Jackson

Landau II, VIII

$$\underline{F} = q\left(\underline{E} + \underline{v} \times \underline{B}\right)$$

$$\operatorname{div} \underline{E} = \frac{1}{\varepsilon_0} \varrho \qquad \operatorname{div} \underline{B} = 0$$

$$\operatorname{rot} \underline{E} = -\frac{\partial \underline{B}}{\partial t} \qquad \operatorname{rot} \underline{B} = \mu_0\left(\underline{j} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

$$\downarrow \operatorname{div}$$

$$0 = \operatorname{div} \underline{j} + \varepsilon_0 \frac{\partial}{\partial t}\left(\operatorname{div} \underline{E}\right) = \operatorname{div} \underline{j} + \frac{\partial \varrho}{\partial t}$$

kontinuitási egyenlet

$$\int_V d^3x \, \frac{\partial \varrho}{\partial t} + \int_V \operatorname{div} \underline{j} \, d^3x = 0$$

$$\frac{d}{dt} \underbrace{\int_V d^3x \cdot \varrho}_{Q} + \int_F \underline{j} \, d\underline{F} = 0$$

Homogén egyenletek kielégítése

$\operatorname{div} \underline{B} = 0 \rightarrow \underline{B} = \operatorname{rot} \underline{A}$ vektorpotenciál

$$\operatorname{rot} \underline{E} = -\frac{\partial \underline{B}}{\partial t} = -\operatorname{rot} \frac{\partial \underline{A}}{\partial t}$$

$$\operatorname{rot}\left(\underline{E} + \frac{\partial \underline{A}}{\partial t}\right) = 0 \rightarrow \underline{E} + \frac{\partial \underline{A}}{\partial t} = -\operatorname{grad} \phi$$

skalárpotenciál

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \operatorname{grad} \phi$$

$$\underline{\hat{A}} = \underline{A} + \operatorname{grad}\Lambda \quad \Rightarrow \quad \underline{\hat{B}} = \operatorname{rot}\underline{\hat{A}} = \operatorname{rot}\underline{A} = \underline{B}$$

$$\underline{\hat{E}} = -\frac{\partial \underline{\hat{A}}}{\partial t} - \operatorname{grad}\hat{\phi} = -\frac{\partial \underline{A}}{\partial t} - \operatorname{grad}\underbrace{\left(\frac{\partial \Lambda}{\partial t} + \hat{\phi}\right)}_{\phi}$$

következmény: $\underline{\hat{E}} = \underline{E}$

mértéktranszformáció: $\underline{\hat{A}} = \underline{A} + \operatorname{grad}\Lambda$

$$\hat{\phi} = \phi - \frac{\partial \Lambda}{\partial t}$$

$\underline{E}$; $\underline{B}$ mértékinvariáns

Mértéket rögzítő feltételek:

1) Coulomb-mérték $\operatorname{div}\underline{A} = 0$

2) Sugárzási mérték $\phi = 0$

3) Lorentz-mérték $\varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t} + \operatorname{div}\underline{A} = 0$

Hullámegyenletek:

$$\operatorname{rot}\underline{B} = \mu_0\left(\underline{j} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

$$\operatorname{rot}\operatorname{rot}\underline{A} = \mu_0\left(\underline{j} - \varepsilon_0\left(\frac{\partial^2 \underline{A}}{\partial t^2} + \operatorname{grad}\frac{\partial \phi}{\partial t}\right)\right)$$

$$\left(\operatorname{rot}\operatorname{rot}\underline{A}\right)_i = \varepsilon_{ijk}\,\partial_j \left(\operatorname{rot}\underline{A}\right)_k = \varepsilon_{ijk}\,\partial_j\left(\varepsilon_{kmn}\,\partial_m A_n\right) =$$

$$= \varepsilon_{ijk}\,\varepsilon_{mnk}\,\partial_j \partial_m A_n = \left(\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}\right)\partial_j \partial_m A_n =$$

$$= \partial_j \partial_i A_j - \partial_j \partial_j A_i = \partial_i\left(\partial_j A_j\right) - \partial_j \partial_j A_i =$$

$$= \partial_i (\operatorname{div} \underline{A}) - \Delta A_i = \left(\operatorname{grad}(\operatorname{div} \underline{A}) - \Delta \underline{A}\right)_i$$

$$\operatorname{grad}(\operatorname{div} \underline{A}) - \Delta \underline{A} = \mu_0 \underline{j} - \varepsilon_0 \mu_0 \frac{\partial^2 \underline{A}}{\partial t^2} - \varepsilon_0 \mu_0 \operatorname{grad} \frac{\partial \phi}{\partial t}$$

$$\underbrace{\left(\varepsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} - \Delta\right)}_{\Box} \underline{A} = \mu_0 \underline{j} - \operatorname{grad} \underbrace{\left(\varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t} + \operatorname{div} \underline{A}\right)}_{0}$$

Lorenz-mértékben

$$\varepsilon_0 \mu_0 = \frac{1}{c^2}$$

$$\operatorname{div} \underline{E} = \frac{1}{\varepsilon_0} \varrho \quad \rightarrow \quad \operatorname{div}\left(-\frac{\partial \underline{A}}{\partial t} - \operatorname{grad} \phi\right) = \frac{1}{\varepsilon_0} \varrho$$

$$-\Delta \phi = \frac{\partial}{\partial t} \underbrace{(\operatorname{div} \underline{A})}_{-\varepsilon_0 \mu_0 \frac{\partial \phi}{\partial t}} + \frac{1}{\varepsilon_0} \varrho$$

← Lorenz-mértékben

$$\underbrace{\varepsilon_0 \mu_0 \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi}_{\Box \phi} = \frac{1}{\varepsilon_0} \varrho$$

Coulomb-mértékben: $\Delta \phi = -\frac{1}{\varepsilon_0} \varrho \rightarrow \phi$

$$\Box \underline{A} = \mu_0 \underbrace{\left(\underline{j} - \varepsilon_0 \operatorname{grad} \frac{\partial \phi}{\partial t}\right)}_{\underline{j}_T}$$

$$\operatorname{div} \underline{j}_T = \operatorname{div} \underline{j} - \varepsilon_0 \Delta \frac{\partial \phi}{\partial t} = \operatorname{div} \underline{j} + \frac{\partial \varrho}{\partial t} = 0$$

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \qquad \Box f(\underline{x},t) = s(\underline{x},t)$$

lineáris

$$s = \sum_i \lambda_i s_i \rightarrow f = \sum_i \lambda_i f_i \qquad \Box f_i = s_i$$

$$f(\underline{x},t) = \int d^3\hat{x} \int d\hat{t}\ \underbrace{G(\underline{x},t,\hat{\underline{x}},\hat{t})}\ s(\hat{\underline{x}},\hat{t})$$

Green-függvény

$$\int d^3\hat{x} \int d\hat{t}\ \Box_{x,t} G(\underline{x},t,\hat{\underline{x}},\hat{t})\, s(\hat{\underline{x}},\hat{t}) = s(\underline{x},t) =$$

$$= \int d^3\hat{x} \int d\hat{t}\ \delta^{(3)}(\underline{x}-\hat{\underline{x}})\,\delta(t-\hat{t})\, s(\hat{\underline{x}},t')$$

$$\Box_{\underline{x},t} G(\underline{x},t,\hat{\underline{x}},\hat{t}) = \delta^{(3)}(\underline{x}-\hat{\underline{x}})\,\delta(t-\hat{t})$$

$$\rightarrow \quad G(\underline{x}-\hat{\underline{x}},\, t-\hat{t})$$

$$f(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\ \tilde{f}(\underline{k},\omega)\ e^{i(\underline{k}\underline{x}-\omega t)}$$

$$s(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\ \tilde{s}(\underline{k},\omega)\ e^{i(\underline{k}\underline{x}-\omega t)}$$

$$f(x) = \int \frac{dk}{2\pi}\ \tilde{f}(k)\ e^{ikx} \qquad \tilde{f}(k) = \int dx\ f(x)\ e^{-ikx}$$

$$f(x) = \int \frac{dk}{2\pi} \int dx\ f(x)\, e^{ik(x-x)} = \int dx \underbrace{\left[\int \frac{dk}{2\pi}\ e^{ik(x-x)}\right]} f(x)$$

$$\delta(x-x) = \int \frac{dk}{2\pi}\ e^{ik(x-x)}$$

$$\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \tilde{f}(\underline{k},\omega) \underbrace{\Box\, e^{i(\underline{k}\underline{x}-\omega t)}}_{\left(-\frac{\omega^2}{c^2}+\underline{k}^2\right) e^{i(\underline{k}\underline{x}-\omega t)}} = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \tilde{j}(\underline{k},\omega)\, e^{i(\underline{k}\underline{x}-\omega t)}$$

$$\left(\underline{k}^2 - \frac{\omega^2}{c^2}\right) \tilde{f}(\underline{k},\omega) = \tilde{j}(\underline{k},\omega)$$

$$f(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \tilde{j}(\underline{k},\omega)\, e^{i(\underline{k}\underline{x}-\omega t)}$$

, also: $\tilde{j}(\underline{k},\omega) = \int d^3x' \int dt'\, j(\underline{x}',t')\, e^{-i(\underline{k}\underline{x}'-\omega t')}$

$$f(\underline{x},t) = \int d^3x' \int dt' \underbrace{\int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \frac{1}{\underline{k}^2 - \frac{\omega^2}{c^2}}\, e^{i\left(\underline{k}(\underline{x}-\underline{x}')-\omega(t-t')\right)}}_{G(\underline{x}-\underline{x}',t-t')} \cdot j(\underline{x}',t')$$

$$\frac{1}{\underline{k}^2 - \frac{\omega^2}{c^2}} = \frac{c^2}{\underline{k}^2c^2-\omega^2} = \frac{c^2}{(\underline{k}c+\omega)(\underline{k}c-\omega)} = \frac{a}{\underline{k}c+\omega} + \frac{b}{\underline{k}c-\omega}$$

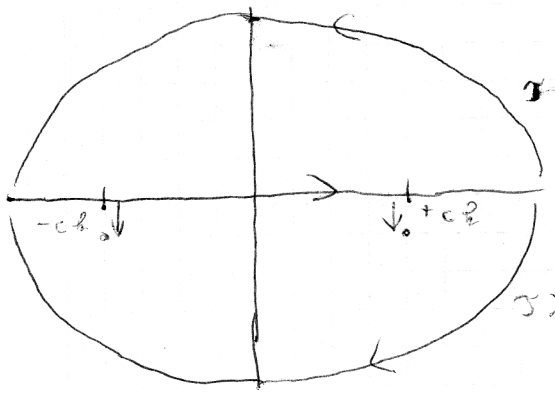
$$c^2 = (\underline{k}c-\omega)\,a + (\underline{k}c+\omega)\,b$$

$$a+b = \frac{c}{\underline{k}} \qquad b-a = 0 \quad \Rightarrow \quad a=b=\frac{c}{2\underline{k}}$$

$$\frac{1}{\underline{k}^2 - \frac{\omega^2}{c^2}} = \frac{c}{2\underline{k}}\left(\frac{1}{\omega+\underline{k}c} - \frac{1}{\omega-\underline{k}c}\right)$$

$$G(\underline{x}-\underline{x}',t-t') = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, \frac{c}{2\underline{k}}\left(\frac{1}{\omega+c\underline{k}} - \frac{1}{\omega-c\underline{k}}\right) e^{i\left(\underline{k}(\underline{x}-\underline{x}')-\omega(t-t')\right)}$$

$\tau = t - t'$ $\qquad \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{c}{2k} \left(\frac{1}{\omega + ck} - \frac{1}{\omega - ck} \right) e^{-i\omega\tau} = ?$



$\omega = \omega' + i\omega''$

$e^{-i\omega\tau} = e^{-i\omega'\tau} \cdot e^{\omega''\tau}$

$\omega'' > 0$ lesz: $\omega''\tau \to -\infty$

Kauzalitás: $G_R(x - x', \tau) = 0$, ha $\tau < 0$ (retardált)

$\Rightarrow$ pólushely lefelé kell eltolni

$$\oint_{C_{z_0}} \frac{f(z)}{z - z_0} dz = 2\pi i f(z_0)$$

$\varepsilon > 0 \quad \tau > 0$

$$\oint \frac{d\omega}{2\pi} \cdot \frac{c}{2k} \left(\frac{1}{\omega + ck + i\varepsilon} - \frac{1}{\omega - ck + i\varepsilon} \right) e^{-i\omega\tau} =$$

$\varepsilon \to 0$

$$= -\frac{c}{2k} \cdot \frac{2\pi i}{2\pi} \left(e^{-i(-ck - i\varepsilon)\tau} - e^{-i(ck - i\varepsilon)\tau} \right) =$$

$$= \frac{c}{2ki} \left(e^{ick\tau} - e^{-ick\tau} \right)$$

$$G_R(\underline{x}-\underline{\dot{x}}, \underbrace{\tau}_{0}) = \int \frac{d^3k}{(2\pi)^3} \frac{c}{2ki} e^{i\underline{k}(\underline{x}-\underline{\dot{x}})} \left(e^{ick\tau} - e^{-ick\tau} \right)$$

$$d^3k = k^2 \, dk \, d(\cos\vartheta_k) \, d\varphi_k$$

$$\underline{k}(\underline{x}-\underline{\dot{x}}) = k\,|\underline{x}-\underline{\dot{x}}| \cos\vartheta_k$$

$$G_R(\underline{x}-\underline{\dot{x}}, \underbrace{\tau}_{0}) = \int_0^\infty \frac{k^2 dk}{(2\pi)^2} \int_{-1}^{1} d(\cos\vartheta_k) \frac{c}{2ki} e^{ik|\underline{x}-\underline{\dot{x}}|\cos\vartheta_k} \left(e^{ick\tau} - e^{-ick\tau} \right) =$$

$$= \int_0^\infty \frac{k^2 dk}{(2\pi)^2} \frac{c}{2ki} \cdot \frac{1}{ik|\underline{x}-\underline{\dot{x}}|} \left(e^{ik|\underline{x}-\underline{\dot{x}}|} - e^{-ik|\underline{x}-\underline{\dot{x}}|} \right) \left(e^{ick\tau} - e^{-ick\tau} \right) =$$

$$= -\frac{c}{4\pi} \frac{1}{|\underline{x}-\underline{\dot{x}}|} \int_0^\infty \frac{dk}{2\pi} \left(e^{ik(|\underline{x}-\underline{\dot{x}}|+c\tau)} - e^{ik(|\underline{x}-\underline{\dot{x}}|-c\tau)} + \text{c.c.} \right) =$$

↑ komplex konjugált

$$= -\frac{c}{4\pi} \frac{1}{|\underline{x}-\underline{\dot{x}}|} \underbrace{\int_{-\infty}^{+\infty} \frac{dk}{2\pi} \left(e^{ik(|\underline{x}-\underline{\dot{x}}|+c\tau)} - e^{ik(|\underline{x}-\underline{\dot{x}}|-c\tau)} \right)}$$

$$\underbrace{\delta(|\underline{x}-\underline{\dot{x}}|+c\tau)}_{0} - \underbrace{\delta(|\underline{x}-\underline{\dot{x}}|-c\tau)}_{0}$$

$$\boxed{G_R(\underline{x}-\underline{\dot{x}}, \underbrace{t-\dot{t}}_{0}) = \frac{c}{4\pi} \frac{1}{|\underline{x}-\underline{\dot{x}}|} \delta(|\underline{x}-\underline{\dot{x}}| - c(t-\dot{t})) =}$$

↑

$$\delta(a\underline{x}) = \frac{1}{a}\delta(\underline{x})$$

$$= \boxed{\frac{1}{4\pi|\underline{x}-\underline{\dot{x}}|} \delta\left((t-\dot{t}) - \frac{|\underline{x}-\underline{\dot{x}}|}{c}\right)}$$

$$f_R(\underline{x},t) = \int d^3\hat{x} \int_{-\infty}^{t} d\hat{t}\; \frac{1}{4\pi|\underline{x}-\hat{\underline{x}}|}\,\delta\left((t-\hat{t}) - \frac{|\underline{x}-\hat{\underline{x}}|}{c}\right) g(\hat{\underline{x}},\hat{t}) =$$

$$= \frac{1}{4\pi}\int d^3\hat{x}\; \frac{g\left(\hat{\underline{x}},\, t - \frac{|\underline{x}-\hat{\underline{x}}|}{c}\right)}{|\underline{x}-\hat{\underline{x}}|}$$

□ $G_1(\underline{x}-\hat{\underline{x}}, t-\hat{t}) = \delta(\underline{x}-\hat{\underline{x}})\,\delta(t-\hat{t})$

□ $G_2(\underline{x}-\hat{\underline{x}}, t-\hat{t}) = \delta(\underline{x}-\hat{\underline{x}})\,\delta(t-\hat{t})$

$\Rightarrow \quad \Box(G_1 - G_2) = 0$

$$f_{ges}(\underline{x},t) = f_R(\underline{x},t) + f_{hom}(\underline{x},t)$$

□ $f_{hom}(\underline{x},t) = 0$

□ $e^{i(\underline{k}\underline{x}-\omega t)} = 0 \quad \Rightarrow \quad \left(-\frac{\omega^2}{c^2} + \underline{k}^2\right) e^{i(\underline{k}\underline{x}-\omega t)} = 0$

$$\Downarrow$$

$$\omega = \pm c\cdot|\underline{k}|$$

$$f(\underline{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[\tilde{f}_1(\underline{k})\, e^{i(\underline{k}\underline{x} - ck t)} + \tilde{f}_2(\underline{k})\, e^{i(\underline{k}\underline{x} + ck t)}\right]$$

$$f(\underline{x},0) = \int \frac{d^3k}{(2\pi)^3}\left[\tilde{f}_1(\underline{k}) + \tilde{f}_2(\underline{k})\right] e^{i\underline{k}\underline{x}}$$

$$\frac{\partial f}{\partial t}(\underline{x},0) = \int \frac{d^3k}{(2\pi)^3}\, ck\left[\tilde{f}_2(\underline{k}) - \tilde{f}_1(\underline{k})\right] e^{i\underline{k}\underline{x}}$$

$$\left.\begin{aligned} \tilde{f}_2(\underline{k}) + \tilde{f}_1(\underline{k}) &= \int d^3\hat{x}\; f(\hat{x},0)\, e^{-i\underline{k}\hat{x}} \\ c\,\underline{k}\left(\tilde{f}_2(\underline{k}) - \tilde{f}_1(\underline{k})\right) &= \int d^3\hat{x}\; \frac{\partial f}{\partial t}(\hat{x},0)\, e^{-i\underline{k}\hat{x}} \end{aligned}\right\} \rightarrow \tilde{f}_1, \tilde{f}_2$$

