0}$$

$\llcorner$ hullámegyenlet

$$\boxed{\frac{\partial^2\delta p}{\partial t^2} - c^2\Delta\,\delta p = 0} \qquad \left.\Big/ \cdot \frac{dp}{d\rho}\right|_{\rho}$$

$\Downarrow$

$$\frac{\partial}{\partial t}\operatorname{grad}\delta\rho + \rho_0\Delta\underline{v} = 0$$

$$\underbrace{\rho_0 \cdot \frac{\partial^2\underline{v}}{\partial t^2} = -c^2 \cdot \frac{\partial}{\partial t}\operatorname{grad}\delta\rho}$$

$$\rho_0 \cdot \frac{\partial^2\underline{v}}{\partial t^2} = +c^2\rho_0 \overset{\text{grad div}}{\Delta}\underline{v} \Rightarrow \boxed{\frac{\partial^2\underline{v}}{\partial t^2} - c^2 \overset{\text{grad div}}{\Delta}\underline{v} = 0}$$

$\Downarrow$

$$\Rightarrow \rho_0 \frac{\partial}{\partial t}\operatorname{rot}\underline{v} = -c^2 \operatorname{rot}\operatorname{grad}\delta\rho = 0$$

$\Downarrow$

$\operatorname{rot}\underline{v} = 0$ (ha kezdetben is)

$$\left.\begin{aligned} &\frac{\partial^2}{\partial t^2}(\operatorname{div}\underline{v}) - c^2\Delta(\operatorname{div}\underline{v}) = 0 \\ &\operatorname{rot}\underline{v} = 0 \end{aligned}\right\}$$

Mi lehet a megoldás?

$$\delta\varrho = \delta\varrho_0 \cdot e^{i(\omega t + \underline{k}\underline{r})} \leftarrow \text{síkhullám}$$

$$\frac{\partial^2 \delta\varrho}{\partial t^2} = -\omega^2 \delta\varrho \quad ; \quad \Delta\delta\varrho = -(k_x^2+k_y^2+k_z^2)\,\delta\varrho = -|\underline{k}|^2 \delta\varrho \Rightarrow -\omega^2 + c^2|\underline{k}|^2 = 0$$

$$\Downarrow$$

$$\boxed{\omega = c|\underline{k}|}$$

$\underline{k}$: hullámszám vektor

$\lambda = \frac{2\pi}{|\underline{k}|}$: hullámhossz

$$\underline{v} = \underline{v}_0 \cdot e^{i(\omega t + \underline{k}\underline{r})}$$

$$\text{rot}\,\underline{v} = (\underline{k} \times \underline{v}_0)\, e^{i(\omega t + \underline{k}\underline{r})} = 0 \Rightarrow \underline{v}_0 \parallel \underline{k} \quad (\text{különben rot}\,\underline{v} \neq 0)$$

$\llcorner$ longitudinális hullám.

$\llcorner$ Kísérlet: gázcső rezegtetése hanggal.

$$\left.\frac{dp}{d\varrho}\right|_{\varrho_0} = ? \qquad pV^\kappa = \text{áll} \Rightarrow \frac{p}{\varrho^\kappa} = \text{áll} \Rightarrow p = \text{áll}\,\varrho^\kappa \Rightarrow \frac{dp}{d\varrho} = \kappa \cdot \text{áll}\,\varrho^{\kappa-1} =$$

$\uparrow$ Adiabatikus

$$= \frac{\kappa}{\varrho}p = \kappa\frac{p}{\varrho} = \boxed{\kappa\frac{R}{M}T}$$

$$\Downarrow$$

$$c^2 = \kappa\frac{R}{M}T$$

Izoterm:

$$p = \varrho\frac{R}{M}T \Rightarrow \boxed{\frac{dp}{d\varrho} = \frac{R}{M}T}$$

Nagy frekv.: $c^2 = \kappa\frac{R}{M}T$

Kis frekv.: $c^2 = \frac{R}{M}T$

A hangsebesség frekvenciafüggő
'diszperzió' (itt pici)