Energia- és impulzusmegmaradás elektromechanikai rendszerekben

$$\underline{f} = \varrho\,\underline{E} + \underline{j}\times\underline{B} \qquad \underline{j} = \varrho\,\underline{v}$$

$$\frac{dW_{mech}}{dt} = \int_V d^3x\; \underline{v}\,\underline{f} = \int_V d^3x\; \underline{v}\,\varrho\,\underline{E} = \int_V d^3x\; \underline{j}\,\underline{E}$$

$$\frac{dW_{mech}}{dt} - \int_V d^3x\; \underline{j}\,\underline{E} = 0$$

$$-\underline{j}\,\underline{E} = -\left(\frac{1}{\mu_0}\,\mathrm{rot}\,\underline{B} - \varepsilon_0\frac{\partial \underline{E}}{\partial t}\right)\underline{E} = -\frac{1}{\mu_0}\,\underline{E}\,\mathrm{rot}\,\underline{B} + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0\,\underline{E}^2\right)$$

$$\mathrm{rot}\,\underline{B} = \mu_0\left(\underline{j} + \varepsilon_0\frac{\partial \underline{E}}{\partial t}\right)$$

$$\underline{E}\,\mathrm{rot}\,\underline{B} = E_i\,\varepsilon_{ijk}\,\partial_j B_k = \partial_j\left(\varepsilon_{ijk}E_i B_k\right) - B_k\,\varepsilon_{ijk}\,\partial_j E_i =$$

$$= -\partial_j\left(\varepsilon_{jik}E_i B_k\right) + B_k\,\varepsilon_{kji}\,\partial_j E_i =$$

$$= -\,\mathrm{div}\left(\underline{E}\times\underline{B}\right) + \underline{B}\,\mathrm{rot}\,\underline{E}$$

$$-\underline{j}\,\underline{E} = \frac{1}{\mu_0}\,\mathrm{div}\left(\underline{E}\times\underline{B}\right) - \frac{1}{\mu_0}\,\underline{B}\underbrace{\mathrm{rot}\,\underline{E}}_{-\frac{\partial \underline{B}}{\partial t}} + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0\,\underline{E}^2\right) =$$

$$= \frac{1}{\mu_0}\,\mathrm{div}\left(\underline{E}\times\underline{B}\right) + \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0\,\underline{E}^2 + \frac{1}{2\mu_0}\,\underline{B}^2\right)$$

$$\frac{d W_{mech}}{dt} + \underbrace{\int_V d^3x \; \frac{\partial}{\partial t}\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right)}_{} + \underbrace{\int_V d^3x \; \operatorname{div}\left(\frac{1}{\mu_0} \underline{E} \times \underline{B}\right)}_{} = 0$$

$$\frac{d}{dt} \int_V d^3x \left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right) \qquad \int_F \frac{1}{\mu_0} (\underline{E} \times \underline{B}) \, d\underline{F}$$

$$\frac{d}{dt}\left(W_{mech} + \int_V d^3x \underbrace{\left(\frac{1}{2}\varepsilon_0 E^2 + \frac{1}{2\mu_0} B^2\right)}_{w_{em}}\right) + \int_F \underbrace{\frac{1}{\mu_0}(\underline{E} \times \underline{B})}_{\underline{S}_{em}} d\underline{F} = 0$$

energiasűrűség — Poynting-vektor

$$\frac{d \underline{P}_{mech}}{dt} = \int_V d^3x \; \underline{f} \qquad \frac{d \underline{P}_{mech}}{dt} - \int_V d^3x \; \underline{f} = 0$$

$$-\underline{f} = -\rho \underline{E} - \underline{j} \times \underline{B} = -\varepsilon_0 \underline{E} \operatorname{div} \underline{E} - \left(\frac{1}{\mu_0} \operatorname{rot} \underline{B} - \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right) \times \underline{B} =$$

$$\operatorname{div} \underline{E} = \frac{1}{\varepsilon_0}\rho \qquad \operatorname{rot} \underline{B} = \mu_0\left(\underline{j} + \varepsilon_0 \frac{\partial \underline{E}}{\partial t}\right)$$

$$= \varepsilon_0 \underbrace{\frac{\partial \underline{E}}{\partial t} \times \underline{B}} - \varepsilon_0 \underline{E} \operatorname{div} \underline{E} + \frac{1}{\mu_0} \underline{B} \times \operatorname{rot} \underline{B} =$$

$$\frac{\partial}{\partial t}(\underline{E} \times \underline{B}) - \underline{E} \times \frac{\partial \underline{B}}{\partial t} = \frac{\partial}{\partial t}(\underline{E} \times \underline{B}) + \underline{E} \times \operatorname{rot} \underline{E}$$

$$\operatorname{rot} \underline{E} = -\frac{\partial \underline{B}}{\partial t}$$

$$= \frac{\partial}{\partial t}\left(\varepsilon_0 (\underline{E} \times \underline{B})\right) + \varepsilon_0 \underline{E} \times \operatorname{rot} \underline{E} - \varepsilon_0 \underline{E} \operatorname{div} \underline{E} + \frac{1}{\mu_0} \underline{B} \times \operatorname{rot} \underline{B} =$$

$$= \frac{\partial}{\partial t}\left(\varepsilon_0\left(\underline{E}\times\underline{B}\right)\right) + \varepsilon_0\underbrace{\left(\underline{E}\times \mathrm{rot}\,\underline{E} - \underline{E}\,\mathrm{div}\,\underline{E}\right)}_{\text{ugyanez alakú tagok}} + \frac{1}{\mu_0}\underbrace{\left(\underline{B}\times\mathrm{rot}\,\underline{B} - \underline{B}\,\mathrm{div}\,\underline{B}\right)}_{\text{ugyanez alakú tagok}}$$

$\uparrow$ $\mathrm{div}\,\underline{B} = 0$

$$\left(\underline{E}\times\mathrm{rot}\,\underline{E} - \underline{E}\,\mathrm{div}\,\underline{E}\right)_i = \varepsilon_{ijk}E_j\varepsilon_{kmn}\partial_m E_n - E_i\partial_j E_j =$$

$$= \underbrace{\varepsilon_{ijk}\varepsilon_{mnk}}_{\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm}}\cdot E_j\partial_m E_n - E_i\partial_j E_j =$$

$$= E_j\partial_i E_j - E_j\partial_j E_i - E_i\partial_j E_j =$$

$$= \partial_i\left(\frac{1}{2}E_jE_j\right) - \partial_j\left(E_iE_j\right) = \partial_j\left(\frac{1}{2}\underline{E}^2\cdot\delta_{ij}\right) - \partial_j\left(E_iE_j\right) =$$

$$= \partial_j\left(\frac{1}{2}\underline{E}^2\cdot\delta_{ij} - E_iE_j\right) = \left(\mathrm{div}\left(\frac{1}{2}\underline{E}^2\cdot\underline{\underline{I}} - \underline{E}\circ\underline{E}\right)\right)_i$$

$$-\underline{f} = \frac{\partial}{\partial t}\left(\varepsilon_0\,\underline{E}\times\underline{B}\right) + \mathrm{div}\left(\left(\frac{1}{2}\varepsilon_0\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2\right)\underline{\underline{I}} - \varepsilon_0\underline{E}\circ\underline{E} - \frac{1}{\mu_0}\underline{B}\circ\underline{B}\right)$$

$$\frac{d}{dt}\left(\underline{P}_{mech} + \int_V d^3x\,\underbrace{\left(\varepsilon_0\,\underline{E}\times\underline{B}\right)}_{\underline{g}_{em}}\right) + \int_F \underbrace{\left(\left(\frac{1}{2}\varepsilon_0\underline{E}^2 + \frac{1}{2\mu_0}\underline{B}^2\right)\underline{\underline{I}} - \varepsilon_0\underline{E}\circ\underline{E} - \frac{1}{\mu_0}\underline{B}\circ\underline{B}\right)}_{\underline{\underline{\Pi}}_{em}} d\underline{F} = 0$$

$\underline{g}_{em}$ — impulzussűrűség

$\underline{\underline{\Pi}}_{em}$ — Maxwell-féle feszültségtenzor

$$\underline{g}_{em} = \varepsilon_0\mu_0\left(\frac{1}{\mu_0}\underline{E}\times\underline{B}\right) = \frac{1}{c^2}\underline{S}_{em}$$

$$\underline{\underline{\Pi}}_{em} = w_{em}\underline{\underline{I}} - \varepsilon_0\underline{E}\circ\underline{E} - \frac{1}{\mu_0}\underline{B}\circ\underline{B}$$

Végtelen térbeli megoldás spektrálanalízise

$$f_R(\underline{x},t) = \frac{1}{4\pi}\int d^3\underline{x}' \; \frac{\varrho\left(\underline{x}', t-\frac{|\underline{x}-\underline{x}'|}{c}\right)}{|\underline{x}-\underline{x}'|}$$

ω frekvenciájú gerjesztés: $\varrho(\underline{x},t) = \varrho_\omega(\underline{x})\, e^{-i\omega t}$

$$f_R(\underline{x},t) = \frac{1}{4\pi}\int d^3\underline{x}' \; \frac{\varrho_\omega(\underline{x}')}{|\underline{x}-\underline{x}'|}\, e^{-i\omega\left(t-\frac{|\underline{x}-\underline{x}'|}{c}\right)}$$

$$\Rightarrow \quad f_R(\underline{x},t) = f_{R,\omega}(\underline{x})\, e^{-i\omega t}$$

$$\boxed{f_{R,\omega}(\underline{x}) = \frac{1}{4\pi}\int d^3\underline{x}' \; \frac{\varrho_\omega(\underline{x}')}{|\underline{x}-\underline{x}'|}\, e^{ik|\underline{x}-\underline{x}'|}} \qquad k = \frac{\omega}{c}$$

spec.: $\omega = 0$

$$f_{R,0}(\underline{x}) = \frac{1}{4\pi}\int d^3\underline{x}' \; \frac{\varrho_0(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

általános gerjesztés:

$$\varrho(\underline{x},t) = \int \frac{d\omega}{2\pi}\, \varrho_\omega(\underline{x})\, e^{-i\omega t}$$

$$\downarrow$$

$$f_R(\underline{x},t) = \int \frac{d\omega}{2\pi}\, f_{R,\omega}(\underline{x})\, e^{-i\omega t}$$

Energiasűrűség:

$$w_{el} = \frac{1}{2}\varepsilon_0 \underline{E}^2 \qquad \overline{w_{el}} = \lim_{T\to\infty} \frac{1}{T} \int_{-\frac{T}{2}}^{+\frac{T}{2}} w_{el}\, dt$$

ω frekvencia:

$$\underline{E}(\underline{x},t) = \underline{E}_\omega(\underline{x})\, e^{-i\omega t}$$ komplex → ne használjuk

Helyette: $\underline{E}(\underline{x},t) = \frac{1}{2}\left(\underline{E}_\omega(\underline{x})\, e^{-i\omega t} + \underline{E}^*_\omega(\underline{x})\, e^{i\omega t}\right)$

fenti valós rész

$$w_{el} = \frac{1}{2}\varepsilon_0 \cdot \frac{1}{4}\left(\underline{E}_\omega(\underline{x})\, e^{-i\omega t} + \underline{E}^*_\omega(\underline{x})\, e^{i\omega t}\right)^2 =$$

$$= \frac{1}{8}\varepsilon_0 \left(\underline{E}^2_\omega(\underline{x})\, e^{-i2\omega t} + \underline{E}^{*2}_\omega(\underline{x})\, e^{i2\omega t} + 2\left|\underline{E}_\omega(\underline{x})\right|^2\right)$$

$$\overline{w_{el}} = \frac{1}{2}\cdot\left(\frac{1}{2}\varepsilon_0 \left|\underline{E}_\omega(\underline{x})\right|^2\right)$$

Fig: valós fv. Fourier-spektruma úgy adott, hogy $-\omega$ is

$$f(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{f}(\omega)\, e^{-i\omega t}$$

$$f^*(t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{f}^*(\omega)\, e^{i\omega t} = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{f}^*(-\omega)\, e^{-i\omega t}$$

$\uparrow$ $\omega \to -\omega$

$$f(t) = f^*(t) \Rightarrow \tilde{f}(\omega) = \tilde{f}^*(-\omega) \Rightarrow \tilde{f}(-\omega) = \tilde{f}^*(\omega)$$

Elektrostatikai peremérték feladat

$$\Delta f(\underline{x}) = -g(\underline{x})$$

+ peremfeltételek:

1. Dirichlet: $f(x)$ adott $\quad x \in F$

2. Neumann: $\frac{\partial f(x)}{\partial n} = \underline{n} \nabla f(\underline{x})$ adott $\quad x \in F$

Green-tétel

$f(\underline{x})$, $g(\underline{x})$ kétszeresen

$$\int_V d^3x \, \nabla (f \nabla g) = \int_F d\underline{F} (f \nabla g) \quad \text{Gauss-tétel } f\nabla g\text{-re}$$

$$\boxed{\int_V d^3x \, (f \Delta g + \nabla f \cdot \nabla g) = \int_F d\underline{F} (f \nabla g)} \quad \text{1. Green-azonosság}$$

$$- \int_V d^3x \, (g \Delta f + \nabla g \cdot \nabla f) = \int_F d\underline{F} (g \nabla f)$$

$$\boxed{\int_V d^3x \, (f \Delta g - g \Delta f) = \int_F d\underline{F} (f \nabla g - g \nabla f)}$$

II Green-azonosság = Green-tétel

Egyértelműség:

$$\Delta f_1 = -g$$
$$\Delta f_2 = -g$$
$$u = f_1 - f_2 \Rightarrow \Delta u = 0$$

felületen: 1. $u = 0$ vagy 2. $\frac{\partial u}{\partial n} = 0$

$$\int_V d^3x \left(u \underbrace{\Delta u}_{0} + \nabla u \cdot \nabla u \right) = \int_F d\underline{F} \; u \nabla u$$

$$\int_V d^3x \, (\nabla u)^2 = \int_F dF \; u \, \underline{n} \nabla u = \underbrace{\int_F dF \; u \frac{\partial u}{\partial n}}_{0}$$

$$\Rightarrow \nabla u = 0 \rightarrow u = \text{állandó}$$

1. $u = 0$ a felületen $\Rightarrow u = 0$ mindenhol $\Rightarrow$ f egyértelmű
2. u - állandó $\Rightarrow$ f additív állandó erejéig meghatározott

Green - függvény: $\Delta_{\underline{x}} G(x, \hat{x}) = -\delta^{(3)}(\underline{x} - \underline{\hat{x}})$

végtelen térfogatban:

$$G^{(\infty)}(x, \hat{x}) = \frac{1}{4\pi} \cdot \frac{1}{|\underline{x} - \underline{\hat{x}}|} \qquad f(x) = \int_V d^3\hat{x} \; G^{(\infty)}(x, \hat{x}) \, g(\hat{x})$$

$$\int_V d^3\hat{x} \left[f(\hat{x}) \underbrace{\Delta_{\hat{x}} G}_{-\delta^{(3)}(\underline{x} - \underline{\hat{x}})} - G \underbrace{\Delta f(\hat{x})}_{-g(\hat{x})} \right] = \int_F d\underline{\hat{F}} \left[f(\underline{\hat{x}}) \nabla_{\underline{\hat{x}}} G - G \nabla f(\hat{x}) \right]$$

$$-f(x) + \int_V d^3\hat{x} \; G(x, \hat{x}) \cdot g(\hat{x}) = \int_F dF' \left[f(\hat{x}) \frac{\partial G}{\partial \hat{n}} - G \frac{\partial f}{\partial n}(\hat{x}) \right]$$

$$\boxed{f(x) = \int_V d^3\hat{x} \; G(x, \hat{x}) \cdot g(\hat{x}) + \int_F dF' \left[G(x, \hat{x}) \frac{\partial f}{\partial \hat{n}}(\hat{x}) - f(\hat{x}) \frac{\partial G}{\partial \hat{n}}(x, \hat{x}) \right]}$$

Peremfeltétel G-re:

1. Dirichlet: $G_D(x,\hat{x}) = 0 \qquad \hat{x} \in F$

$$\boxed{\varphi_D(x) = \int_V d^3\hat{x}\; G_D(x,\hat{x})\cdot s(\hat{x}) - \int_F dF\; \varphi_D(\hat{x})\; \frac{\partial G_D}{\partial n}(x,\hat{x})}$$

Áll.: G_D szimmetrikus: $G_D(x,\hat{x}) = G_D(\hat{x},x)$

$$\int_V d^3y \left[G_D(x,y)\underbrace{\Delta_y G_D(\hat{x},y)}_{-\delta^{(3)}(\hat{x}-y)} - G_D(\hat{x},y)\underbrace{\Delta_y G_D(x,y)}_{-\delta^{(3)}(x-y)} \right] = 0$$

$$\Rightarrow \; -G_D(x,\hat{x}) + G_D(\hat{x},x) = 0$$

2. Neumann: $\frac{\partial G_N}{\partial n}(x,\hat{x}) = -\frac{1}{F} \qquad \hat{x} \in F$

↑ nincs rá 0?

$$\int_F \frac{\partial G_N}{\partial n}\, dF = \int_F dF\; \nabla_{\hat{x}} G_N = \int_V d^3\hat{x}\; \underbrace{\Delta_{\hat{x}} G_N}_{-\delta^{(3)}(x-\hat{x})} = -1$$

$$\boxed{\varphi_N(x) = \int_V d^3\hat{x}\; G_N(x,\hat{x})\cdot s(\hat{x}) + \int_F dF\; G_N(x,\hat{x})\frac{\partial \varphi_N}{\partial n}(\hat{x}) + \frac{1}{F}\int_F dF\; \varphi_N(\hat{x})}$$

↑ "0" feltételű

$G_N(x,\hat{x}) = G_N(\hat{x},x)$ feltételű

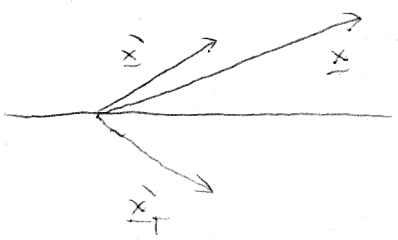
↑

G_N additív állandója függhet x-től

additív állandó valódi

$$\left.\begin{array}{l} \Delta_{\underline{x}'} G(x,x') = -\delta^{(3)}(x-x') \\ \Updownarrow \quad G(x,x') = G(x',x) \end{array}\right\} \Rightarrow \Delta_{\underline{x}} G(x,x') = -\delta^{(3)}(x-x')$$

Tükrözés módszer



$$\Delta_{\underline{x}} G_D(x,x') = -\delta^{(3)}(x-x')$$

$$G_D(x,x') = \frac{1}{4\pi} \cdot \frac{1}{|\underline{x}-\underline{x}'|} + H(x,x')$$

$$\Delta_{\underline{x}} H(x,x') = 0 \qquad H(x,x') = -\frac{1}{4\pi} \cdot \frac{1}{|\underline{x}-\underline{x}'_T|}$$

$$\Rightarrow \Delta_{\underline{x}} H(x,x') = -\delta^{(3)}(x-x'_T) = 0 \quad , \text{ mert } \underline{x} \neq \underline{x}'_T$$

Laplace - egyenlet megoldása a változók szétválasztásával

$$\Delta f = 0 \qquad f(r,\vartheta,\varphi) \quad \text{gömbi koordináták}$$

$$0 = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2}\left[\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial f}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 f}{\partial\varphi^2}\right]$$

$$f(r,\vartheta,\varphi) = \frac{u(r)}{r} P(\vartheta) Q(\varphi)$$

$$0 = \frac{1}{r}\frac{d^2u}{dr^2} \cdot PQ + \frac{u(r)}{r^3}\left[\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP}{d\vartheta}\right)\cdot Q + \frac{1}{\sin^2\vartheta}\cdot\frac{d^2Q}{d\varphi^2}\cdot P\right]$$

$$0 = \underbrace{\frac{r^2}{u}\cdot\frac{d^2u}{dr^2}}_{\ell(\ell+1)} + \underbrace{\frac{1}{P\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP}{d\vartheta}\right) + \frac{1}{Q\sin^2\vartheta}\cdot\frac{d^2Q}{d\varphi^2}}_{-\ell(\ell+1)}$$

$\ell(\ell+1)$ ← állandók → $-\ell(\ell+1)$

$$\frac{d^2 u}{dr^2} = \ell(\ell+1)\frac{u}{r^2} \qquad u(r) \sim r^{\alpha}$$

$$\alpha(\alpha-1) r^{\alpha-2} = \ell(\ell+1) r^{\alpha-2} \Rightarrow \alpha_1 = -\ell \qquad \alpha_2 = \ell+1$$

$$u(r) = a \cdot r^{-\ell} + b\, r^{\ell+1}$$

$$\frac{1}{P}\cdot\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\frac{\sin^2\vartheta}{\sin\vartheta}\frac{dP}{d\vartheta}\right) + \frac{1}{\sin^2\vartheta}\cdot\frac{1}{Q}\frac{d^2Q}{d\varphi^2} = -\ell(\ell+1)$$

$$\frac{1}{P}\frac{d}{d(\cos\vartheta)}\left((1-\cos^2\vartheta)\frac{dP}{d\cos\vartheta}\right) + \frac{1}{1-\cos^2\vartheta}\cdot\underbrace{\frac{1}{Q}\frac{d^2Q}{d\varphi^2}}_{-m^2} = -\ell(\ell+1)$$

$$\frac{d^2Q}{d\varphi^2} = -m^2 \cdot Q$$

$$\downarrow$$

$$Q = e^{\pm i m \varphi} \qquad Q(\varphi+2\pi) = Q(\varphi) \Rightarrow m \in \mathbb{Z}$$

$x = \cos\vartheta$

$$\boxed{\frac{d}{dx}\left((1-x^2)\frac{dP}{dx}\right) + \left(\ell(\ell+1) - \frac{m^2}{1-x^2}\right)P(x) = 0}$$

általánosított Legendre-egyenlet

$m = 0$ hengerszimmetrikus eset → Legendre-egyenlet

$x \in [-1,1]$-en konvergens megoldás csak $\ell \in \{0,1,2,\dots\}$ esetén létezik, ekkor is csak egy!

$$\boxed{P_\ell(x) = \frac{1}{2^\ell \cdot \ell!}\frac{d^\ell}{dx^\ell}\left(x^2-1\right)^\ell}$$ Legendre-polinomok

ortogonalitás:

$$\boxed{\int_{-1}^{1} dx\; P_\ell(x)\, P_{\ell'}(x) = \frac{2}{2\ell+1}\,\delta_{\ell,\ell'}}$$

$$P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{1}{2}\left(3x^2-1\right) \quad \dots$$

teljesség: tetsz. $[-1,1]$-en értelmezett f kifejthető

$$f(x) = \sum_{\ell=0}^{\infty} a_\ell P_\ell(x) \qquad / \cdot P_{\ell'}(x) \qquad / \int_{-1}^{1} dx$$

$$\int_{-1}^{1} f(x) P_{\ell'}(x)\, dx = \sum_{\ell=0}^{\infty} a_\ell \frac{2}{2\ell+1} \delta_{\ell\ell'} = \frac{2}{2\ell'+1} a_{\ell'}$$

$$a_\ell = \frac{2\ell+1}{2} \int_{-1}^{1} f(x) P_\ell(x)\, dx$$

<u>hengerszimmetrikus megoldás:</u>

$$\boxed{f(r,\vartheta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^{-\ell-1} + b_\ell r^{\ell} \right) P_\ell(\cos\vartheta)}$$

alkalmazás: földelt fémgömb homogén E_0 térben

$$\Phi(r,\vartheta) = \sum_{\ell=0}^{\infty} \left(a_\ell r^{-\ell-1} + b_\ell r^{\ell} \right) P_\ell(\cos\vartheta)$$

$$\Phi(R,\vartheta) = 0$$

$r \gg R$ esetén: $\Phi \approx -\underline{E}_0 \underline{x} = -E_0 r \cos\vartheta = -E_0 r P_1(\cos\vartheta)$