$c' = \frac{dp}{d\rho}$ – függ az állapotváltozástól

↳ Adiabatikus: $\frac{p}{\rho^{\kappa}} = \text{áll}$ $\quad \omega \to \infty$

↳ Izoterm: $\frac{p}{\rho} = \text{áll}$ $\quad \omega \to 0$

extrém esetek

⇓

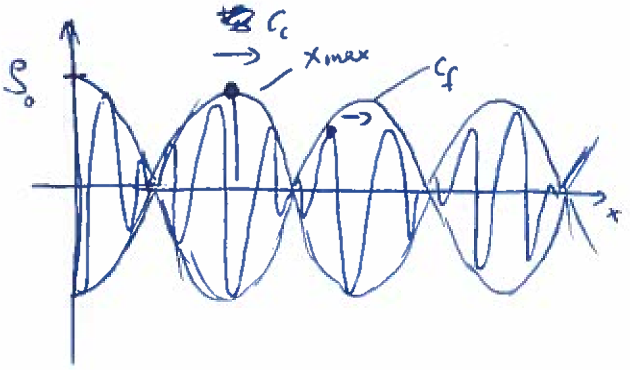
$c(\omega)$ – diszperzió

Általánosabban: $\omega(k)$ $\qquad \frac{\omega}{k} = c$

$$\rho = \rho_0 \sin(\omega_1 t + k_1 x) + \rho_0 \sin(\omega_2 t + k_2 x) =$$

// $\sin(\alpha + \beta) = 2 \cdot \cos\frac{\alpha - \beta}{2} \sin\frac{\alpha + \beta}{2}$

$$= 2\rho_0 \cdot \cos\left(\frac{\omega_1 - \omega_2}{2} t + \frac{k_1 - k_2}{2} x\right) \cdot \sin\left(\frac{\omega_1 + \omega_2}{2} t + \frac{k_1 + k_2}{2} x\right)$$



$x_{max} \approx c_c \cdot t$

$$\Rightarrow \boxed{c_c = \frac{\omega_1 - \omega_2}{k_1 - k_2} \approx \frac{d\omega}{dk}}$$ csoportsebesség

$$\boxed{c_f = \frac{\omega_1 + \omega_2}{k_1 + k_2} \approx \frac{\omega}{k}}$$ fázissebesség

---

Hangtan = Optika (hang longitudinális, fény transzverzális)

Szilárdtestekben:

$$\rho \ddot{\underline{u}} = \text{div}\, \hat{\sigma} + \underline{f} \qquad \hat{\sigma}(\hat{\varepsilon}) = 2\mu \hat{\varepsilon} + \lambda \hat{I} \cdot Sp(\hat{\varepsilon})$$

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial_j u_i + \partial_i u_j\right)$$

$$\Downarrow$$

$$\sigma_{ij} = \mu\left(\partial_j u_i + \partial_i u_j\right) + \lambda\, \delta_{ij}\, \partial_l u_l$$

$$(\operatorname{div} \hat{\sigma})_j = \partial_i \sigma_{ij} = \mu\, \partial_i \partial_j u_i + \mu\, \partial_i^2 u_j + \lambda\, \partial_j \partial_l u_l =$$

$$= (\mu+\lambda)\, \partial_j \partial_i u_i + \mu\, \partial_i^2 u_j$$

$$\Downarrow$$

$$\rho \ddot{\underline{u}} = (\lambda+\mu)\, \operatorname{grad}(\operatorname{div} \underline{u}) + \mu\, \Delta \underline{u} + \underline{f}$$ ← homogén anyag esetén

↳ Hullámok esetén $\underline{f} \approx 0$, mert a nyugalmi állapothoz képest nézzük

div:

$$\rho\, \partial_t^2 (\operatorname{div} \underline{u}) = (\lambda+\mu)\, \Delta(\operatorname{div} \underline{u}) + \mu\, \Delta(\operatorname{div} \underline{u}) = (\lambda+2\mu)\, \Delta(\operatorname{div} \underline{u})$$

$$\underline{\partial_t^2 (\operatorname{div} \underline{u}) = c_L^2\, \Delta(\operatorname{div} \underline{u})} \qquad c_L^2 = \frac{\lambda+2\mu}{\rho}$$

↳ relatív térfogatváltozás

rot:

$$\rho\, \partial_t^2 (\operatorname{rot} \underline{u}) = \mu \cdot \Delta(\operatorname{rot} \underline{u})$$

$$\underline{\partial_t^2 (\operatorname{rot} \underline{u}) = c_T^2\, \Delta(\operatorname{rot} \underline{u})} \qquad c_T^2 = \frac{\mu}{\rho}$$

$\underline{u} = \underline{u}_L + \underline{u}_T$ felbontás: $\operatorname{rot} \underline{u}_L = 0$; $\operatorname{div} \underline{u}_T = 0$

$$\mathrm{rot}\left(\underline{u}_0 \cdot e^{i(\omega t + \underline{k}\underline{r})}\right) = i\left(\underline{k} \times \underline{u}_0\right) e^{i(\omega t + \underline{k}\underline{r})} = 0$$

$\Rightarrow$ $\underline{k} \parallel \underline{u}_0$ $\rightarrow$ longitudinális

$$\mathrm{div}\left(\underline{u}_0 \cdot e^{i(\omega t + \underline{k}\underline{r})}\right) = i\underline{k}\underline{u}_0 e^{i(\omega t + \underline{k}\underline{r})} = 0$$

$\Rightarrow$ $\underline{u}_0 \perp \underline{k}$ $\rightarrow$ transzverzális

↳ Hang terjedési sebességéből λ és μ meghatározható.

Hogyan csatolódik a szilárdtestek rezgése a hanghullámokhoz?

$$\rho \underline{\ddot{u}} = \mathrm{div}\, \hat{\sigma}$$

Minden közegben érvényes

A határfelületre merőleges feszültségkomponensek folytonosan mennek át.

↳ A hullámok összecsatolódnak

(Fresnel-formulákhoz hasonló képleteket lehet kapni)

↓

A hang nyomáshullám

1D-ben:

$$\rho \cdot \frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x} \sigma(x,t) \qquad \sigma(x,t) = E \frac{\partial u}{\partial x}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} \qquad c^2 = \frac{E}{\rho}$$

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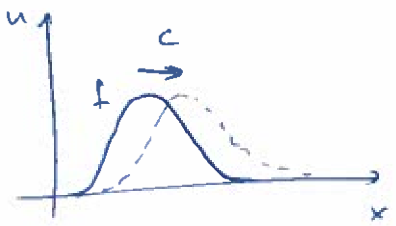
$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$ Hullámegyenlet

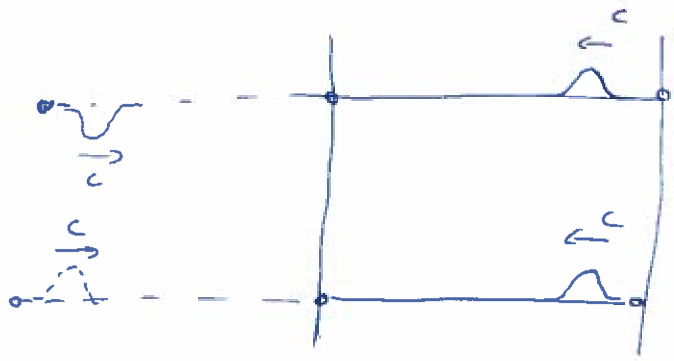
$u = f(x \pm ct)$ — megoldás



Mi van a szélén?

$u = 0$

$\sigma = 0$
$\frac{\partial u}{\partial x} = 0$



„rögzített vég"

„szabad vég"

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$$\rho \frac{\partial^2 u}{\partial t^2} = E \frac{\partial^2 u}{\partial x^2} + \underbrace{f}_{\rho g}$$

(legalsó pont: $M \frac{d^2 u(\ell,t)}{dt^2} = \underline{Mg} - E \frac{\partial u}{\partial x}(\ell,t) \cdot A$

legfelső $\quad u(0,t) = 0$

$$u(x,t) = b \cdot \sin(\omega t + \varphi) \cdot \sin(kx) \qquad \omega^2 = \frac{E}{\rho} k^2$$

$$-\rho \omega^2 u = -E k^2 u$$

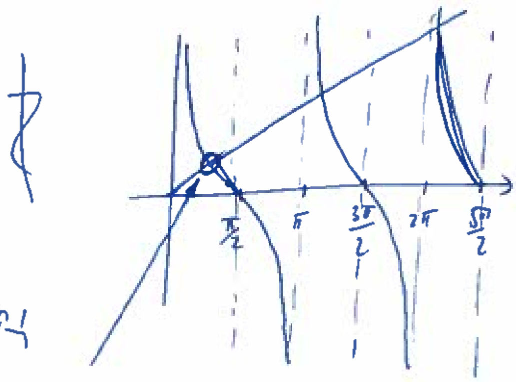
$$M\omega^2 \sin(k\ell) = AEk \cos(k\ell) \quad \text{(határon)}$$

⇓

$$M \frac{\cancel{E}}{\rho} k^{\cancel{2}} \sin(k\ell) = A \cancel{E} \cancel{k} \cos(k\ell)$$

$$M \frac{1}{\rho \ell} (k\ell) = A \frac{\cos(k\ell)}{\sin(k\ell)} \Rightarrow$$

$$\frac{M}{m}(k\ell) = \underline{\mathrm{ctg}\,(k\ell)} \xrightarrow{\text{grafikusan}}$$



↳ csak a legalacsonyabb frekvenciájú megoldás marad meg, a többi kihal.