$r \to \infty \qquad r^{-\ell-1} \approx 0$

$$\sum_{\ell=0}^{\infty} b_\ell r^{\ell} P_\ell(\cos\vartheta) = -E_0 r P_1(\cos\vartheta)$$

$$\Rightarrow \quad b_1 = -E_0 \qquad b_0 = b_2 = b_3 = b_4 = \dots = 0$$

$$\sum_{\ell=0}^{\infty} \left(a_\ell R^{-\ell-1} + b_\ell R^{\ell} \right) P_\ell(\cos\vartheta) = 0 \Rightarrow a_\ell R^{-\ell-1} + b_\ell R^{\ell} = 0$$

$a_\ell = -\delta_\ell \cdot R^{2\ell+1} \Rightarrow a_1 = E_0 R^3 \qquad a_0 = a_2 = a_3 = a_4 = \ldots = 0$

$$\Phi = \left(E_0 \frac{R^3}{r^2} - E_0 r\right) \cos\vartheta = E_0 \left(\frac{R^3}{r^3} - 1\right) \cdot z$$

$z = r\cos\vartheta$

$m \neq 0 \rightarrow$ asszociált Legendre-függvények:

$$P_\ell^m(x) = (-1)^m \left(1-x^2\right)^{\frac{m}{2}} \frac{d^m}{dx^m} P_\ell(x) \qquad 0 \leq m \leq \ell$$

$$P_\ell^{-m}(x) = (-1)^m \cdot \frac{(\ell-m)!}{(\ell+m)!} \cdot P_\ell^m(x) \qquad 0 \leq m \leq \ell$$

Egy képletben:

$$\boxed{P_\ell^m(x) = \frac{(-1)^m}{2^\ell \cdot \ell!} \left(1-x^2\right)^{\frac{m}{2}} \frac{d^{\ell+m}}{dx^{\ell+m}} \left(x^2-1\right)^\ell} \qquad -\ell \leq m \leq \ell$$

ortogonalitás: $\int_{-1}^{1} dx\, P_\ell^m(x) \cdot P_{\ell'}^m(x) = \frac{2}{2\ell+1} \cdot \frac{(\ell+m)!}{(\ell-m)!} \cdot \delta_{\ell\ell'}$

gömbfüggvény: $\boxed{Y_{\ell,m}(\vartheta,\varphi) = \sqrt{\frac{2\ell+1}{4\pi} \cdot \frac{(\ell-m)!}{(\ell+m)!}} \cdot P_\ell^m(\cos\vartheta) \cdot e^{im\varphi}}$

$$\int_{-1}^{1} d(\cos\vartheta) \int_0^{2\pi} d\varphi\, Y_{\ell,m}^*(\vartheta,\varphi)\, Y_{\ell',m'}(\vartheta,\varphi) = \delta_{\ell\ell'}\, \delta_{mm'}$$

↑ ortonormált

↓ jelölés

$$\boxed{\int d\Omega_{\vec{r}}\, Y_{\ell,m}^*(\vec{r})\, Y_{\ell',m'}(\vec{r}) = \delta_{\ell,\ell'}\, \delta_{m,m'}}$$

Fv. kifejtése gömbfüggvények szerint:

$$f(\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}\, Y_{\ell m}(\vartheta,\varphi)$$

$$a_{\ell' m'} = \int d\Omega_{\underline{r}}\; Y^*_{\ell' m'}(\underline{r})\, f(\underline{r})$$

$$a_{\ell m} = \int_{-1}^{1} d(\cos\vartheta') \int_{0}^{2\pi} d\varphi'\; Y^*_{\ell m}(\vartheta',\varphi')\, f(\vartheta',\varphi')$$

$\downarrow$

visszaírva:

$$f(\vartheta,\varphi) = \int_{-1}^{1} d(\cos\vartheta') \int_{0}^{2\pi} d\varphi' \underbrace{\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\vartheta',\varphi')\, Y_{\ell m}(\vartheta,\varphi)}_{\delta(\cos\vartheta-\cos\vartheta')\,\delta(\varphi-\varphi')} \cdot f(\vartheta',\varphi')$$

teljességi reláció:

$$\delta(\cos\vartheta-\cos\vartheta')\,\delta(\varphi-\varphi') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\vartheta',\varphi')\, Y_{\ell m}(\vartheta,\varphi)$$

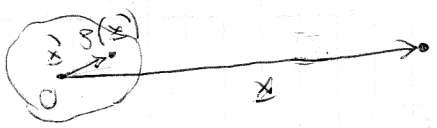
<u>addíciós tétel:</u>

$$P_\ell(\underbrace{\underline{r}\cdot\underline{r}'}_{\cos\gamma}) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\underline{r}')\, Y_{\ell m}(\underline{r})$$

$\Delta f = 0$ általános megoldása:

$$f(r,\vartheta,\varphi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(a_{\ell m}\, r^{-\ell-1} + b_{\ell m}\, r^{\ell}\right) \cdot Y_{\ell m}(\vartheta,\varphi)$$

Multipól sorfejtés



$$\Phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho(\underline{x}')}{|\underline{x}-\underline{x}'|} \qquad |\underline{x}| = r \quad |\underline{x}'| = r' \qquad r \gg r'$$

$$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{\sqrt{(\underline{x}-\underline{x}')^2}} = \frac{1}{\sqrt{r^2+r'^2-2\underline{x}\underline{x}'}} = \frac{1}{\sqrt{r^2+r'^2-2rr'\cdot\underline{n}\underline{n}'}}$$

$$\underline{x} = r\cdot\underline{n} \qquad \underline{x}' = r'\cdot\underline{n}'$$

$$\frac{1}{|\underline{x}-\underline{x}'|} = \frac{1}{r}\cdot\frac{1}{\sqrt{1+\frac{r'^2}{r^2}-2\frac{r'}{r}\underline{n}\underline{n}'}} \to \left(\frac{r'}{r}\right)^2 \text{ rendig sorfejtjük}$$

$$\frac{1}{\sqrt{1+y}} \approx 1-\frac{1}{2}y+\frac{3}{8}y^2$$

$$\frac{1}{|\underline{x}-\underline{x}'|} \approx \frac{1}{r}\left[1-\frac{1}{2}\left(\frac{r'^2}{r^2}-2\frac{r'}{r}\underline{n}\underline{n}'\right)+\frac{3}{8}\left(\frac{r'^2}{r^2}-2\frac{r'}{r}\underline{n}\underline{n}'\right)^2\right] =$$

$$= \frac{1}{r}+\frac{r'}{r^2}\underline{n}\underline{n}'+\frac{1}{2}\left(3\frac{r'^2}{r^3}(\underline{n}\underline{n}')^2-\frac{r'^2}{r^3}\right) =$$

$$= \frac{1}{r}+\frac{\underline{n}}{r^2}\cdot\underline{x}'+\frac{1}{2r^3}\left(3(\underline{n}\underline{x}')^2-r'^2\right)$$

$$3(\underline{n}\underline{x}')^2-r'^2 = n_i n_j\left(3x'_i x'_j - r'^2\cdot\delta_{ij}\right) \quad \text{faktorizáció}$$

$$\Phi(\underline{x}) \approx \frac{1}{4\pi\varepsilon_0}\int d^3x'\left(\frac{1}{r}+\frac{\underline{n}}{r^2}\underline{x}'+\frac{n_i n_j}{2r^3}\left(3x'_i x'_j - r'^2\cdot\delta_{ij}\right)\right)\rho(\underline{x}') =$$

$$= \frac{1}{4\pi\varepsilon_0}\cdot\left[\frac{Q}{r}+\frac{\underline{n}\cdot\underline{p}}{r^2}+\frac{n_i n_j}{2r^3}Q_{ij}\right]$$

, ahol: $Q = \int d^3x\, \varrho(\underline{x})$ $\qquad \underline{p} = \int d^3x\, \varrho(\underline{x}) \cdot \underline{x}$ dipólmomentum

$$Q_{ij} = \int d^3x \left(3x_i x_j - r^2 \delta_{ij}\right) \varrho(\underline{x})$$ kvadrupólmomentum

Ismétlésre:

$$\Phi(\underline{x}) \approx \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\underline{n}\cdot\underline{p}}{r^2} + \frac{\underline{n}\,\underline{\underline{Q}}\,\underline{n}}{2r^3}\right]$$

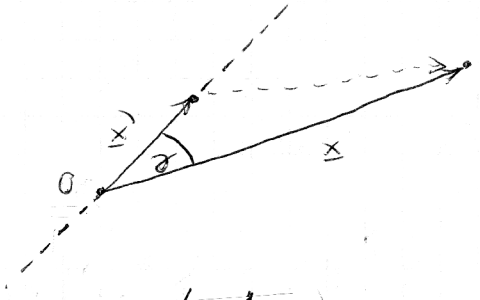
$$\underline{\underline{Q}} = \int d^3x \left(3\,\underline{x}\circ\underline{x} - r^2 \cdot \underline{\underline{I}}\right)\varrho(\underline{x})$$

$Q_{ij} = Q_{ji}$ szimmetrikus

$$\mathrm{Sp}\,\underline{\underline{Q}} = Q_{ii} = \int d^3x \left(3\underbrace{x_i x_i}_{r^2} - r^2 \underbrace{\delta_{ii}}_{3}\right)\varrho(\underline{x}) = 0$$

nyomtalan

Sorfejtés gömbfüggvények segítségével:



$\underline{n}\cdot\underline{n}' = \cos\gamma$

$$\Delta_{\underline{x}}\left(\underbrace{\frac{1}{|\underline{x}-\underline{x}'|}}\right) = 0 \quad , \text{ha} \quad \underline{x} \neq \underline{x}'$$

$\underline{x}$ irányú tengelyre szimmetrikus megoldás

$$\Rightarrow \frac{1}{|\underline{x}-\underline{x}'|} = \sum_{\ell=0}^{\infty} \frac{a_\ell}{r^{\ell+1}} P_\ell(\cos\gamma) \qquad r' < r$$

$a_\ell = 0 \quad \forall \ell$ az $r \to \infty$-beli viselkedés alapján

spec.: $\gamma = 0 \Rightarrow P_\ell(1) = 1$

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{|\underline{r} - \underline{r}'|} = \frac{1}{r - r'} = \frac{1}{r} \cdot \frac{1}{1 - \frac{r'}{r}} = \frac{1}{r} \cdot \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell =$$

$$= \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} = \sum_{\ell=0}^{\infty} \frac{a_\ell}{r^{\ell+1}} \Rightarrow a_\ell = r'^\ell$$

$$\boxed{\frac{1}{|\underline{x} - \underline{x}'|} = \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\underline{n} \cdot \underline{n}')} \qquad r' < r$$

$\downarrow$ addíciós tétel

$$\boxed{\frac{1}{|\underline{x} - \underline{x}'|} = \frac{4\pi}{2\ell+1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{r^{\ell+1}} Y_{\ell m}(\underline{n}) \cdot r'^\ell \, Y^*_{\ell m}(\underline{n}')} \qquad r' < r$$

$$\Delta_{\underline{x}}(\dots) = 0 \qquad \Delta_{\underline{x}'}(\dots) = 0$$

$$\Rightarrow \Delta_{\underline{x}}\left(\frac{1}{|\underline{x} - \underline{x}'|}\right) = \Delta_{\underline{x}'}\left(\frac{1}{|\underline{x} - \underline{x}'|}\right) = 0 \quad , \text{ ha } \underline{x} \neq \underline{x}'$$

$\underline{x} = \underline{x}'$ esetén a sor **divergens**

$$\phi(\underline{x}) = \frac{1}{4\pi\varepsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \cdot \frac{1}{r^{\ell+1}} Y_{\ell m}(\underline{n}) \underbrace{\int d^3x' \, r'^\ell \, Y^*_{\ell m}(\underline{n}') \, \varrho(\underline{x}')}_{q_{\ell m}}$$

multipólusmomentum

$\ell = 0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$

$$q_{00} = \int d^3x' \frac{1}{\sqrt{4\pi}} \rho(\vec{x}') = \frac{Q}{\sqrt{4\pi}} \Rightarrow \phi_0 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{Q}{r}$$

$\ell = 1 \qquad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\vartheta \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\vartheta\, e^{i\varphi} = Y^*_{1,-1}$

$$q_{10} = \int d^3x' \rho(\vec{x}') r' \sqrt{\frac{3}{4\pi}} \cos\vartheta' = \sqrt{\frac{3}{4\pi}} \int d^3x' \rho(\vec{x}') z' = \sqrt{\frac{3}{4\pi}} \cdot \uparrow_z$$

$$q_{11} = -\int d^3x' \rho(\vec{x}') r' \sqrt{\frac{3}{8\pi}} \sin\vartheta' (\cos\varphi' - i \sin\varphi') =$$

$$= -\sqrt{\frac{3}{8\pi}} \int d^3x' \rho(\vec{x}') (x' - i y') = -\sqrt{\frac{3}{8\pi}} (\uparrow_x - i \cdot \uparrow_y) = -q^*_{1,-1}$$

$$\phi_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \left[\frac{4\pi}{3} \sqrt{\frac{3}{4\pi}} \cos\vartheta \sqrt{\frac{3}{4\pi}} \uparrow_z + \frac{4\pi}{3} \sqrt{\frac{3}{8\pi}} \sin\vartheta (\cos\varphi + i \sin\varphi) \cdot \right.$$

$$\left. \cdot \sqrt{\frac{3}{8\pi}} (\uparrow_x - i \uparrow_y) + \frac{4\pi}{3} \sqrt{\frac{3}{8\pi}} \sin\vartheta (\cos\varphi - i \sin\varphi)(\uparrow_x + i \uparrow_y) \right] =$$

$$= \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \left[\uparrow_z \cos\vartheta + \frac{1}{2} \sin\vartheta \underbrace{\left((\cos\varphi + i \sin\varphi)(\uparrow_x - i \uparrow_y) + (\cos\varphi - i \sin\varphi)(\uparrow_x + i \uparrow_y) \right)}_{2 \uparrow_x \cos\varphi + 2 \uparrow_y \sin\varphi} \right]$$

$$\phi_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{r^2} \underbrace{\left[\uparrow_x \sin\vartheta \cos\varphi + \uparrow_y \sin\vartheta \sin\varphi + \uparrow_z \cos\vartheta \right]}_{\vec{n} \cdot \vec{\uparrow}}$$

$$\phi_1 = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\vec{n}\vec{\uparrow}}{r^2}$$

Elektrosztatika anyagi közegben

$$\nabla \underline{e}(\underline{x}) = \frac{1}{\varepsilon_0}\left(\eta_{sz}(\underline{x}) + \eta_h(\underline{x})\right) \qquad \nabla \times \underline{e}(\underline{x}) = 0$$

$$\eta_h(\underline{x}) = \sum_n \sum_{j_n} q_{j_n}\, \delta(\underline{x} - \underline{x}_n - \underline{x}_{j_n})$$

$\underline{x}_n$: rendezés molekulák; $\underline{x}_{j_n}$: tiltott elhatározások

$$\sum_{j_n} q_{j_n} = 0$$

átlagolás: $\underline{E}(\underline{x}) = \langle \underline{e} \rangle(\underline{x}) = \int d^3\underline{x}'\, f(\underline{x} - \underline{x}')\, \underline{e}(\underline{x}')$

f sima, izotróp: $f(\underline{x} - \underline{x}') = \tilde{f}(|\underline{x} - \underline{x}'|)$

$$\int d^3\underline{x}'\, f(\underline{x} - \underline{x}') = 1$$

$\underline{x}'' := \underline{x} - \underline{x}' \Rightarrow \langle \underline{e} \rangle(\underline{x}) = \int d^3\underline{x}''\, f(\underline{x}'')\, \underline{e}(\underline{x} - \underline{x}'')$

$$\langle \underbrace{\nabla \times \underline{e}}_{0} \rangle = \int d^3\underline{x}''\, f(\underline{x}'')\, \nabla_{\underline{x}} \times \underline{e}(\underline{x} - \underline{x}'') =$$

$$= \nabla_{\underline{x}} \times \int d^3\underline{x}''\, f(\underline{x}'')\, \underline{e}(\underline{x} - \underline{x}'') = \nabla \times \underline{E}(\underline{x}) = 0$$

hasonlóan: $\langle \nabla \underline{e} \rangle = \nabla \underline{E}$

$$\varrho(\underline{x}) := \langle \eta_{sz} \rangle(\underline{x})$$

$$\langle \eta_h \rangle(\underline{x}) = \int d^3\underline{x}''\, f(\underline{x}'')\, \eta_e(\underline{x} - \underline{x}'') =$$

$$= \sum_n \sum_{j_n} q_{j_n} \int d^3\underline{x}''\, f(\underline{x}'')\, \delta(\underline{x} - \underline{x}'' - \underline{x}_n - \underline{x}_{j_n})$$

$$\langle \eta_e \rangle (\underline{x}) = \sum_n \sum_{j_n} q_{j_n} f(\underline{x} - \underline{x}_n - \underline{x}_{j_n}) .$$

sorfejtés: $f(\underline{x} - \underline{x}_n - \underline{x}_{j_n}) \approx f(\underline{x} - \underline{x}_n) - \underline{x}_{j_n} \underline{\nabla}_{\underline{x}} f(\underline{x} - \underline{x}_n) + \dots$

$$\langle \eta_e \rangle (\underline{x}) = \sum_n \underbrace{\left(\sum_{j_n} q_{j_n} \right)}_{0} f(\underline{x} - \underline{x}_n) - \sum_n \underbrace{\left(\sum_{j_n} q_{j_n} \underline{x}_{j_n} \right)}_{\underline{p}_n} \underline{\nabla}_{\underline{x}} f(\underline{x} - \underline{x}_n)$$

$$\langle \eta_e \rangle (\underline{x}) = - \sum_n \underline{p}_n \underline{\nabla}_{\underline{x}} f(\underline{x} - \underline{x}_n) = - \underline{\nabla}_{\underline{x}} \left(\sum_n \underline{p}_n f(\underline{x} - \underline{x}_n) \right) =$$

$$= - \underline{\nabla}_{\underline{x}} \left(\sum_n \underline{p}_n \int d^3 x'' \, f(\underline{x}'') \, \delta(\underline{x} - \underline{x}_n - \underline{x}'') \right) =$$

$$= - \underline{\nabla}_{\underline{x}} \underbrace{\left(\int d^3 x'' \, f(\underline{x}'') \sum_n \underline{p}_n \delta(\underline{x} - \underline{x}_n - \underline{x}'') \right)}_{\underline{P}(\underline{x})}$$

$\underline{P}(\underline{x})$ polarizációs vektor

$$\langle \eta_e \rangle (\underline{x}) = - \nabla P(\underline{x})$$

$$\downarrow$$

$$\underline{\nabla} \underline{E} = \frac{1}{\varepsilon_0} (\rho - \underline{\nabla} \underline{P}) \Rightarrow \underline{\nabla} \underbrace{(\varepsilon_0 \underline{E} + \underline{P})}_{\underline{D}} = \rho$$

$\underline{\nabla} \times \underline{E} = 0$ el. eltol.

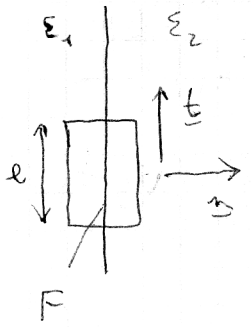
$\underline{E} \longmapsto \underline{P}$ indukált dipólmomentum

lineáris modell: $\underline{P} = \varepsilon_0 \underline{\underline{\chi}}_e \underline{E}$ $\underline{\underline{\chi}}_e$ el. szuszceptibilitás

izotróp közegben: $\underline{\underline{\chi}}_e = \chi_e \underline{\underline{I}}$

$$\underline{D} = \varepsilon_0 \underbrace{(1 + \chi_e)}_{\varepsilon} \underline{E} \qquad \varepsilon = \varepsilon_0 \varepsilon_r$$

Határfeltételek:



$$\operatorname{div} \underline{D} = \varrho \Rightarrow \underbrace{\oint \underline{D}\, d\underline{F}}_{(D_{2n} - D_{1n})F} = \underbrace{\int d^3x\, \varrho(\underline{x})}_{\sigma_{fel} F}$$

$$\boxed{D_{2n} - D_{1n} = \sigma_{fel}}$$

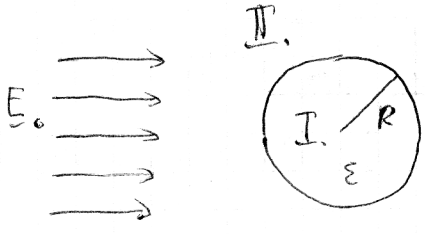
$$\operatorname{rot} \underline{E} = 0 \Rightarrow \underbrace{\oint \underline{E}(\underline{x})\, d\underline{x} = 0}_{(E_{2t} - E_{1t})\, e} \rightarrow \boxed{E_{1t} = E_{2t}}$$

$$\underline{E} = -\underline{\nabla}\phi \qquad \boxed{\phi_1(\underline{x}) = \phi_2(\underline{x})}$$

$$D_{i,n} = -\varepsilon_i \cdot \frac{\partial \phi_i}{\partial n}(\underline{x}) \qquad i \in \{1,2\}$$

$$\boxed{\varepsilon_1 \frac{\partial \phi_1}{\partial n} - \varepsilon_2 \frac{\partial \phi_2}{\partial n} = \sigma_{fel}}$$

alkalmazás: ε dielektromos állandójú gömb homogén E_0 térben



$$\phi_I(R,\vartheta) = \phi_{II}(R,\vartheta)$$

$$\varepsilon \frac{\partial \phi_I}{\partial r}(R,\vartheta) = \varepsilon_0 \frac{\partial \phi_{II}}{\partial n}(R,\vartheta)$$

ill. $r \to \infty$-re: $\phi_{II} \approx -E_0 r P_1(\cos\vartheta)$

$$\left.\begin{aligned} \phi_I &= A r P_1(\cos\vartheta) \\ \phi_{II} &= \left(-E_0 r + \frac{B}{r^2}\right) P_1(\cos\vartheta) \end{aligned}\right\} \text{próbafüggvény}$$

$$A R = -E_0 R + \frac{B}{R^2} \qquad \varepsilon A = \varepsilon_0 \left(-E_0 - \frac{2B}{R^3}\right)$$

$$\Downarrow \qquad \Downarrow$$

$$A = \frac{B}{R^3} - E_0 \qquad \varepsilon_r A = -E_0 - \frac{2B}{R^3}$$

$$\varepsilon_r \left(\frac{B}{R^3} - E_0\right) = -E_0 - \frac{2B}{R^3} \Rightarrow (2+\varepsilon_r)\frac{B}{R^3} = (\varepsilon_r - 1) E_0$$

$$B = \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 E_0 \qquad A = \left(\frac{\varepsilon_r - 1}{\varepsilon_r + 2} - 1\right) E_0 = -\frac{3}{\varepsilon_r + 2} E_0$$

$$\phi_I = -\underline{E}_1 \cdot \underline{x}$$

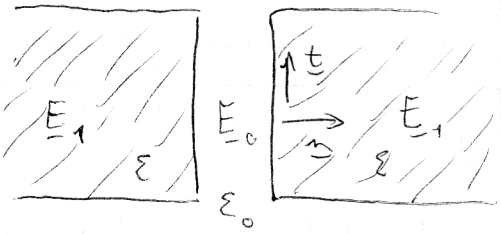
$$\boxed{\underline{E}_1 = \frac{3}{\varepsilon_r + 2} \underline{E}_0}$$

$$\phi_{II} = -\underline{E}_0 \underline{x} + \frac{1}{4\pi\varepsilon_0} \cdot \frac{\underline{n}\,\underline{\mu}}{r^2}$$

$$\boxed{\underline{\mu} = 4\pi\varepsilon_0 \frac{\varepsilon_r - 1}{\varepsilon_r + 2} R^3 \underline{E}_0}$$

$$\underline{n} = \frac{\underline{x}}{|\underline{x}|}$$

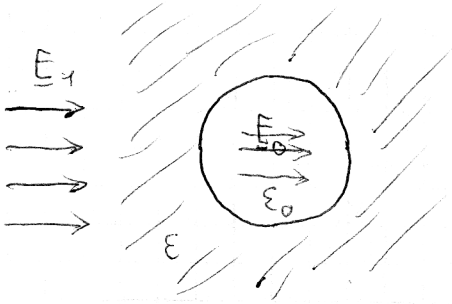
<u>tnyagcleli el. térrész rendre</u> : $\underline{E}_1$ kelyett $\underline{E}_0$-t nézzük



$$\varepsilon_0 E_{0n} = \varepsilon E_{1n} \Rightarrow E_{0n} = \varepsilon_r \cdot E_{1n}$$

$$E_{0t} = E_{1t}$$

$$\underline{E}_0 = \varepsilon_r \underline{n}(\underline{n}\,\underline{E}_1) + \underline{t}(\underline{t}\,\underline{E}_1)$$



$$\varepsilon_0 \to \varepsilon \qquad \varepsilon \to \varepsilon_0 \qquad \varepsilon_r \to \frac{1}{\varepsilon_r}$$

$$\underline{E}_0 \to \underline{E}_1 \qquad \underline{E}_r \to \underline{E}_0$$

$$\underline{E}_0 = \frac{3}{\frac{1}{\varepsilon_r} + 2} \underline{E}_1 = \frac{3\varepsilon_r}{1 + 2\varepsilon_r} \underline{E}_1$$

Polarizálható test elektrosztatikus energiája

$$\delta W_e = \int d^3x\ \phi \cdot \delta \varrho = \int d^3x\ \phi \cdot \nabla(\delta \underline{D}) = \qquad \nabla(\delta \underline{D}) = \delta \varrho$$

$$= \int d^3x\ \nabla(\phi\, \delta \underline{D}) - \int d^3x\ \nabla\phi \cdot \delta \underline{D} \qquad \underline{E} = -\nabla\phi$$

$$\int d\underline{F}\,(\phi\, \delta \underline{D}) = 0 \quad \leftarrow \text{felület a } \infty\text{-ben}$$

$$\boxed{\delta W_e = \int d^3x\ \underline{E} \cdot \delta \underline{D}}$$ igaz állt. esetben

ha $\underline{D} = \varepsilon \underline{E} \Rightarrow \delta \underline{D} = \varepsilon\, \delta \underline{E} \Rightarrow \underline{E}\, \delta \underline{D} = \delta\left(\frac{1}{2} \varepsilon \underline{E}^2\right)$

$$\delta W_e = \delta\left(\int d^3x \left(\frac{1}{2} \varepsilon \underline{E}^2\right)\right)$$

$$\Downarrow$$

$$\boxed{W_e = \int d^3x \left(\frac{1}{2} \underline{E}\, \underline{D}\right)} \qquad \boxed{w_e = \frac{1}{2} \underline{E} \cdot \underline{D}}$$

visszajátszás a formára:

$$W_e = \frac{1}{2} \int d^3x\ \underline{E}\, \underline{D} = -\frac{1}{2} \int d^3x\ \nabla\phi \cdot \underline{D} =$$

$$= -\frac{1}{2} \underbrace{\int d^3x\ \nabla(\phi \cdot \underline{D})}_{0} + \frac{1}{2} \int d^3x\ \phi\, \underbrace{\nabla \underline{D}}_{\varrho}$$

$$\boxed{W_e = \frac{1}{2} \int d^3x\ \phi \cdot \varrho}$$

$$\delta W_e = \frac{1}{2} \int d^3x\ \phi\, \delta \varrho + \frac{1}{2} \int d^3x\ \varrho\, \delta\phi$$

Ezt a kiindulóegyenlettel összevetve: $\boxed{\int d^3x\ \phi\, \delta \varrho = \int d^3x\ \varrho\, \delta\phi}$

Clausius - Mossotti - egyenlet

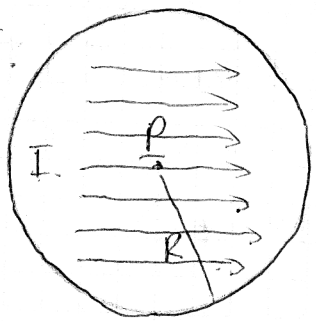
$\underline{p} = \gamma \cdot \underline{e}$ $\quad$ $\underline{P} = \varepsilon_0 \chi_e \underline{E}$ $\quad$ kapcsolat?

$\underline{e} = \underline{E} + \underline{e}_{\text{körüli}} - \underline{E}_{\underline{P}}$ $\quad$ folyós rácsban: $\underline{e}_{\text{körüli}} = 0$

$\underline{e}_{\text{körüli}}$: körüli molekulák tere

$\underline{E}_{\underline{P}}$: elhúzódások $\underline{P}$-vel figyelembe vett sűrített tere

II.



$$\Phi_I = -\underline{E}_{\underline{P}} \cdot \underline{x} = -\underline{E}_{\underline{P}} \, r \, P_1(\cos\vartheta)$$

$$\Phi_{II} = \frac{C}{r^2} P_1(\cos\vartheta) = \frac{1}{4\pi\varepsilon_0} \cdot \frac{\underline{p} \cdot \underline{n}}{r^2}$$

dipól

$$C = \frac{1}{4\pi\varepsilon_0} \cdot \underline{p}$$

$$\underline{p} = \frac{4}{3}\pi R^3 \cdot \underline{P} \quad \Rightarrow \quad \Phi_{II} = \frac{1}{3\varepsilon_0} R^3 \frac{\underline{P} \cdot \underline{x}}{r^3}$$

$$\Phi_I(R) = \Phi_{II}(R) \quad \Rightarrow \quad \underline{E}_{\underline{P}} = -\frac{\underline{P}}{3\varepsilon_0} \quad \Rightarrow \quad \underline{e} = \underline{E} + \frac{\underline{P}}{3\varepsilon_0}$$

egységnyi térfogatban N molekula

$$\underline{P} = N \cdot \underline{p} = N \cdot \gamma \cdot \underline{e} = N\gamma\left(\underline{E} + \frac{\underline{P}}{3\varepsilon_0}\right)$$

$$\varepsilon_0 \chi_e \underline{E} = N\gamma\left(1 + \frac{1}{3}\chi_e\right)\underline{E}$$

$$\boxed{\varepsilon_0 \frac{\chi_e}{1 + \frac{\chi_e}{3}} = N\gamma}$$

makroszkópikus $\quad$ mikroszkópikus

Molekuláris polarizálhatóság modelljei

1) indukált dipólnyomaték → rugalmasan kötött töltések

$$0 = q\underline{E} - m\omega_0^2 \cdot \underline{x} \quad \Rightarrow \quad \underline{p} = q\underline{x} = \frac{q^2}{m\omega_0^2}\underline{E}$$

$$\boxed{\gamma = \frac{q^2}{m\omega_0^2}} \rightarrow \text{több állapot esetén: } \gamma = \sum_i \frac{q_i^2}{m_i \omega_i^2}$$

2) orientációs polarizálhatóság → elektret-modell

$\underline{E} \rightarrow$ $\uparrow$ / $\rightarrow$ $\uparrow$ $\qquad W = e^{-\frac{W}{k_B T}}$ Boltzmann-tényező

$$W = q\,\phi(\underline{x}+\underline{l}) - q\,\phi(\underline{x}) \approx \underbrace{q\underline{l}}_{\uparrow}\underbrace{\nabla\phi(\underline{x})}_{-\underline{E}} = -\underline{p}\underline{E}$$

$$w(\vartheta,\varphi) = e^{\frac{\underline{p}\underline{E}\cos\vartheta}{k_B T}} \qquad \alpha := \frac{pE}{k_B T} \qquad x = \cos\vartheta$$

$$\langle p_z \rangle = p\langle\cos\vartheta\rangle = p\frac{\int d\Omega\, w\cdot\cos\vartheta}{\int d\Omega \cdot w} =$$

$$= p\frac{\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\vartheta)\,\cos\vartheta \cdot e^{\alpha\cdot\cos\vartheta}}{\int_0^{2\pi} d\varphi \int_{-1}^{1} d(\cos\vartheta)\cdot e^{\alpha\cos\vartheta}} =$$

$$= p\frac{\int_{-1}^{1} dx \cdot x\, e^{\alpha x}}{\int_{-1}^{1} dx \cdot e^{\alpha x}} = p\cdot\frac{\frac{d}{d\alpha}\int_{-1}^{1} dx\cdot e^{\alpha x}}{\int_{-1}^{1} dx\cdot e^{\alpha x}} = p\cdot\frac{d}{d\alpha}\left(\ln\int_{-1}^{1} dx\, e^{\alpha x}\right)$$

$$\langle \uparrow_z \rangle = \uparrow \frac{d}{d\alpha}\left(\ln\left(\frac{1}{\alpha}\left(e^{\alpha}-e^{-\alpha}\right)\right)\right) =$$

$$= \uparrow \frac{d}{d\alpha}\left(-\ln\alpha + \ln\left(e^{\alpha}-e^{-\alpha}\right)\right) = \uparrow\left(-\frac{1}{\alpha} + \frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}}\right)$$

sorfejtés: $\frac{e^{\alpha}+e^{-\alpha}}{e^{\alpha}-e^{-\alpha}} = \frac{2\cdot\left(1+\frac{1}{2}\alpha^2\right)}{2\left(\alpha+\frac{1}{6}\alpha^3\right)} = \frac{1}{\alpha}\cdot\frac{1+\frac{1}{2}\alpha^2}{1+\frac{1}{6}\alpha^2} =$

$$= \frac{1}{\alpha}\left(1+\frac{1}{2}\alpha^2\right)\left(1-\frac{1}{6}\alpha^2\right) = \frac{1}{\alpha}\left(1+\frac{1}{3}\alpha^2\right) = \frac{1}{\alpha}+\frac{1}{3}\alpha$$

$$\langle \uparrow_z \rangle = \frac{1}{3}\alpha\cdot\uparrow = \frac{\uparrow^2 E}{3k_B T} \Rightarrow \boxed{\chi = \frac{\uparrow^2}{3k_B T}}$$

Magnetosztatika

$\operatorname{div}\underline{B} = 0 \rightarrow \underline{B} = \operatorname{rot}\underline{A}$ $\qquad \operatorname{div}\underline{A} = 0$

$\operatorname{rot}\underline{B} = \mu_0 \underline{j} \rightarrow \operatorname{grad}(\underbrace{\operatorname{div}\underline{A}}_{0}) - \Delta\underline{A} = \mu_0 \underline{j}$

$\Delta\underline{A} = -\mu_0 \underline{j}$ → végtelen térfogatban:

$$\boxed{\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\underline{j}(\underline{x}')}{|\underline{x}-\underline{x}'|}} \qquad \operatorname{div}\underline{A}(\underline{x}) \overset{?}{=} 0$$

$$\nabla_{\underline{x}}\underline{A} = \frac{\mu_0}{4\pi}\int d^3x' \, \underline{j}(\underline{x}')\,\nabla_{\underline{x}}\frac{1}{|\underline{x}-\underline{x}'|} =$$

$$= -\frac{\mu_0}{4\pi}\int d^3x' \, \underline{j}(\underline{x}')\,\nabla_{\underline{x}'}\frac{1}{|\underline{x}-\underline{x}'|} =$$

$$= -\frac{\mu_0}{4\pi}\underbrace{\int d^3x' \, \nabla_{\underline{x}'}\left(\frac{\underline{j}(\underline{x}')}{|\underline{x}-\underline{x}'|}\right)}_{0} + \frac{\mu_0}{4\pi}\int d^3x' \, \frac{\nabla_{\underline{x}'}\underline{j}(\underline{x}')}{|\underline{x}-\underline{x}'|}$$

$$\nabla_{\underline{x}} \underline{A} = \frac{\mu_0}{4\pi} \int d^3x' \frac{\nabla_{\underline{x}'} \underline{j}(\underline{x}')}{|\underline{x} - \underline{x}'|} = 0, \text{ ha } \operatorname{div} \underline{j} = 0$$

$$B_i(\underline{x}) = \varepsilon_{ijk}\, \partial_j A_k(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \varepsilon_{ijk}\, \partial_j \left(\frac{1}{|\underline{x} - \underline{x}'|} \right) \cdot j_k(\underline{x}') =$$

$$= -\frac{\mu_0}{4\pi} \int d^3x' \, \varepsilon_{ijk} \frac{(\underline{x} - \underline{x}')_j}{|\underline{x} - \underline{x}'|^3} \, j_k(\underline{x}') =$$

$$= -\frac{\mu_0}{4\pi} \int d^3x' \frac{\left((\underline{x} - \underline{x}') \times \underline{j}(\underline{x}') \right)_i}{|\underline{x} - \underline{x}'|^3} \Rightarrow \boxed{\underline{B}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\underline{j}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}}$$

"vékony áramvezető": $\underline{j} = I \cdot \underline{t}\, \delta^{(2)}_{\Gamma}(\underline{x}' - \underline{x}_0)$

$$\int dF \cdot \underline{j} = I \cdot \underline{t} \qquad d^3x = d\underline{x}_0 \cdot dF$$

$$\boxed{\underline{B}(\underline{x}) = \frac{\mu_0 I}{4\pi} \int \frac{d\underline{x}_0 \times (\underline{x} - \underline{x}_0)}{|\underline{x} - \underline{x}_0|^3}} \quad \text{Biot - Savart}$$

Multipól sorfejtés

$$\frac{1}{|\underline{x} - \underline{x}'|} = \frac{1}{r} + \frac{\underline{r}}{r^2} \cdot \underline{x}' + \dots$$

$$\underline{A}^{(0)}(\underline{x}) = \frac{\mu_0}{4\pi} \cdot \frac{1}{r} \int d^3x' \, \underline{j}(\underline{x}') = 0 \leftarrow \text{állítás}$$

azonosság: $\int d^3x' \, \nabla_{\underline{x}'} \left(f(\underline{x}')\, g(\underline{x}')\, \underline{j}(\underline{x}') \right) = 0$

$$\int d^3\underline{x}' \left(\underline{j}\cdot g\,\underline{\nabla}_{\underline{x}'} f + \underline{j} f\,\underline{\nabla}_{\underline{x}'} g + f g \underbrace{\underline{\nabla}_{\underline{x}'}\underline{j}}_{0} \right) = 0$$

$$\int d^3x' \left(j_i\, g\,\frac{\partial f}{\partial x_i'} + j_i f \frac{\partial g}{\partial x_i'} \right) = 0$$

$$f := x_k' \qquad g := 1 \quad \Rightarrow \int d^3x'\; j_k(\underline{x}') = 0$$

$$\underline{A}^{(1)}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{(\underline{n}\cdot\underline{x}')\,\underline{j}(\underline{x}')}{r^2}$$

$$A_k^{(1)}(\underline{x}) = \frac{\mu_0}{4\pi}\cdot\frac{n_l}{r^2}\int d^3x' \cdot x_l'\, j_k(\underline{x}')$$

$$\int d^3x' \cdot x_l' \cdot j_k(\underline{x}') = \frac{1}{2}\underbrace{\int d^3x' \left(x_l' j_k + x_k' j_l \right)}_{\text{simetrico}} + \frac{1}{2}\underbrace{\int d^3x' \left(x_l' j_k - x_k' j_l \right)}_{\text{antisimetrico}}$$

$$f := x_k' \qquad g := x_l'$$

$$\int d^3x' \left(j_i\, x_l'\,\delta_{ki} + j_i\, x_k'\,\delta_{li} \right) = 0$$

$$\int d^3x' \left(j_k\, x_l' + j_l\, x_k' \right) = 0 \qquad \text{simetrico tog}$$

$$A_k^{(1)}(\underline{x}) = \frac{\mu_0}{4\pi}\,\frac{n_l}{r^2}\cdot\frac{1}{2}\int d^3x' \left(x_l' j_k - x_k' j_l \right)$$

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi}\cdot\frac{1}{2r^2}\int d^3x' \underbrace{\left[(\underline{n}\cdot\underline{x}')\,\underline{j} - (\underline{n}\cdot\underline{j})\,\underline{x}' \right]}_{\underline{n}\times(\underline{j}\times\underline{x}')}$$

$$\underline{A}^{(1)}(\underline{x}) = -\frac{\mu_0}{4\pi}\frac{\underline{n}}{r^2} \times \underbrace{\frac{1}{2}\int d^3x' \left(\underline{\dot{x}} \times \underline{j}(\underline{\dot{x}})\right)}_{\underline{m}}$$

$\underline{m}$: mágneses dipólmomentum

$$\boxed{\underline{A}^{(1)}(\underline{x}) = \frac{\mu_0}{4\pi}\frac{\underline{m}\times\underline{n}}{r^2}} \qquad \boxed{\underline{m} = \frac{1}{2}\int d^3x' \left(\underline{\dot{x}} \times \underline{j}(\underline{\dot{x}})\right)}$$

Példa: M_i tömegű, q_i töltésű, $\underline{v}_i$ sebességű részecskék

$$\underline{j}(\underline{\dot{x}}) = \sum_i q_i \underline{v}_i \delta(\underline{\dot{x}} - \underline{x}_i)$$

$$\boxed{\underline{m} = \frac{1}{2}\sum_i q_i \int d^3x' \, \underline{\dot{x}} \times \underline{v}_i \, \delta(\underline{\dot{x}} - \underline{x}_i)} = \frac{1}{2}\sum_i q_i \underline{x}_i \times \underline{v}_i =$$

$$= \frac{1}{2}\sum_i \frac{q_i}{M_i}\left(\underline{x}_i \times \underline{p}_i\right) = \boxed{\sum_i \frac{q_i}{2M_i}\underline{L}_i}$$

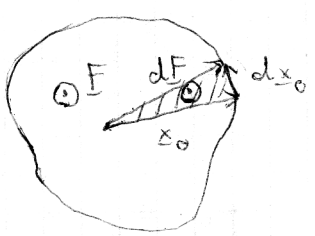
e^- : $L_z = n\hbar \quad n \in \mathbb{Z} \qquad m_z = \underbrace{\frac{e\hbar}{2M}}_{\mu_B} \cdot n$

μ_B Bohr-magneton

vékony áramvezető: $\underline{j}(\underline{\dot{x}}) = I \cdot \underline{t}(\underline{x}_0)\,\delta_1^{(2)}(\underline{\dot{x}} - \underline{x}_0)$

$$\boxed{\underline{m} = \frac{1}{2} I \int \underline{x}_0 \times d\underline{x}_0 = \underline{I}\cdot\underline{F}}$$

$$d\underline{F} = \frac{1}{2}\underline{x}_0 \times d\underline{x}_0$$



Mágnesezhető anyagi közegben

$\underline{M} = \sum_i N_i \cdot m_i$ mágnesezettség

$\Downarrow$

egységnyi térfogatban

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\underline{j}(\underline{x}')}{|\underline{x} - \underline{x}'|} + \frac{\mu_0}{4\pi} \int d^3x' \frac{\underline{M}(\underline{x}') \times (\underline{x} - \underline{x}')}{|\underline{x} - \underline{x}'|^3}$$

szabad áramok — közegbeli mágn. dipólusok

$$\frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3} = \underline{\nabla}_{\underline{x}'} \left(\frac{1}{|\underline{x} - \underline{x}'|} \right)$$

$$\left(\int d^3x' \, \underline{M}(\underline{x}') \times \underline{\nabla}_{\underline{x}'} \frac{1}{|\underline{x} - \underline{x}'|} \right)_i = \varepsilon_{ijk} \int d^3x' \, M_j(\underline{x}') \, \partial'_k \left(\frac{1}{|\underline{x} - \underline{x}'|} \right) =$$