<u>$m \ll M$ eset:</u>

$$\mathrm{ctg}(x)=\frac{\cos(x)}{\sin(x)}=\frac{1-\frac{x^2}{2}+\dots}{x-\frac{x^3}{6}+\dots}\approx\frac{1}{x}\,\frac{1-\frac{x^2}{2}}{1-\frac{x^2}{6}}\approx\frac{1}{x}\left(1-\frac{x^2}{2}\right)\left(1+\frac{x^2}{6}\right)\approx$$

$$\approx\frac{1}{x}\left(1-\frac{x^2}{3}\right)=\frac{1}{x}-\frac{x}{3}$$

$$\Downarrow$$

$$\frac{M}{m}(kl)=\frac{1}{kl}-\frac{kl}{3}$$

$$\left(\frac{M}{m}+\frac{1}{3}\right)(kl)=\frac{1}{kl}\quad\Rightarrow\quad kl^2=\frac{3m}{3M+m}=\frac{m}{M+\frac{m}{3}}$$

$$\Rightarrow\ \omega^2=\frac{E}{\rho}\,\frac{1}{l^2}\,\frac{m}{M+\frac{m}{3}}=\frac{E}{l}\cdot\frac{m}{\rho l}\,\frac{1}{M+\frac{m}{3}}$$

$$\boxed{\omega^2=\frac{E\cdot A}{l}\,\frac{1}{M+\frac{m}{3}}=\frac{D}{M+\frac{m}{3}}}$$

Leeső rugó

$$\rho\,\frac{\partial^2u}{\partial t^2}=E\,\frac{\partial^2u}{\partial x^2}+\rho g\qquad u_\infty(x)=\frac{\rho g}{E}\left(lx-\frac{x^2}{2}\right)$$

$$\frac{\partial u}{\partial x}(0)=0\qquad\frac{\partial u}{\partial x}(l)=0$$

$$u(x,t)=u_\infty(x)+u'(x,t)\qquad u'(x,0)=0$$

$$\Downarrow$$

$$\boxed{\rho\,\frac{\partial^2u'}{\partial t^2}=E\,\frac{\partial^2u'}{\partial x^2}}$$

$$\frac{\partial u}{\partial x}(l)=\frac{\partial u_\infty}{\partial x}(l)+\frac{\partial u'}{\partial x}(l)\overset{!}{=}0$$

$$\frac{\partial u}{\partial x}(0)=\underbrace{\frac{\partial u_\infty}{\partial x}(0)}_{\rho g l}+\frac{\partial u'}{\partial x}(0)\overset{!}{=}0$$

$$\frac{\partial u'}{\partial x}(l)=0\qquad \boxed{+\ u'(x,0)=0}$$

$$\frac{\partial u'}{\partial x}(0)=-\frac{\rho g l}{\dots}$$

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$0 < x < \ell$

$f(x - ct)$ $\qquad c^2 = \frac{E}{\rho}$

$\operatorname{tg} \alpha = + \frac{\rho g \ell}{E}$

$P(1) \; N P(\tau) \Delta\tau \cdot \left(1 - P(\tau)\Delta\tau\right)^{N-1}$

$\cancel{P(2) \quad P(\tau)\, 2\Delta\tau}$

$P(2) = \frac{N \cdot (N-1)}{2} P(\tau) (2\Delta\tau)^2 \cdot \left(1 - P(\tau) 2 \Delta\tau\right)^{N-2} \cdot \frac{3}{4}$

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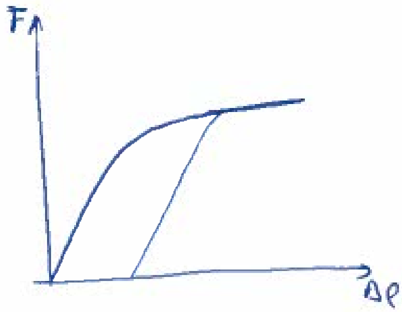
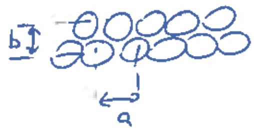


Elméleti folyásfeszültség



$$\tau = \frac{a}{b} \cdot \frac{\mu}{\pi} \left| \sin\left(\frac{\pi x}{a}\right) \right|$$

$$K = \frac{a}{b} \frac{\mu}{\pi}$$

$$\tau_{max} \approx \sim \frac{\mu}{\pi} \sim 100 \text{ MPa}$$

Diszlokációk

Örvények a rugalmas térben