$\uparrow$ parc. int.

$$= - \varepsilon_{ijk} \int d^3x' \frac{1}{|\underline{x} - \underline{x}'|} \partial'_k M_j(\underline{x}') = \int d^3x' \frac{\varepsilon_{ikj} \partial'_k M_j(\underline{x}')}{|\underline{x} - \underline{x}'|} =$$

$$= \left(\int d^3x' \frac{\mathrm{rot}\, \underline{M}(\underline{x}')}{|\underline{x} - \underline{x}'|} \right)_i$$

$\rightarrow$ összevonható az 1. taggal:

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \frac{\overbrace{\underline{j}(\underline{x}') + \mathrm{rot}\, \underline{M}(\underline{x}')}^{\underline{j}_{\mathrm{eff}}(\underline{x}')}}{|\underline{x} - \underline{x}'|}$$

$$\mathrm{div}\, \underline{B} = 0 \qquad \mathrm{rot}\, \underline{B} = \mu_0 \underline{j}_{\mathrm{eff}} = \mu_0 (\underline{j} + \mathrm{rot}\, \underline{M})$$

$$\Rightarrow \mathrm{rot} \underbrace{\left(\frac{1}{\mu_0} \underline{B} - \underline{M} \right)}_{\underline{H}} = \underline{j}$$

$\underline{H}$ mágneses térerősség

$\operatorname{div} \underline{B} = 0 \qquad \operatorname{rot} \underline{H} = \underline{j} \qquad \underline{B} = \mu_0 (\underline{H} + \underline{M})$

lineáris modell: $\underline{M} = \chi_m \cdot \underline{H}$

↳ mágneses szuszceptibilitás

- paramágn. $\chi_m > 0$
- diamágn. $\chi_m < 0$

$\underline{B} = \mu_0 \underbrace{(1 + \chi_m)}_{\mu_r} \underline{H} \qquad \mu = \mu_0 \mu_r$ mágneses permeabilitás

spec. eset: $\underline{j} = 0$

$\operatorname{rot} \underline{H} = 0 \qquad \operatorname{div} \underline{B} = \mu_0 (\operatorname{div} \underline{H} + \operatorname{div} \underline{M}) = 0$

$\operatorname{div} \underline{H} = - \operatorname{div} \underline{M} = \rho_{mágn.}$ analóg az elektrosztatikával

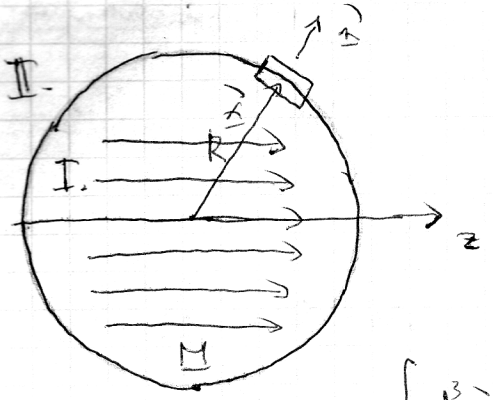
$\underline{H} = - \operatorname{grad} \phi_{mágn.} \qquad \Delta \phi_{mágn.} = - \rho_{mágn.} = \operatorname{div} \underline{M}$

$$\phi_{mágn.}(\underline{x}) = + \frac{1}{4\pi} \int d^3x' \, \frac{\rho_{mágn.}(\underline{x}')}{|\underline{x} - \underline{x}'|} = - \frac{1}{4\pi} \int d^3x' \, (\nabla_{\underline{x}'} \underline{M}) \cdot \frac{1}{|\underline{x} - \underline{x}'|} =$$

$$= \frac{1}{4\pi} \int d^3x' \, \underline{M}(\underline{x}') \underbrace{\nabla_{\underline{x}'} \left(\frac{1}{|\underline{x} - \underline{x}'|} \right)}_{- \nabla_{\underline{x}} \frac{1}{|\underline{x} - \underline{x}'|} = \frac{\underline{x} - \underline{x}'}{|\underline{x} - \underline{x}'|^3}} = - \frac{1}{4\pi} \nabla_{\underline{x}} \left(\int d^3x' \, \frac{\underline{M}(\underline{x}')}{|\underline{x} - \underline{x}'|} \right)$$

$\underline{M}(\underline{x}') = \underline{m} \, \delta(\underline{x}') \Rightarrow \phi_{mágn.} = - \frac{1}{4\pi} \nabla_{\underline{x}} \left(\frac{\underline{m}}{|\underline{x}|} \right) = \frac{1}{4\pi} \cdot \frac{\underline{m} \, \underline{x}}{|\underline{x}|^3}$

$\Rightarrow$ mágneses dipólus analóg az elektromossal



$\rho_{mágn} = -\operatorname{div} \underline{M}$

$\sigma_{mágn} = ?$ felületi töltéssűrűség

kis, lapos téglatestre:

$$\int d^3\grave{x}\ \rho_m(\underline{\grave{x}}) = -\int d^3\grave{x}\ \operatorname{div}\underline{M}(\underline{\grave{x}}) =$$

$$= -\oint d\underline{\grave{F}}\cdot\underline{M}(\underline{\grave{x}}) = +\int d\grave{F}\cdot \underbrace{\underline{\grave{n}}\,\underline{M}(\underline{\grave{x}})}_{\sigma_{mágn}(\underline{\grave{x}})}$$

csak az alsó lap ad járulékot

$$\boxed{\sigma_{mágn}(\underline{\grave{x}}) = \underline{\grave{n}}\,\underline{M}(\underline{\grave{x}})}$$

$$\Phi_{mágn}(\underline{x}) = \frac{1}{4\pi}\int d\grave{F}\ \frac{\underline{\grave{n}}\cdot\underline{M}}{|\underline{x}-\underline{\grave{x}}|} = \frac{1}{4\pi}\int R^2\, d\grave{\Omega}\ \frac{M\cdot\cos\grave{\vartheta}}{|\underline{x}-\underline{\grave{x}}|} =$$

$$= \frac{MR^2}{4\pi}\int d\grave{\Omega}\ \underbrace{P_1(\cos\grave{\vartheta})}_{\sqrt{\frac{4\pi}{3}}Y_{10}(\underline{\grave{n}})}\cdot\sum_\ell \frac{4\pi}{2\ell+1}\cdot\frac{r_<^\ell}{r_>^{\ell+1}}\sum_m Y^*_{\ell m}(\underline{\grave{n}})\,Y_{\ell m}(\underline{n}) =$$

$$= \frac{MR^2}{4\pi}\sqrt{\frac{4\pi}{3}}\sum_{\ell,m}\frac{4\pi}{2\ell+1}\,\frac{r_<^\ell}{r_>^{\ell+1}}\,Y_{\ell m}(\underline{n})\underbrace{\int d\grave{\Omega}\ Y_{10}(\underline{\grave{n}})\,Y^*_{\ell m}(\underline{\grave{n}})}_{\delta_{1\ell}\,\delta_{0m}} =$$

$$= \frac{MR^2}{4\pi}\cdot\frac{4\pi}{3}\cdot\frac{r_<}{r_>^2}\sqrt{\frac{4\pi}{3}}\,Y_{10}(\underline{n}) = \frac{MR^2}{3}\,\frac{r_<}{r_>^2}\,P_1(\cos\vartheta)$$

$r_< = \min(|\underline{x}|, |\underline{\grave{x}}|) = \min(r, R)$ $\qquad$ $r_> = \max(r, R)$

külső tér: $\Phi_{II} = \frac{MR^2}{3} \cdot \frac{R}{r^2} \cos\vartheta = \frac{1}{4\pi} \frac{\underline{m} \cdot \underline{n}}{r^2}$ dipólus

$$\underline{m} = \frac{4\pi}{3} R^3 \underline{M}$$

belső tér: $\Phi_I = \frac{MR^2}{3} \cdot \frac{r}{R^2} \cos\vartheta = \frac{M}{3} \cdot z = \frac{\underline{M} \cdot \underline{x}}{3}$ homogén tér

$$\underline{H}_I = -\frac{\underline{M}}{3} \qquad \underline{B}_I = \mu_0 (\underline{H}_I + \underline{M}) = \frac{2\mu_0}{3} \underline{M}$$

Szuperponáljunk erre külső, homogén $\underline{B}_0$ teret $\quad \underline{H}_0 = \frac{1}{\mu_0} \underline{B}_0$

$$\underline{B}_{ként} = \underline{B}_0 + \frac{2\mu_0}{3} \underline{M}$$

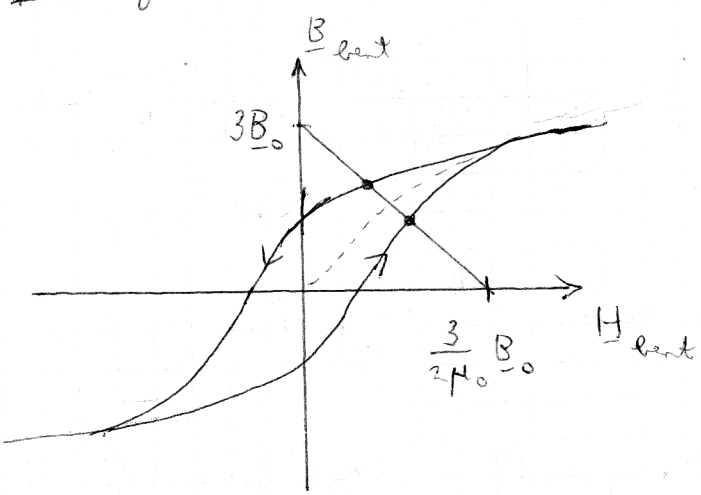
$$+ \quad \underline{H}_{ként} = \frac{1}{\mu_0} \underline{B}_0 - \frac{1}{3} \underline{M} \quad / \cdot 2\mu_0$$

$$\boxed{\underline{B}_{ként} + 2\mu_0 \underline{H}_{ként} = 3 \underline{B}_0}$$

Levezetésben nem használtuk $\underline{M}$ és $\underline{B}_0$ közötti kapcsolatra von. semmit

$\Rightarrow$ ferromágnesre is igaz (csak a geometriától függ)

ferromágnes



Szupravezetők

Meissner - effektus: $\underline{B} = 0 \Rightarrow \underline{M} = -\underline{H}$

$\chi_m = -1$ tökéletes diamágnes

1938 London - fivérek: árnyat nálliló részecskék szabadon gyorsulnak

$$m^* \frac{\partial \underline{v}}{\partial t} = e^* \cdot \underline{E} \qquad \underline{j} = n^* \cdot e^* \cdot \underline{v}$$

$$\Rightarrow \frac{\partial \underline{j}}{\partial t} = n^* \cdot \frac{e^{*2}}{m^*} \cdot \underline{E}$$

Coulomb - mértékben: $\varrho = 0 \Rightarrow \Phi = 0 \Rightarrow \underline{E} = -\frac{\partial \underline{A}}{\partial t}$

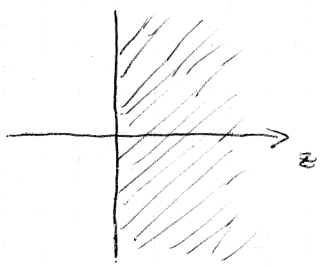
$$\boxed{\underline{j} = -n^* \cdot \frac{e^{*2}}{m^*} \cdot \underline{A}}$$

$\operatorname{rot} \underline{H} = \underline{j} + \frac{\partial \underline{D}}{\partial t}$ ← elhanyagolható $\qquad \underline{B} = \mu \underline{H} = \operatorname{rot} \underline{A}$

$$\underbrace{\operatorname{rot}(\operatorname{rot} \underline{A})}_{\operatorname{grad}(\operatorname{div} \underline{A}) - \Delta \underline{A}} = \mu \cdot \underline{j} \qquad \Rightarrow \Delta \underline{A} = \underbrace{\mu n^* \cdot \frac{e^{*2}}{m^*}}_{\frac{1}{\Lambda^2}} \cdot \underline{A}$$

$\operatorname{div} \underline{A} = 0$ (vesz. az egyenlet teljesül $\underline{E}$-re és $\underline{B}$-re)

$$\boxed{\Delta \underline{A} = \frac{1}{\Lambda^2} \underline{A}} \Rightarrow \underline{A}(z) = \underline{A}(0) \cdot e^{-\frac{z}{\Lambda}}$$



Λ a mágneses tér behatolási mélysége

$$\frac{1}{\Lambda^2} = \mu n^* \cdot \frac{e^{*2}}{m^*} = \mu n \cdot \frac{e^2}{m}$$

nálliló részecskék: Cooper - párok

$$e^* = 2e \qquad m^* = 2m \qquad n^* = \frac{1}{2} n$$

Mozgási indukció

$$\Phi_m = \int_F \underline{B}\, d\underline{F}$$

$$\frac{d\Phi_m}{dt} = \int_F d\underline{F} \underbrace{\left(\frac{\partial \underline{B}}{\partial t} + (\underline{v}\nabla)\underline{B} \right)}_{\frac{d\underline{B}}{dt}}$$

ahol $\underline{v}$ = állandó

$$\left(rot(\underline{v}\times\underline{B})\right)_i = \varepsilon_{ijk}\,\partial_j\left(\varepsilon_{klm}\, v_l\, B_m\right) =$$

$$= \underbrace{\varepsilon_{ijk}\,\varepsilon_{lmk}}_{\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}}\, v_l\,\partial_j B_m = v_i\,\partial_j B_j - v_j\,\partial_j B_i =$$

$$= \left(\underline{v}\,\underbrace{div\,\underline{B}}_{0} - (\underline{v}\nabla)\underline{B}\right)_i \Rightarrow (\underline{v}\nabla)\underline{B} = -rot(\underline{v}\times\underline{B})$$

$$\frac{\partial \underline{B}}{\partial t} = -rot\,\underline{E}$$

$$\boxed{\frac{d\Phi_m}{dt} = -\int_F d\underline{F}\, rot\left(\underline{E}+\underline{v}\times\underline{B}\right) = -\int_S \left(\underline{E}+\underline{v}\times\underline{B}\right) d\underline{x}}$$

Az áramhúrkör rögzített rendszerben: $\frac{d\Phi_m}{dt} = -\int_S \underline{\dot{E}}\, d\underline{x}$

, ahol $\Phi_m = \int_F \underline{\dot{B}}\, d\underline{F}$ + feltevés: $\underline{\dot{B}} = \underline{B}$

Galilei-transzformáció:

$$\underline{\dot{E}} = \underline{E} + \underline{v}\times\underline{B}$$

$$\underline{\dot{B}} = \underline{B}$$

$\Rightarrow$ Lorentz-erő:

$$\underline{F} = \underline{\dot{F}} = q\,\underline{\dot{E}} = q\left(\underline{E}+\underline{v}\times\underline{B}\right)$$

Mágneses tér energiája

$$\delta W_m = -\int d^3x \, \underbrace{j E}_{\text{energiadisszipáció teljesítménysűrűsége}} \delta t$$

Coulomb-mértékben: $\varrho = 0 \Rightarrow \Phi = 0 \Rightarrow \underline{E} = -\frac{\partial \underline{A}}{\partial t}$

$$\delta \underline{A} = -\underline{E}\,\delta t \Rightarrow \boxed{\delta W_m = \int d^3x \, j \, \delta \underline{A}}$$

spec. eset: <u>vékony áramvezető</u> $\quad j(\underline{x}) = I \cdot \underline{t}(\underline{x}_0)\, \delta^{(2)}_{\perp}(\underline{x} - \underline{x}_0)$

$$\boxed{\delta W_m = I \cdot \oint_S \delta \underline{A}(\underline{x}_0)\, d\underline{x}_0 = I \cdot \int_F \underbrace{\mathrm{rot}(\delta \underline{A})}_{\delta \underline{B}} d\underline{F} = I \cdot \delta \Phi_m}$$

$$\delta W_m = \int d^3x \, (\mathrm{rot}\, \underline{H}) \cdot \delta \underline{A}$$

$$\mathrm{div}(\underline{H} \times \delta \underline{A}) = \partial_i (\varepsilon_{ijk} H_j \delta A_k) = \varepsilon_{ijk} \delta A_k \partial_i H_j + \varepsilon_{ijk} H_j \partial_i \delta A_k =$$

$$= \delta A_k \cdot \varepsilon_{kij} \partial_i H_j - H_j \varepsilon_{jik} \partial_i (\delta A_k) = \delta \underline{A}\, \mathrm{rot}\, \underline{H} - \underline{H}\, \mathrm{rot}(\delta \underline{A})$$

$$\boxed{\delta W_m = \underbrace{\int d^3x \, \mathrm{div}(\underline{H} \times \delta \underline{A})}_{0} + \int d^3x \, \underline{H} \underbrace{(\mathrm{rot}\, \delta \underline{A})}_{\delta \underline{B}} = \int d^3x \, \underline{H}\, \delta \underline{B}}$$

igaz általában is

<u>ha $\underline{B} = \mu \underline{H}$</u> $\Rightarrow \underline{H}\, \delta \underline{B} = \mu \underline{H}\, \delta \underline{H} = \delta\left(\frac{1}{2} \mu \underline{H}^2\right)$

$$\delta W_m = \delta \int d^3x \left(\frac{1}{2} \mu \underline{H}^2\right) \Rightarrow \boxed{W_m = \int d^3x \left(\frac{1}{2} \underline{H}\, \underline{B}\right)}$$

$$\boxed{w_m = \frac{1}{2} \underline{H}\, \underline{B}}$$

visszajátcsás a forrásra:

$$W_m = \frac{1}{2}\int d^3x\ \underline{H}\,\underline{B} = \frac{1}{2}\int d^3x\ \underline{H}\ \mathrm{rot}\,\underline{A} =$$

$$= \frac{1}{2}\int d^3x\ \underline{A}\,\underbrace{\mathrm{rot}\,\underline{H}}_{\dot{\underline{E}}} - \frac{1}{2}\underbrace{\int d^3x\ \mathrm{div}(\underline{H}\times\underline{A})}_{0}$$

$$\boxed{W_m = \frac{1}{2}\int d^3x\ \underline{A}\,\dot{\underline{E}}}$$

Időfüggő elektromos polarizálhatóság

$$m\left(\ddot{\underline{x}} + \alpha\dot{\underline{x}} + \omega_0^2\underline{x}\right) = q\underline{E}(t) \qquad / \cdot e^{i\omega t} \qquad / \int dt$$

$$m\left(-\omega^2 - i\alpha\omega + \omega_0^2\right)\tilde{\underline{x}}(\omega) = q\,\tilde{\underline{E}}(\omega)$$

$$\tilde{\underline{p}}(\omega) = q\,\tilde{\underline{x}}(\omega) = \underbrace{\frac{q^2}{m\left(\omega_0^2 - \omega^2 - i\alpha\omega\right)}}_{\tilde{\gamma}(\omega)}\tilde{\underline{E}}(\omega)$$

$$\Rightarrow \boxed{\varepsilon_0\tilde{\chi}_e(\omega) = N\tilde{\gamma}(\omega) = \frac{Nq^2}{m\left(\omega_0^2 - \omega^2 - i\alpha\omega\right)}}$$

$$\boxed{\tilde{\underline{P}}(\omega) = \varepsilon_0\tilde{\chi}_e(\omega)\,\tilde{\underline{E}}(\omega)}$$

$$\underline{P}(t) = \int \frac{d\omega}{2\pi}\,\varepsilon_0\tilde{\chi}_e(\omega)\,\underbrace{\tilde{\underline{E}}(\omega)}_{\int d\tau\ \underline{E}(\tau)e^{i\omega\tau}}\,e^{-i\omega t}$$

$$\underline{P}(t) = \int d\tau\ \underline{E}(\tau)\ \varepsilon_0 \underbrace{\int \frac{d\omega}{2\pi}\ \tilde{\chi}_e(\omega)\ e^{-i\omega(t-\tau)}}_{\chi_e(t-\tau)}$$

$$\boxed{\underline{P}(t) = \int_{-\infty}^{+\infty} d\tau\ \varepsilon_0\, \chi_e(t-\tau)\, \underline{E}(\tau)}$$

Kauzalitás: $\chi_e(\tau) = 0$, ha $\tau < 0$

↓ a későbbiek az $\alpha > 0$ esetén teljesül

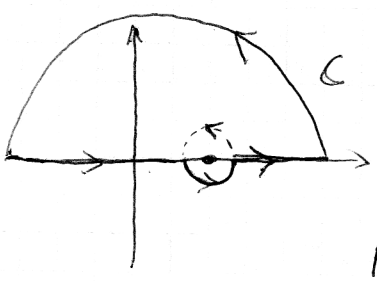
Következmény: ált. esetben:

$$\tilde{\chi}_e(\omega) = \int_0^{\infty} dt\ \chi_e(t)\ e^{i\omega t}$$

$$\omega = \omega' + i\omega'' \qquad e^{i\omega t} = e^{i\omega' t} \cdot e^{-\omega'' t}$$

$t > 0 \Rightarrow \omega'' > 0$ esetén $\tilde{\chi}_e(\omega)$ analitikus

$\omega \in \mathbb{R}$ $\qquad \tilde{\chi}_e(\omega) = \frac{1}{2\pi i} \cdot \oint_C \frac{\tilde{\chi}_e(\omega')}{\omega' - \omega}\, d\omega'$



Következtetés:

$$P\int_{-\infty}^{+\infty} \frac{\tilde{\chi}_e(\omega')}{\omega' - \omega}\, d\omega' := \lim_{\varepsilon \to 0} \left(\int_{-\infty}^{\omega-\varepsilon} + \int_{\omega+\varepsilon}^{+\infty} \right) \frac{\tilde{\chi}_e(\omega')}{\omega' - \omega}\, d\omega'$$

$$\int_{\text{félkör}} \frac{\tilde{\chi}_e(\omega')}{\omega - \omega'}\, d\omega' = \frac{1}{2} \oint_{\text{teljes kör}} \frac{\tilde{\chi}_e(\omega')}{\omega - \omega'}\, d\omega' = \frac{1}{2} \cdot 2\pi i \cdot \tilde{\chi}_e(\omega)$$

$$\chi_e(\omega) = \frac{1}{2\pi i} P \int_{-\infty}^{+\infty} \frac{\chi_e(\omega')}{\omega' - \omega} d\omega' + \frac{1}{2}\chi_e(\omega)$$

$$\chi_e(\omega) = \frac{1}{\pi i} P \int_{-\infty}^{+\infty} \frac{\chi_e(\omega')}{\omega' - \omega} d\omega'$$

$$\mathrm{Re}\,\chi_e(\omega) = \frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Im}\,\chi_e(\omega')}{\omega' - \omega} d\omega'$$

$$\mathrm{Im}\,\chi_e(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{+\infty} \frac{\mathrm{Re}\,\chi_e(\omega')}{\omega' - \omega} d\omega'$$

Kramers - Kronig - relációk

Közegbeli Maxwell-egyenletek

$\rho_{össz} = \rho_{sz} + \rho_{pol}$

$\vec{j}_{össz} = \vec{j}_{sz} + \vec{j}_{pol} + \vec{j}_{mágn}$

$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\vec{j} = 0$ tagonként teljesül

$\vec{j}_{mágn} = \mathrm{rot}\,\vec{M} \Rightarrow \mathrm{div}\,\vec{j}_{mágn} = 0$

$\rho_{pol} = -\mathrm{div}\,\vec{P} \Rightarrow \frac{\partial \rho_{pol}}{\partial t} + \mathrm{div}\,\vec{j}_{pol} = \mathrm{div}\left(\vec{j}_{pol} - \frac{\partial \vec{P}}{\partial t}\right) = 0$

$\Rightarrow \vec{j}_{pol} = \frac{\partial \vec{P}}{\partial t}$

$$\mathrm{rot}\,\vec{B} = \mu_0\left(\vec{j}_{sz} + \mathrm{rot}\,\vec{M} + \frac{\partial \vec{P}}{\partial t} + \varepsilon_0 \frac{\partial \vec{E}}{\partial t}\right)$$

$$\mathrm{rot}\underbrace{\left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right)}_{\vec{H}} = \vec{j}_{sz} + \frac{\partial}{\partial t}\underbrace{\left(\varepsilon_0 \vec{E} + \vec{P}\right)}_{\vec{D}}$$

$$\operatorname{div}\underline{E} = \frac{1}{\varepsilon_0}\left(\varrho_{sz} - \operatorname{div}\underline{P}\right) \Rightarrow \operatorname{div}\underbrace{\left(\varepsilon_0\underline{E} + \underline{P}\right)}_{\underline{D}} = \varrho_{sz}$$

+ Maxwell egyenletek: $\operatorname{div}\underline{B} = 0 \qquad \operatorname{rot}\underline{E} = -\frac{\partial \underline{B}}{\partial t}$

Időbeli Fourier-transzformált: $\frac{\partial}{\partial t} \to -i\omega$

$$\operatorname{rot}\tilde{\underline{H}}(\underline{x},\omega) = \tilde{\underline{j}}_{sz}(\underline{x},\omega) - i\omega\,\tilde{\underline{D}}(\underline{x},\omega) \qquad \operatorname{div}\tilde{\underline{B}}(\underline{x},\omega) = 0$$

$$\operatorname{rot}\tilde{\underline{E}}(\underline{x},\omega) = i\omega\,\tilde{\underline{B}}(\underline{x},\omega) \qquad \operatorname{div}\tilde{\underline{D}}(\underline{x},\omega) = \tilde{\varrho}_{sz}(\underline{x},\omega)$$

+ anyagi egyenletek: $\tilde{\underline{D}}(\underline{x},\omega) = \tilde{\varepsilon}(\omega)\,\tilde{\underline{E}}(\underline{x},\omega)$

$$\tilde{\underline{B}}(\underline{x},\omega) = \tilde{\mu}(\omega)\,\tilde{\underline{H}}(\underline{x},\omega)$$

Energiasűrűségek: (rögzített ω frekvencián)

$$\underline{E}(\underline{x},t) = \operatorname{Re}\left(\underline{E}(\underline{x})\,e^{-i\omega t}\right) = \frac{1}{2}\left[\underline{E}(\underline{x})\,e^{-i\omega t} + \underline{E}^*(\underline{x})\,e^{i\omega t}\right]$$

energiadisszipáció teljesítménysűrűsége:

$$\overline{\underline{j}(\underline{x},t)\,\underline{E}(\underline{x},t)} = \overline{\frac{1}{4}\left(\underline{j}(\underline{x})\,e^{-i\omega t} + \underline{j}^*(\underline{x})\,e^{i\omega t}\right)\left(\underline{E}(\underline{x})\,e^{-i\omega t} + \underline{E}^*(\underline{x})\,e^{i\omega t}\right)}$$

$$= \frac{1}{4}\left(\underline{j}(\underline{x})\,\underline{E}^*(\underline{x}) + \underline{j}^*(\underline{x})\,\underline{E}(\underline{x})\right) = \frac{1}{2}\operatorname{Re}\left(\underline{j}^*(\underline{x})\,\underline{E}(\underline{x})\right)$$

$$-\frac{1}{2}\underline{j}^*(\underline{x})\,\underline{E}(\underline{x}) = -\frac{1}{2}\left(\operatorname{rot}\underline{H}^*(\underline{x}) - i\omega\,\underline{D}^*(\underline{x})\right)\underline{E}(\underline{x}) =$$

$$= -\frac{1}{2}\underline{E}\operatorname{rot}\underline{H}^* + \frac{1}{2}i\omega\,\underline{E}\,\underline{D}^*$$

$$\operatorname{div}\left(\underline{E}\times\underline{H}^*\right) = \underline{H}^*\operatorname{rot}\underline{E} - \underline{E}\operatorname{rot}\underline{H}^*$$

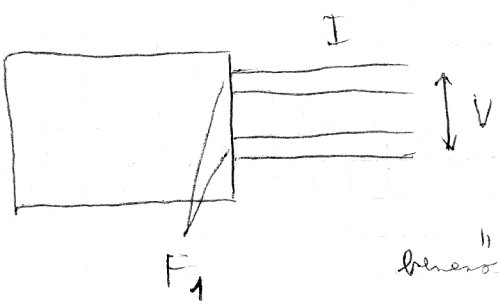
$$-\frac{1}{2}\underline{j}^*\underline{E} = -\frac{1}{2}\underline{H}^*\underbrace{\operatorname{rot}\underline{E}}_{i\omega\underline{B}} + \frac{1}{2}\operatorname{div}\left(\underline{E}\times\underline{H}^*\right) + \frac{1}{2}i\omega\underline{E}\,\underline{D}^*$$

$$\int_V d^3x\ \frac{1}{2}i\omega\left(\underline{E}\,\underline{D}^* - \underline{B}\,\underline{H}^*\right) + \oint_F d\underline{F}\left(\frac{1}{2}\underline{E}\times\underline{H}^*\right) = -\int_V d^3x\ \frac{1}{2}\underline{j}^*\underline{E}$$

$$w_e = \frac{1}{4}\underline{E}\,\underline{D}^* \qquad w_m = \frac{1}{4}\underline{B}\,\underline{H}^* \qquad \underline{S} = \frac{1}{2}\underline{E}\times\underline{H}^*$$

$$\int_V d^3x\left(\frac{1}{2}\underline{j}^*\underline{E}\right) + \int_V d^3x\ 2i\omega\left(w_e - w_m\right) + \oint_F \underline{S}\,d\underline{F} = 0$$

Kétpólus



bemenő teljesítmény:

$$\frac{1}{2}I^*V = -\int_{F_1}\underline{S}\,d\underline{F} = \int_V d^3x\ \frac{1}{2}\underline{j}^*\underline{E} + \int_V d^3x\ 2i\omega\left(w_e - w_m\right) +$$

$$+\underbrace{\int_{F-F_1}\underline{S}\,d\underline{F}}_{=0}$$

sugárzási veszteség kvázistacionárius esetben elhanyagolják

bemeneti impedancia: $Z = R - iX$

$$V = Z\cdot I = (R - iX)I$$

$$\frac{1}{2}I^*V = \frac{1}{2}|I|^2(R - iX)$$

$$|I|^2 (R - iX) = \int_V d^3x \; \underline{j}^* \underline{E} + i 4\omega \int_V d^3x \; (w_e - w_m)$$

+ feltevések: $\underline{j} = \sigma \underline{E}$, w_e és w_m valós

$$\boxed{R = \frac{1}{|I|^2} \int_V d^3x \; \sigma |\underline{E}|^2} \qquad \boxed{X = \frac{1}{|I|^2} \int_V d^3x \; 4\omega (w_m - w_e)}$$

1. Ohmikus ellenállás

$$R = \frac{1}{|I|^2} \cdot F \cdot \ell \cdot \sigma |\underline{E}|^2 \qquad |I| = |\underline{j}| \cdot F \qquad \underline{E} = \sigma \underline{j}$$

$$\boxed{R = \frac{1}{F^2 |\underline{j}|^2} \cdot F \cdot \ell \cdot \sigma \frac{|\underline{j}|^2}{\sigma^2} = \frac{\ell}{F \sigma}}$$

2. Kapacitás

$$X_c = -\frac{1}{|I|^2} \int_V d^3x \; 4\omega \cdot w_e = -\frac{\omega}{|I|^2} \int_V d^3x \; \underline{E} \, \underline{D}^* =$$

$$= +\frac{\omega}{|I|^2} \int_V d^3x \; (\underline{\nabla} \phi) \, \underline{D}^* = \frac{\omega}{|I|^2} \underbrace{\int_V d^3x \; \underline{\nabla} (\phi \, \underline{D}^*)}_{0} - \frac{\omega}{|I|^2} \int_V d^3x \; \phi \underbrace{\underline{\nabla} \, \underline{D}^*}_{\rho^*}$$

$$\boxed{X_c = -\frac{\omega}{|I|^2} \int d^3x \; \phi(\underline{x}) \, \rho^*(\underline{x})}$$

+ feltevés: $\phi = $ áll. $\Rightarrow \quad X_c = -\frac{\omega}{|I|^2} \cdot \phi \cdot Q^*$

$$\left.\begin{array}{l} \phi = \dfrac{Q}{C} \Rightarrow X_c = -\dfrac{\omega}{C} \cdot \dfrac{|Q|^2}{|I|^2} \\ \dfrac{dQ(t)}{dt} = I(t) \quad \Rightarrow I = -i\omega \cdot Q \end{array}\right\} \Rightarrow \boxed{X_c = -\frac{1}{\omega C}}$$

3. Induktivitás

$$X_L = \frac{1}{|I|^2} \int d^3x \; 4\omega \cdot W_m = \frac{\omega}{|I|^2} \int d^3x \; \underline{B}\,\underline{H}^* = \frac{\omega}{|I|^2} \int d^3x \; (\mathrm{rot}\,\underline{A})\,\underline{H}^*$$

$$\mathrm{div}\,(\underline{H}^* \times \underline{A}) = \underline{H}^* \,\mathrm{rot}\,\underline{A} - \underline{A}\,\mathrm{rot}\,\underline{H}^*$$

$$= \frac{\omega}{|I|^2} \underbrace{\int_V d^3x \; \mathrm{div}\,(\underline{H}^* \times \underline{A})}_{0} + \frac{\omega}{|I|^2} \int_V d^3x \; \underline{A}\,\underbrace{\mathrm{rot}\,\underline{H}^*}_{\underline{j}^*}$$

$$\boxed{X_L = \frac{\omega}{|I|^2} \int_V d^3x \; \underline{A}\,\underline{j}^*}$$

+ feltétel: keresztmetszet $\underline{A} = \text{áll.}$

$$X_L = \frac{\omega}{|I|^2} \cdot I^* \oint \underline{A}\, d\underline{x} = \frac{\omega}{I} \int \underbrace{\mathrm{rot}\,\underline{A}}_{\underline{B}}\, d\underline{F} = \omega \cdot \frac{\Phi_m}{I}$$

$$\Phi_m = L \cdot I \quad \Rightarrow \quad \boxed{X_L = \omega L}$$

Kvázistacionárius négypólus mezők

$\underline{j} = \sigma \underline{E}$ $\quad \varrho = 0 \rightarrow$ belsőben $\Phi = 0$

$$\mathrm{rot}\,\underline{H}(\underline{x}) = \sigma \underline{E}(\underline{x}) - i\omega\,\varepsilon(\omega)\,\underline{E}(\underline{x})$$

$\sigma \gg \omega\,|\varepsilon(\omega)|$ $\qquad \mathrm{rot}\,\underline{B}(\underline{x}) = \mu\sigma\,\underline{E}(\underline{x})$

$$\underline{E}(\underline{x},t) = -\frac{\partial \underline{A}(\underline{x},t)}{\partial t} \quad \Rightarrow \quad \underline{E}(\underline{x}) = i\omega\,\underline{A}(\underline{x})$$

$$\underline{B}(\underline{x}) = \mathrm{rot}\,\underline{A}(\underline{x})$$

$$\text{rot}(\text{rot}\,\underline{A}) = \text{grad}\underbrace{(\text{div}\,\underline{A})}_{0} - \Delta\underline{A} = i\omega\mu\sigma\underline{A}$$

$$\Rightarrow \boxed{\Delta\underline{A} = -i\omega\mu\sigma\underline{A}}$$

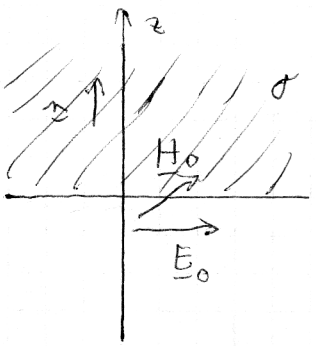
$\underline{E} = i\omega\underline{A}$ $\quad \underline{B} = \text{rot}\,\underline{A}$ $\quad \rightarrow$ ugyanez az egyenlet teljesül rájuk

$\text{rot}\,\underline{H} = \sigma\underline{E}$ $\qquad \text{div}\,\underline{B} = 0$

$\text{rot}\,\underline{E} = i\omega\underline{B}$ $\qquad \text{div}\,\underline{Q} = 0$

$$\frac{1}{\sigma}\,\text{rot}(\text{rot}\,\underline{H}) = \frac{1}{\sigma}\left(\text{grad}\underbrace{(\text{div}\,\underline{H})}_{0} - \Delta\underline{H}\right) = i\omega\mu\underline{H}$$

$$\Rightarrow \boxed{\Delta\underline{H} = -i\omega\mu\sigma\underline{H}}$$



$\underline{H}(z)$

$$\frac{d^2\underline{H}}{dz^2} = -i\omega\mu\sigma\underline{H}$$

$$\underline{H}(z) = \underline{H}_0 \cdot e^{ikz}$$

$$k^2 = i\omega\mu\sigma = \omega\mu\sigma \cdot e^{i\frac{\pi}{2}} \qquad k = \pm\sqrt{\omega\mu\sigma}\, e^{i\frac{\pi}{4}}$$

$$k = \pm\sqrt{\omega\mu\sigma}\,\frac{1+i}{\sqrt{2}} = \pm\frac{1+i}{\delta}$$

$$\delta = \sqrt{\frac{2}{\omega\mu\sigma}} \quad \text{behatolási mélység.}$$

$$\boxed{\underline{H} = \underline{H}_0 \cdot e^{(i-1)\frac{z}{\delta}}} \qquad \text{"lecsengő" megoldás}$$

$$\underline{E} = \frac{1}{\sigma}\,\underline{\nabla} \times \underline{H} = \frac{1}{\sigma}\,\underline{\nabla}\, e^{(i-1)\frac{z}{\delta}} \times \underline{H}_0 =$$

$$= \underbrace{\left(\frac{i-1}{\sigma\delta}\cdot \underline{n} \times \underline{H}_0\right)}_{\underline{E}_0} \cdot e^{(i-1)\frac{z}{\delta}} = \underline{E}_0\, e^{(i-1)\frac{z}{\delta}}$$

Torzómentes, homogén részletes közegben terjedő síkhullámok

$\mathrm{rot}\,\underline{H} = -i\omega\,\underline{D}$ $\quad \mathrm{div}\,\underline{B} = 0$ $\quad \underline{B} = \mu\,\underline{H}$ $\quad \mu$ valós

$\mathrm{rot}\,\underline{E} = i\omega\,\underline{B}$ $\quad \mathrm{div}\,\underline{D} = 0$ $\quad \underline{D} = \varepsilon(\omega)\,\underline{E}$ $\quad \varepsilon(\omega)$ komplex

$$\mathrm{rot}\,\mathrm{rot}\,\underline{H} = \underbrace{\mathrm{grad}(\mathrm{div}\,\underline{H})}_{0} - \Delta\underline{H} = -i\omega\,\varepsilon(\omega)\,\mathrm{rot}\,\underline{E} =$$

$$= (-i\omega)\,\varepsilon(\omega)\,(i\omega\,\underline{B}) = \varepsilon(\omega)\,\mu\,\omega^2 \cdot \underline{H}$$

$$\boxed{\Delta\underline{H} = -\varepsilon(\omega)\,\mu\,\omega^2 \cdot \underline{H}}$$ ugyanez $\underline{E}$-re

$$\underline{H} = \underline{H}_0\, e^{i(\underline{k}\,\underline{x} - \omega t)} \qquad \underline{E} = \underline{E}_0\, e^{i(\underline{k}\,\underline{x} - \omega t)}$$

$$\Rightarrow \quad \underline{k}^2 = \omega^2\,\varepsilon(\omega)\,\mu \qquad \varepsilon = \varepsilon' + i\varepsilon'' \qquad \varepsilon'', k'' \text{ hiesi}$$

$$\underline{k} = \underline{k}' + i\underline{k}''$$

$$\downarrow$$

$$k'^2 - k''^2 = \varepsilon'\mu\omega^2 \qquad 2\underline{k}'\underline{k}'' = \varepsilon''\mu\omega^2$$

$$v_f = \frac{\omega}{k'} = \frac{1}{\sqrt{\varepsilon'\mu}} \qquad c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$$

$$n = \frac{c}{v_\phi} = \sqrt{\frac{\varepsilon \cdot \mu}{\varepsilon_0 \cdot \mu_0}} = \sqrt{\varepsilon_r \cdot \mu_r} \approx \sqrt{\varepsilon_r} \quad , \text{ neet } \mu_r \approx 1$$

$$k'' = \frac{\varepsilon'' \mu \omega^2}{2k'} = \frac{1}{2} \varepsilon'' \mu v_\phi \omega$$

$$\boxed{\frac{k''}{k'} = \frac{1}{2} \varepsilon'' \mu v_\phi^2 = \frac{1}{2} \varepsilon'' \mu \left(\frac{1}{\sqrt{\varepsilon' \mu}}\right)^2 = \frac{1}{2} \frac{\varepsilon''}{\varepsilon'}}$$

$$\operatorname{div} \underline{E} = 0 \Rightarrow \vec{n} \underline{E}_0 = 0 \qquad \vec{n} \perp \underline{E}_0$$

$$\operatorname{div} \underline{H} = 0 \Rightarrow \vec{n} \underline{H}_0 = 0 \qquad \vec{n} \perp \underline{H}_0$$

$$\operatorname{rot} \underline{E} = i\omega \underline{B} \Rightarrow \underline{H} = \frac{1}{i\omega\mu} \nabla \times \underline{E} =$$

$$= \underbrace{\frac{ik}{i\omega\mu} \vec{n} \times \underline{E}_0}_{\underline{H}_0} e^{i(k \vec{n} \underline{x} - \omega t)}$$

$$\Rightarrow \boxed{\underline{H}_0 = \frac{k}{\omega \mu} \vec{n} \times \underline{E}_0} \qquad \underline{H}_0 \perp \underline{E}_0$$

$$\boxed{\underline{H}_0 = \frac{k}{\omega \mu} \underline{E}_0 = \frac{k' + ik''}{\omega \mu} \underline{E}_0 = \frac{k'}{\omega \mu} \left(1 + i \frac{k''}{k'}\right) \underline{E}_0 =}$$

$$= \frac{\sqrt{\varepsilon \mu}}{\mu} \left(1 + i \frac{\varepsilon''}{2\varepsilon'}\right) \underline{E}_0 = \boxed{\sqrt{\frac{\varepsilon'}{\mu}} \underline{E}_0 \cdot e^{i \frac{\varepsilon''}{2\varepsilon'}}}$$

$$H_0 = \varepsilon' \frac{1}{\sqrt{\varepsilon' \mu}} E_0 \cdot e^{i \frac{\varepsilon''}{2\varepsilon'}} = \varepsilon' \cdot v_\phi \cdot E_0 \cdot e^{i \frac{\varepsilon''}{2\varepsilon'}}$$

$$\underline{S} = \frac{1}{2} \underline{E} \times \underline{H}^* = \frac{1}{2} \varepsilon' \cdot v_\phi \cdot |E_0|^2 \cdot e^{i \frac{\varepsilon''}{2\varepsilon'}} \cdot \vec{n}$$

$$W = \frac{1}{4}\left(\underline{E}\,\underline{D}^* + \underline{B}\,\underline{H}^*\right) = \frac{1}{4}\,\varepsilon'|E_0|^2 + \frac{1}{4}\,\mu\varepsilon'^2 v_f^2 |E_0|^2 = \frac{1}{2}\,\varepsilon'|E_0|^2$$

energiasűrűség sebessége:

$$\underline{v}_W = \frac{\mathrm{Re}(\underline{S})}{W} = v_f \cdot \underbrace{\cos\left(\frac{\varepsilon''}{2\varepsilon'}\right)}_{\approx 1} \cdot \underline{n}$$

hullámcsomag:

$$\int d\underline{k}\; C(\underline{k})\, e^{i\left((\underline{k}+i\underline{k}'')\underline{n}\underline{x} - \omega t\right)} = \int d\underline{k}\,|C(\underline{k})| \cdot e^{\overbrace{i\left(\underline{k}\underline{n}\underline{x} - \omega t + \delta(\underline{k})\right)}^{\text{fázis}}} \cdot e^{-\underline{k}''\underline{n}\underline{x}}$$

$\rightarrow$ csúcsok: $\frac{d}{d\underline{k}}\left(\underline{k}\underline{n}\underline{x} - \omega(\underline{k})\,t + \delta(\underline{k})\right) = 0$

$$\underline{n}\underline{x} - \frac{d\omega(\underline{k})}{d\underline{k}}\,t + \frac{d\delta(\underline{k})}{d\underline{k}} = 0$$

$\Rightarrow$ csoportsebesség: $\boxed{v_{cs} = \frac{d\omega}{d\underline{k}} = \frac{d}{d\underline{k}}\left(\underline{k}\,v_f\right) = v_f + \underline{k}\,\frac{dv_f}{d\underline{k}} =}$

$$= v_f + \underline{k}\,\frac{dv_f}{d\omega}\cdot\frac{d\omega}{d\underline{k}} = v_f + \underline{k}\left(-\frac{c}{n^2}\frac{dn}{d\omega}\right) v_{cs} =$$

$$\boxed{= v_f\left(1 - \underline{k}\,v_{cs}\,\frac{1}{n}\,\frac{dn}{d\omega}\right)}$$

$\frac{dn}{d\omega} > 0$ normális diszperzió $\quad v_{cs} < v_f$

$\frac{dn}{d\omega} < 0$ anomális diszperzió $\quad v_{cs} > v_f$

$$v_{cs} = v_f - \hbar v_f v_g \frac{1}{n} \frac{dn}{d\omega}$$

$$v_{cs} = \frac{v_f}{1 + \hbar v_f \frac{1}{n} \frac{dn}{d\omega}}$$

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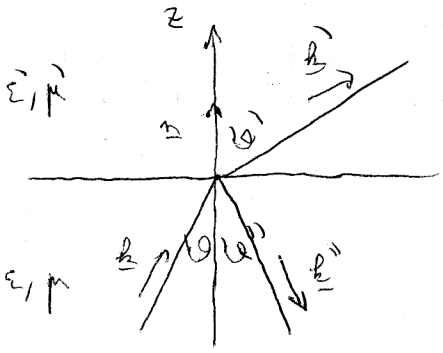
vezetőképesség figyelembevétele: $\underline{j} = \sigma \underline{E}$

$$\mathrm{rot}\, \underline{H} = \underline{j} - i\omega \underline{D} = \sigma \underline{E} - i\omega \varepsilon \underline{E} =$$

$$= -i\omega \cdot \underbrace{\varepsilon \left(1 + i \frac{\sigma}{\varepsilon \omega}\right)}_{\varepsilon_{eff}} \underline{E}$$

$$\varepsilon_{eff} = \varepsilon \cdot \left(1 + i \frac{\sigma}{\varepsilon \omega}\right)$$

Fresnel - képletek



$$\underline{E}(z>0) = \underline{E}_0' \, e^{i(\underline{k}'\underline{x} - \omega' t)}$$

$$\underline{E}(z<0) = \underline{E}_0 \, e^{i(\underline{k}\underline{x} - \omega t)} + \underline{E}_0'' \, e^{i(\underline{k}''\underline{x} - \omega'' t)}$$

peremfeltételek:

$\underline{n}\,\underline{D}$ folyt. $\rightarrow$ $\underline{n}\left(\varepsilon'\,\underline{E}(z=0^+) - \varepsilon\,\underline{E}(z=0^-)\right) = 0$

$\underline{n}\,\underline{B}$ folyt. $\rightarrow$ $\underline{n}\left(\mu'\,\underline{H}(z=0^+) - \mu\,\underline{H}(z=0^-)\right) = 0$

$\underline{n}\times\underline{E}$ folyt. $\rightarrow$ $\underline{n}\times\left(\underline{E}(z=0^+) - \underline{E}(z=0^-)\right) = 0$

$\underline{n}\times\underline{H}$ folyt. $\rightarrow$ $\underline{n}\times\left(\underline{H}(z=0^+) - \underline{H}(z=0^-)\right) = 0$

$\underline{x}=0 \quad \forall t \Rightarrow \omega = \omega' = \omega''$

$t=0 \quad \forall \underline{x}_\perp = (x, y, 0)$

$$\underline{k}\cdot\underline{x}_\perp = \underline{k}'\cdot\underline{x}_\perp = \underline{k}''\cdot\underline{x}_\perp$$

$$k \sin\vartheta = k' \sin\vartheta' = k'' \sin\vartheta''$$

$\omega = \frac{c}{n} k \qquad \omega = \omega'' \Rightarrow k = k'' \Rightarrow \vartheta = \vartheta''$

$\omega = \omega' \qquad \frac{n}{c}\omega \sin\vartheta = \frac{n'}{c}\omega \sin\vartheta'$

$\Rightarrow n \sin\vartheta = n' \sin\vartheta'$

$$\operatorname{rot} \underline{H} = \frac{\partial \underline{D}}{\partial t} \longrightarrow i\underline{k} \times \underline{H}_0 = -i\omega \varepsilon \underline{E}_0$$

$$\hat{\underline{k}} = \frac{\underline{k}}{k}$$

$$\underline{E}_0 = -\frac{1}{\omega \varepsilon} \underline{k} \times \underline{H}_0 = -\frac{k}{\omega \varepsilon} \hat{\underline{k}} \times \underline{H}_0 = -\frac{1}{\varepsilon v_f} \hat{\underline{k}} \times \underline{H}_0$$

$$\operatorname{rot} \underline{E} = -\frac{\partial \underline{B}}{\partial t} \longrightarrow i\underline{k} \times \underline{E}_0 = i\omega \mu \underline{H}_0$$

$$\underline{H}_0 = \frac{1}{\omega \mu} \underline{k} \times \underline{E}_0 = +\frac{k}{\omega \mu} \hat{\underline{k}} \times \underline{E}_0 = \frac{1}{\mu v_f} \hat{\underline{k}} \times \underline{E}_0$$

I. Síkra merőleges polarizáció

$$\underline{E}_0 \underline{n} = 0 \Rightarrow \underline{E}_0' \underline{n} = \underline{E}_0'' \underline{n} = 0$$

$$\underline{E}_0 + \underline{E}_0'' = \underline{E}_0'$$

$$\underline{n} \times \left(\frac{1}{\mu v_f} \left(\hat{\underline{k}} \times \underline{E}_0 + \hat{\underline{k}}'' \times \underline{E}_0'' \right) - \frac{1}{\mu' v_f'} \hat{\underline{k}}' \times \underline{E}_0' \right) = 0$$

$$\frac{1}{\mu v_f} \left(\underline{n} \times \left(\hat{\underline{k}} \times \underline{E}_0 \right) + \underline{n} \times \left(\hat{\underline{k}}'' \times \underline{E}_0'' \right) \right) = \frac{1}{\mu' v_f'} \underline{n} \times \left(\hat{\underline{k}}' \times \underline{E}_0' \right)$$

$$\frac{1}{\mu v_f} \left(\underbrace{\left(\underline{n} \underline{E}_0 \right)}_{0} \hat{\underline{k}} - \underbrace{\left(\underline{n} \hat{\underline{k}} \right)}_{\cos\vartheta} \underline{E}_0 + \underbrace{\left(\underline{n} \underline{E}_0'' \right)}_{0} \hat{\underline{k}}'' - \underbrace{\left(\underline{n} \hat{\underline{k}}'' \right)}_{-\cos\vartheta'' = -\cos\vartheta} \underline{E}_0'' \right) =$$

$$= \frac{1}{\mu' v_f'} \left(\underbrace{\left(\underline{n} \underline{E}_0' \right)}_{0} \hat{\underline{k}}' - \underbrace{\left(\underline{n} \hat{\underline{k}}' \right)}_{\cos\vartheta'} \underline{E}_0' \right)$$

$$\frac{1}{\mu v_f}\left(\underline{E}_0'' - \underline{E}_0\right)\cos\vartheta = -\frac{1}{\mu' v_f'}\underline{E}_0'\cos\vartheta'$$

$$v_f = \frac{1}{\sqrt{\varepsilon\mu}}$$

$$\underline{E}_0 + \underline{E}_0'' = \underline{E}_0'$$

$$\sqrt{\frac{\varepsilon}{\mu}}\cos\vartheta\left(\underline{E}_0 - \underline{E}_0''\right) = \sqrt{\frac{\varepsilon'}{\mu'}}\cos\vartheta'\,\underline{E}_0'$$

$$\underline{E}_0'' = \frac{\sqrt{\frac{\varepsilon}{\mu}}\cos\vartheta - \sqrt{\frac{\varepsilon'}{\mu'}}\cos\vartheta'}{\sqrt{\frac{\varepsilon}{\mu}}\cos\vartheta + \sqrt{\frac{\varepsilon'}{\mu'}}\cos\vartheta'}\,\underline{E}_0$$

$$\underline{E}_0' = \frac{2\sqrt{\frac{\varepsilon}{\mu}}\cos\vartheta}{\sqrt{\frac{\varepsilon}{\mu}}\cos\vartheta + \sqrt{\frac{\varepsilon'}{\mu'}}\cos\vartheta'}\,\underline{E}_0$$

II. Šikbeli ploščici

$$\underline{n}\times\underline{H}_0 = \underline{n}\times\underline{H}_0' = \underline{n}\times\underline{H}_0'' = 0$$

$$\underline{H}_0 + \underline{H}_0'' = \underline{H}_0'$$

$$\underline{n}\times\left(\left(-\frac{1}{\varepsilon v_f}\right)\left(\hat{\underline{k}}\times\underline{E}_0 + \hat{\underline{k}}''\times\underline{E}_0''\right) - \left(-\frac{1}{\varepsilon' v_f'}\right)\hat{\underline{k}}'\times\underline{H}_0'\right) = 0$$

$$\frac{1}{\varepsilon v_f}\left(\underline{n}\times\left(\hat{\underline{k}}\times\underline{H}_0\right)+\underline{n}\times\left(\hat{\underline{k}}''\times\underline{H}_0''\right)\right)=\frac{1}{\varepsilon' v_f'}\underline{n}\times\left(\hat{\underline{k}}'\times\underline{H}_0'\right)$$

$$\frac{1}{\varepsilon v_f}\Big(\underbrace{(\underline{n}\,\underline{H}_0)}_{0}\hat{\underline{k}}-\underbrace{(\underline{n}\,\hat{\underline{k}})}_{\cos\vartheta}\underline{H}_0+\underbrace{(\underline{n}\,\underline{H}_0'')}_{0}\hat{\underline{k}}''-\underbrace{(\underline{n}\,\hat{\underline{k}}'')}_{-\cos\vartheta}\underline{H}_0''\Big)=$$

$$=\frac{1}{\varepsilon' v_f'}\Big(\underbrace{(\underline{n}\,\underline{H}_0')}_{0}\hat{\underline{k}}'-\underbrace{(\underline{n}\,\hat{\underline{k}}')}_{\cos\vartheta'}\underline{H}_0'\Big)$$

$$\frac{1}{\varepsilon v_f}\left(\underline{H}_0''-\underline{H}_0\right)\cos\vartheta=-\frac{1}{\varepsilon' v_f'}\underline{H}_0'\cos\vartheta' \qquad v_f=\frac{1}{\sqrt{\varepsilon\mu}}$$

$$\underline{H}_0+\underline{H}_0''=\underline{H}_0'$$

$$\sqrt{\frac{\mu}{\varepsilon}}\cos\vartheta\left(\underline{H}_0-\underline{H}_0''\right)=\sqrt{\frac{\mu'}{\varepsilon'}}\cos\vartheta'\cdot\underline{H}_0'$$

$$\underline{H}_0''=\frac{\sqrt{\frac{\mu}{\varepsilon}}\cos\vartheta-\sqrt{\frac{\mu'}{\varepsilon'}}\cos\vartheta'}{\sqrt{\frac{\mu}{\varepsilon}}\cos\vartheta+\sqrt{\frac{\mu'}{\varepsilon'}}\cos\vartheta'}\,\underline{H}_0$$

$$\underline{H}_0'=\frac{2\sqrt{\frac{\mu}{\varepsilon}}\cos\vartheta}{\sqrt{\frac{\mu}{\varepsilon}}\cos\vartheta+\sqrt{\frac{\mu'}{\varepsilon'}}\cos\vartheta'}\,\underline{H}_0$$

$\mu,\mu'\approx\mu_0$ hasik , $n=\sqrt{\varepsilon_r}$

$\Rightarrow\sqrt{\frac{\mu}{\varepsilon}}$ helyébe $\frac{1}{n}$ írható mindenhol

Brewster - szög (II. eset)

$$\underline{H}_0'' = 0 \Rightarrow \frac{1}{n}\cos\vartheta = \frac{1}{n'}\cos\vartheta'$$

$$n \sin\vartheta = n' \sin\vartheta'$$

$$\Rightarrow \cos\vartheta \cdot \sin\vartheta = \cos\vartheta' \sin\vartheta' \Rightarrow \sin 2\vartheta = \sin 2\vartheta'$$

$$\vartheta \neq \vartheta' \Rightarrow 2\vartheta + 2\vartheta' = \pi \Rightarrow \boxed{\vartheta' = \frac{\pi}{2} - \vartheta}$$

$$\frac{1}{n}\cos\vartheta = \frac{1}{n'}\underbrace{\cos\left(\frac{\pi}{2} - \vartheta\right)}_{\sin\vartheta} \Rightarrow \boxed{\operatorname{tg}\vartheta = \frac{n'}{n}}$$

I. esetben nincs Brewster - szög:

$$\underline{E}_0'' = 0 \Rightarrow \left.\begin{aligned} n\cos\vartheta &= n'\cos\vartheta' \\ n\sin\vartheta &= n'\sin\vartheta' \end{aligned}\right\} \Rightarrow n = n' \quad \vartheta = \vartheta'$$

Teljes visszaverődés

$$n \sin\vartheta = n' \sin\vartheta'$$

$$\cos\vartheta' = \sqrt{1 - \frac{n^2}{n'^2}\sin^2\vartheta}$$

$$n > n' \quad \frac{n}{n'}\sin\vartheta > 1 \Rightarrow \cos\vartheta' \text{ képzetes}$$

$$\cos\vartheta' = i\sqrt{\frac{n^2}{n'^2}\sin^2\vartheta - 1}$$

$$e^{i\underline{k}\cdot\underline{x}} = e^{ik\sin\vartheta\cdot x + ik\cos\vartheta\cdot z} =$$

$$\underline{k} = \begin{pmatrix} k\sin\vartheta \\ 0 \\ k\cos\vartheta \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$= e^{ik\sin\vartheta\cdot x} \cdot e^{-k\sqrt{\frac{n^2}{n'^2}\sin^2\vartheta - 1}\, z}$$

behatolási mélység:

$$\boxed{\delta = \frac{1}{k\sqrt{\frac{n^2}{n'^2}\sin^2\vartheta - 1}}}$$

Kirchoff-féle elhajlási elmélet

$\Psi(\underline{x},t)$ skalár $\qquad \Box\,\Psi(\underline{x},t) = 0$

$$\Psi(\underline{x},t) = \Psi_\omega(\underline{x})\, e^{-i\omega t} \rightarrow \boxed{\Delta\Psi_\omega(\underline{x}) + k^2\Psi_\omega(\underline{x}) = 0} \qquad k = \frac{\omega}{v_f}$$

Helmholtz-egyenlet

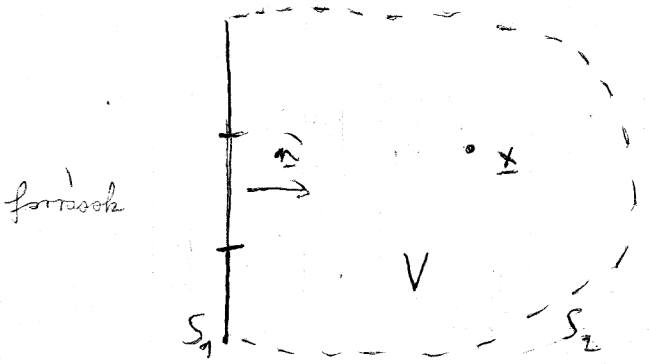
Green-függvény: $\Box\, G(\underline{x},t,\underline{\dot{x}},\dot{t}) = \delta(\underline{x}-\underline{\dot{x}})\,\delta(t-\dot{t})$

$$G(\underline{x},t,\underline{\dot{x}},\dot{t}) = \int \frac{d\omega}{2\pi}\, G_\omega(\underline{x},\underline{\dot{x}})\, e^{-i\omega(t-\dot{t})}$$

$$\Box\, G(\underline{x},t,\underline{\dot{x}},\dot{t}) = \int \frac{d\omega}{2\pi}\, \underbrace{(-k^2-\Delta)\, G_\omega(\underline{x},\underline{\dot{x}})}_{\|}\, e^{-i\omega(t-\dot{t})}$$

$$\delta(\underline{x}-\underline{\dot{x}})\,\delta(t-\dot{t}) = \int \frac{d\omega}{2\pi}\, \underbrace{\delta(\underline{x}-\underline{\dot{x}})}_{\|}\, e^{-i\omega(t-\dot{t})}$$

$$\Rightarrow \boxed{(\Delta + k^2)\, G_\omega(\underline{x},\underline{\dot{x}}) = -\delta(\underline{x}-\underline{\dot{x}})}$$



Green-tétel: $\int_V d^3\hat{x}\,(f\hat{\Delta}g - g\hat{\Delta}f) = \oint_{S_1+S_2} d\hat{F}\,(f\hat{\nabla}g - g\hat{\nabla}f)$

$f(\hat{x}) = \Psi_\omega(\hat{x}) \qquad g(\hat{x}) = G_\omega(x,\hat{x})$

$$\int_V d^3\hat{x}\,\left(\Psi_\omega(\hat{x})\,\underbrace{\hat{\Delta}G_\omega(x,\hat{x})}_{-k^2 G_\omega\,\delta(x-\hat{x})} - G_\omega(x,\hat{x})\,\underbrace{\hat{\Delta}\Psi_\omega(\hat{x})}_{-k^2\Psi_\omega(\hat{x})}\right) =$$

$$= \int_V d^3\hat{x}\,\left(\cancel{-k^2\Psi_\omega(\hat{x})G_\omega} - \Psi_\omega(\hat{x})\,\delta(x-\hat{x}) + \cancel{k^2 G_\omega\Psi_\omega(\hat{x})}\right) =$$

$$= -\Psi_\omega(x) = \oint_{S_1+S_2} d\hat{F}\,(-\hat{n})\left(\Psi_\omega(\hat{x})\hat{\nabla}G_\omega - G_\omega\hat{\nabla}\Psi_\omega(\hat{x})\right)$$

$$\boxed{\Psi_\omega(x) = \oint_{S_1+S_2} d\hat{F}\,\hat{n}\left(\Psi_\omega(\hat{x})\hat{\nabla}G_\omega - G_\omega\hat{\nabla}\Psi_\omega(\hat{x})\right)}$$

Kirchoff: G_ω legyen a reláció tér Green-függvénye

$$G^{(\infty)}(\underline{x}, t, \underline{\dot{x}}, \dot{t}) = \frac{1}{4\pi|\underline{x}-\underline{\dot{x}}|}\,\delta\left(t-\dot{t}-\frac{|\underline{x}-\underline{\dot{x}}|}{v_f}\right) = \qquad R = |\underline{x}-\underline{\dot{x}}|$$

$$= \frac{1}{4\pi R}\,\delta\left(t-\dot{t}-\frac{R}{v_f}\right) = \frac{1}{4\pi R}\int \frac{d\omega}{2\pi}\, e^{-i\omega\left(t-\dot{t}-\frac{R}{v_f}\right)} =$$

$$= \int \frac{d\omega}{2\pi}\,\underbrace{\frac{e^{ikR}}{4\pi R}}\, e^{-i\omega(t-\dot{t})}$$

$$G_\omega^{(\infty)}(\underline{x}, \underline{\dot{x}})$$

$$\boxed{G_\omega^{(\infty)}(\underline{x}, \underline{\dot{x}}) = \frac{e^{ikR}}{4\pi R}}$$

ellenőrzés: $R \neq 0$ $\qquad (\Delta + k^2)\, G_\omega^{(\infty)} = \frac{1}{R}\frac{d^2}{dR^2}\left(R\, G_\omega^{(\infty)}\right) + k^2\, G_\omega^{(\infty)} = 0$

$$\frac{d^2}{dR^2}\left(R\, G_\omega^{(\infty)}\right) = -k^2\left(R \cdot G_\omega^{(\infty)}\right) \Rightarrow R \cdot G_\omega^{(\infty)} \sim e^{ikR}$$

$$\nabla G_\omega^{(\infty)} = \left(ik - \frac{1}{R}\right)\frac{e^{ikR}}{4\pi R}\,\nabla R$$

$$\nabla R = \nabla|\underline{x}-\underline{\dot{x}}| = -\frac{\underline{x}-\underline{\dot{x}}}{|\underline{x}-\underline{\dot{x}}|} = -\frac{\underline{R}}{R} = -\hat{\underline{R}}$$

$$\nabla G_\omega^{(\infty)} = -ik\left(1+\frac{i}{kR}\right)\frac{e^{ikR}}{4\pi R}\,\hat{\underline{R}}$$

S_2-re vett integrál $\to 0$

Kirchoff-integrál:

$$\Psi_\omega^K(\underline{x}) = -\frac{1}{4\pi}\int_{S_1} dF\ \hat{n}\left(ik\left(1+\frac{i}{kR}\right)\Psi_\omega^K(\underline{x})\cdot\hat{R} + \nabla\Psi_\omega^K(\underline{x})\right)\frac{e^{ikR}}{R}$$

probléma: S_1-en nem ismerjük első Ψ_ω és $\frac{\partial\Psi_\omega}{\partial n}$ értékét is (túlhatározottság)

megoldás: Rayleigh → Dirichlet- és Neumann-féle Green-fv.

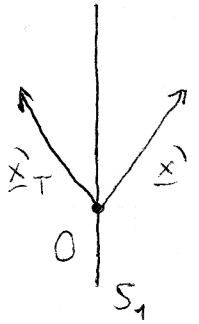
$$G_\omega^D(\underline{x},\underline{\xi}) = 0 \quad , \text{ha } \underline{\xi}\in S_1$$

$$\Psi_\omega^D(\underline{x}) = \int_{S_1} dF\ \Psi_\omega^D(\underline{\xi})\frac{\partial G_\omega^D}{\partial n}(\underline{x},\underline{\xi})$$

$$\frac{\partial G_\omega^N}{\partial n}(\underline{x},\underline{\xi}) = 0 \quad , \text{ha } \underline{\xi}\in S_1$$

$$\Psi_\omega^N(\underline{x}) = -\int_{S_1} dF\ \frac{\partial\Psi_\omega^N}{\partial n}(\underline{\xi})\ G_\omega^N(\underline{x},\underline{\xi})$$

előállítás: tükrözés-módszer



$$R = |\underline{x}-\underline{\xi}|$$

$$R_T = |\underline{x}-\underline{\xi}_T| = |\underline{x}_T-\underline{\xi}|$$

$$G_\omega^{D,N}(\underline{x},\underline{x}') = \frac{1}{4\pi}\left(\frac{e^{ikR}}{R} \mp \frac{e^{ikR_T}}{R_T}\right)$$

$\underline{x}' \in S_1 \Rightarrow R_T = R$

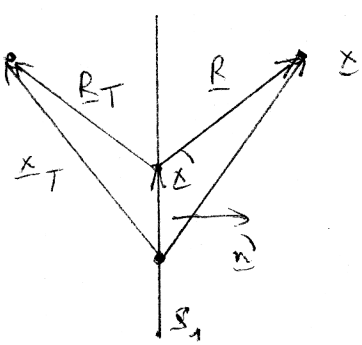
$$G_\omega^{D}(\underline{x},\underline{x}') = 0 \qquad G_\omega^{N} = \frac{1}{2\pi}\cdot\frac{e^{ikR}}{R}$$

$$\frac{\partial G_\omega^{D,N}}{\partial n'}(\underline{x},\underline{x}') = \underline{n}'\cdot\underline{\nabla}' G_\omega^{D,N} =$$

$$= \frac{1}{4\pi}\left[ik\left(1+\frac{i}{kR}\right)\frac{e^{ikR}}{R}\,\underline{\nabla}' R \mp ik\left(1+\frac{i}{kR_T}\right)\frac{e^{ikR_T}}{R_T}\,\underline{\nabla}' R_T\right]\cdot\underline{n}'$$

$$\underline{\nabla}' R = -\hat{\underline{R}} \qquad \underline{\nabla}' R_T = \underline{\nabla}'\left|\underline{x}_T - \underline{x}'\right| = -\frac{\underline{x}_T - \underline{x}'}{\left|\underline{x}_T - \underline{x}'\right|} = -\hat{\underline{R}}_T$$

$\underline{x}' \in S_1 \Rightarrow R_T = R \;\wedge\; \hat{\underline{R}}_T\cdot\underline{n}' = -\hat{\underline{R}}\cdot\underline{n}'$



$$\frac{\partial G_\omega^{D,N}}{\partial n'} = -ik\left(1+\frac{i}{kR}\right)\frac{e^{ikR}}{4\pi R}\left(\hat{\underline{R}} \mp \hat{\underline{R}}_T\right)\cdot\underline{n}'$$

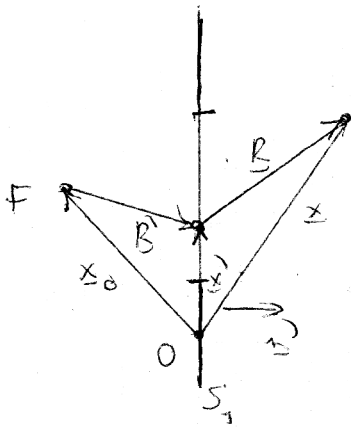
$$\frac{\partial G_\omega^{D}}{\partial n'} = -\frac{ik}{2\pi}\left(1+\frac{i}{kR}\right)\frac{e^{ikR}}{R}\cdot\hat{\underline{R}}\cdot\underline{n}'$$

$$\frac{\partial G_\omega^{N}}{\partial n'} = 0$$

t megoldások:

$$\Psi_\omega^D(\underline{x}) = -\frac{ik}{2\pi}\int_{S_1} dF'\ \Psi_\omega^D(\underline{x}')\left(1+\frac{i}{kR}\right)\frac{e^{ikR}}{R}\cdot\hat{\underline{R}}\,\underline{n}'$$

$$\Psi_\omega^N(\underline{x}) = -\frac{1}{2\pi}\int_{S_1} dF'\ \frac{\partial\Psi_\omega^N}{\partial n'}(\underline{x}')\cdot\frac{e^{ikR}}{R}$$



forgatási forrás $\underline{x}_0$-lal

$$\underline{R} = \underline{x} - \underline{x}' \qquad \underline{R}' = \underline{x}' - \underline{x}_0$$

$$\Psi_\omega(\underline{x}') = A\frac{e^{ikR'}}{R'}$$

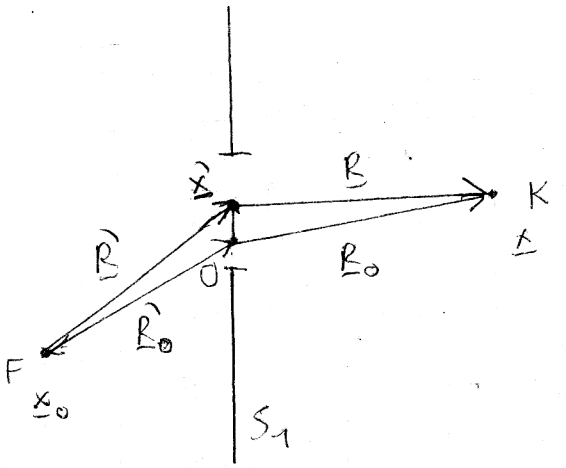
← részek!

$$\frac{\partial\Psi_\omega}{\partial n'}(\underline{x}') = ikA\left(1+\frac{i}{kR'}\right)\frac{e^{ikR'}}{R'}\underbrace{\underline{\nabla}'R'}_{\hat{\underline{R}}'}\cdot\underline{n}'$$

+ feltevés: $kR \gg 1 \qquad kR' \gg 1$

$$\Psi_\omega(\underline{x}) = -\frac{ik}{2\pi}A\int_{\text{részek}} dF'\cdot\frac{e^{ik(R+R')}}{R\,R'}\cdot f(\vartheta,\vartheta')$$

$$f(\vartheta,\vartheta') \simeq \begin{cases} \cos\vartheta = \underline{n}'\hat{\underline{R}} & \text{Dirichlet} \\ \cos\vartheta' = \underline{n}'\hat{\underline{R}}' & \text{Neumann} \\ \frac{1}{2}(\cos\vartheta + \cos\vartheta') & \text{Kirchoff} \end{cases}$$



$$\underline{x}' = \begin{pmatrix} \xi \\ \eta \\ 0 \end{pmatrix}$$

$$|\underline{x}'| \ll R, R'$$

$$\psi_\omega(\underline{x}) = -\frac{ik}{2\pi} A \frac{f(\vartheta,\vartheta')}{R_0 R'_0} \int_{\text{recek}} e^{ik(R+R')} \underbrace{dF}_{d\xi\, d\eta}$$

$$R'^2 = (x_0 - \xi)^2 + (y_0 - \eta)^2 + z_0^2 = R_0'^2 - 2(x_0\xi + y_0\eta) + \xi^2 + \eta^2$$

$$R^2 = (x - \xi)^2 + (y - \eta)^2 + z^2 = R_0^2 - 2(x\xi + y\eta) + \xi^2 + \eta^2$$

$$R = R_0 \sqrt{1 - 2\frac{x\xi + y\eta}{R_0^2} + \frac{\xi^2 + \eta^2}{R_0^2}}$$

$$\sqrt{1 + x} = 1 + \frac{1}{2}x - \frac{1}{4}x^2 + \dots$$

$$R \approx R_0 \left(1 - \frac{x\xi + y\eta}{R_0^2} + \frac{\xi^2 + \eta^2}{2R_0^2} - \frac{(x\xi + y\eta)^2}{2R_0^4}\right)$$

$$R' \approx R'_0 \left(1 - \frac{x_0\xi + y_0\eta}{R_0'^2} + \frac{\xi^2 + \eta^2}{2R_0'^2} - \frac{(x_0\xi + y_0\eta)^2}{2R_0'^4}\right)$$

speciális esetek:

I. Fresnel-diffrakció: valós

$$x_0 = y_0 = x = y = 0 \qquad f(\vartheta, \vartheta') \approx 1$$

$$R \approx R_0 \left(1 + \frac{\xi^2 + \eta^2}{2R_0^2}\right) \qquad R' \approx R'_0 \left(1 + \frac{\xi^2 + \eta^2}{2R'^2_0}\right)$$

$$\Psi_\omega(\underline{x}) = -\frac{ik}{2\pi} A \frac{e^{ik(R_0 + R'_0)}}{R_0 R'_0} \int_{\text{rések}} d\xi\, d\eta\, e^{ik\frac{1}{2}\left(\frac{1}{R_0} + \frac{1}{R'_0}\right)(\xi^2 + \eta^2)}$$

II Fraunhofer-diffrakció: messziről, fordán

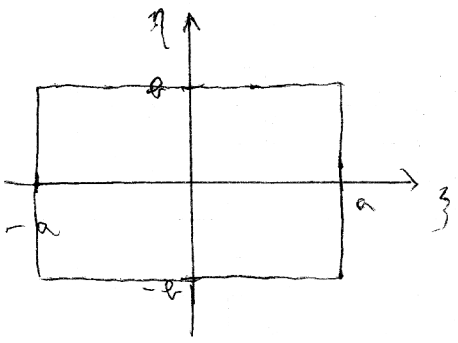
$$R \approx R_0 \left(1 - \frac{x\xi + y\eta}{R_0^2}\right) \qquad R' \approx R'_0 \left(1 - \frac{x_0\xi + y_0\eta}{R'^2_0}\right)$$

új változók: $\ell = \frac{x}{R_0} \qquad m = \frac{y}{R_0}$

$$\ell_0 = -\frac{x_0}{R'_0} \qquad m_0 = -\frac{y_0}{R'_0}$$

$$\Psi_\omega(\underline{x}) = -\frac{ik}{2\pi} A\, f(\vartheta, \vartheta') \frac{e^{ik(R_0 + R'_0)}}{R_0 R'_0} \int_{\text{rések}} d\xi\, d\eta\, e^{-ik\left((\ell - \ell_0)\xi + (m - m_0)\eta\right)}$$

Téglalap alakú nyílás



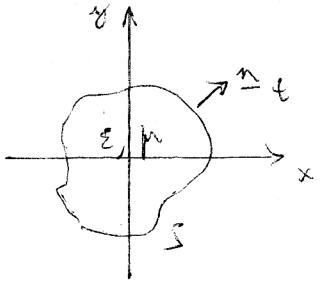
$$\int_{-a}^{a} d\xi\; e^{-ik_x(\ell-\ell_0)\xi} = \;!$$

$$= \frac{e^{-ik_x(\ell-\ell_0)a} - e^{ik_x(\ell-\ell_0)a}}{-ik_x(\ell-\ell_0)} =$$

$$= \frac{2}{k_x(\ell-\ell_0)} \sin k_x(\ell-\ell_0)a$$

$$\Psi_\omega(x) = -\frac{2ik}{\pi} A\, f(\omega,\vartheta)\, \frac{e^{ik(R_0+R_0')}}{R_0 R_0'} \cdot \frac{\sin k_x(\ell-\ell_0)a}{k_x(\ell-\ell_0)} \cdot \frac{\sin k_y(m-m_0)b}{k_y(m-m_0)}$$

Általános keresztmetszetű hullámvezető



$$\underline{E}(x,t) = \underline{E}(x,y)\, e^{i(kz-\omega t)}$$

$$\underline{E} = \underline{E}_\ell + \underline{E}_t = E_\ell \cdot \underline{e}_z + \underline{E}_t$$

$$\nabla = \underline{e}_z \partial_z + \nabla_t = ik \cdot \underline{e}_z + \nabla_t$$

Hullámegyenlet: $\square \underline{E} = \varepsilon\mu \frac{\partial^2 \underline{E}}{\partial t^2} - \Delta \underline{E} =$

$$= \left(-\varepsilon\mu\omega^2 + k^2 - \Delta_t\right) \underline{E} = 0$$

$$\Delta_t \underline{E} = -\gamma^2 \underline{E} \qquad \gamma^2 = \varepsilon\mu\omega^2 - k^2$$

ua. teljesül $\underline{B}$-ra is

Maxwell-egyenletek:

$$\nabla \times \underline{E} = i\omega \underline{B} \quad \Rightarrow \quad \nabla \underline{B} = 0$$

$$\nabla \times \underline{B} = -i\omega\varepsilon\mu \underline{E} \quad \Rightarrow \quad \nabla \underline{E} = 0$$

$$\nabla \times \underline{E} = (ik \underline{e}_z + \nabla_t) \times (E_e \underline{e}_z + \underline{E}_t) =$$

$$= ik \underline{e}_z \times \underline{E}_t + \nabla_t \times (E_e \underline{e}_z) + \nabla_t \times \underline{E}_t$$

$\nabla_t E_e \times \underline{e}_z$ → xy-síkban

$\nabla_t \times \underline{E}_t \parallel \underline{e}_z$

$$(\nabla \times \underline{E})_e = \nabla_t \times \underline{E}_t$$

$$(\nabla \times \underline{E})_t = ik \underline{e}_z \times \underline{E}_t - \underline{e}_z \times \nabla_t E_e$$

Longitudinális Maxwell:

$$\nabla_t \times \underline{E}_t = i\omega \underline{B}_e$$

$$\nabla_t \times \underline{B}_t = -i\omega\varepsilon\mu \underline{E}_e$$

Transzverzális Maxwell:

1) $ik \underline{e}_z \times \underline{E}_t - \underline{e}_z \times \nabla_t E_e = i\omega \underline{B}_t$

2) $ik \underline{e}_z \times \underline{B}_t - \underline{e}_z \times \nabla_t B_e = -i\omega\varepsilon\mu \underline{E}_t$

Tegyünk ki $\underline{E}_t$-t és $\underline{B}_t$-t E_e-lel és B_e-lel.

Ekkor 1)-ből helyettesítünk $\underline{B}_t$-t 2)-be.

$$ik\,\underline{e}_z \times \frac{1}{i\omega}\left(ik\,\underline{e}_z \times \underline{E}_t - \underline{e}_z \times \underline{\nabla}_t E_e\right) - \underline{e}_z \times \underline{\nabla}_t B_e =$$

$$= -i\omega\varepsilon\mu\,\underline{E}_t \qquad /\cdot(i\omega)$$

$$-k^2\,\underline{e}_z \times (\underline{e}_z \times \underline{E}_t) - ik\,\underline{e}_z \times (\underline{e}_z \times \underline{\nabla}_t E_e) - i\omega\,\underline{e}_z \times \underline{\nabla}_t B_e =$$

$$= +\omega^2\varepsilon\mu\,\underline{E}_t$$

$$\underline{e}_z \times (\underline{e}_z \times \underline{E}_t) = \underbrace{(\underline{e}_z \underline{E}_t)}_{0}\,\underline{e}_z - \underbrace{(\underline{e}_z \underline{e}_z)}_{1}\,\underline{E}_t = -\underline{E}_t$$

$$k^2\,\underline{E}_t + ik\,\underline{\nabla}_t E_e - i\omega\,\underline{e}_z \times \underline{\nabla}_t B_e = \varepsilon\mu\omega^2\,\underline{E}_t$$

$$\underbrace{(\varepsilon\mu\omega^2 - k^2)}_{\gamma^2}\,\underline{E}_t = ik\,\underline{\nabla}_t E_e - i\omega\,\underline{e}_z \times \underline{\nabla}_t B_e$$

$\underline{E}_t$ kiszámolása fejezhető ki

$$\boxed{\begin{aligned}\underline{E}_t &= \frac{1}{\gamma^2}\left(ik\,\underline{\nabla}_t E_e - i\omega\,\underline{e}_z \times \underline{\nabla}_t B_e\right)\\ \underline{B}_t &= \frac{1}{\gamma^2}\left(ik\,\underline{\nabla}_t B_e + i\omega\varepsilon\mu\,\underline{e}_z \times \underline{\nabla}_t E_e\right)\end{aligned}}$$

Határfeltételek: $\underline{n}_t \times \underline{E} = 0 \qquad \underline{n}_t \underline{B} = 0$

$\Rightarrow$ S-en: $E_e = 0$ és $\underline{n}_t \underline{B}_t = \frac{ik}{\gamma^2}\underbrace{\underline{n}_t \underline{\nabla}_t B_e}_{\frac{\partial B_e}{\partial n_t}} = 0$

$$\frac{\partial B_e}{\partial n_t} = 0$$

Mesosorts'lyok:

I. TM hullámok: $E_z = \Psi \qquad B_z = 0$

$\Delta_t \Psi = -\gamma^2 \Psi \qquad$ S-en: $\Psi = 0$

II. TE hullámok: $E_z = 0 \qquad B_z = \Psi$

$\Delta_t \Psi = -\gamma^2 \Psi \qquad$ S-en: $\frac{\partial \Psi}{\partial n_t} = 0$

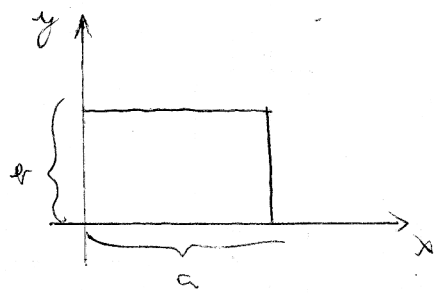
$\boxed{\gamma^2 = \varepsilon \mu \omega^2 - k^2}$ disp. rel.

$0 = \varepsilon \mu \, 2\omega \, d\omega - 2k \, dk$

$\frac{\omega}{k} \cdot \frac{d\omega}{dk} = \frac{1}{\varepsilon \mu}$ $\qquad \boxed{v_f \cdot v_{cs} = \frac{1}{\varepsilon \mu}}$

Példa: TM $\qquad \Psi(0,y) = \Psi(a,y) = 0$

$\Psi(x,0) = \Psi(x,b) = 0$



$\Psi(x,y) = A \sin(k_x x) \sin(k_y y)$

$k_x = \frac{\pi}{a} m \qquad k_y = \frac{\pi}{b} n$

$m, n = 1, 2, 3, \ldots$

$\Delta_t \Psi = (-k_x^2 - k_y^2) \Psi = -\gamma^2 \cdot \Psi$

$\Rightarrow \gamma^2 = k_x^2 + k_y^2 = \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$

$\boxed{k^2 = \varepsilon \mu \omega^2 - \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right)}$

Energiaáramlás a hullámvezetőben

$$S_z = \frac{1}{2}\left(\underline{E} \times \underline{H}^*\right)_z = \frac{1}{2}\left(\underline{E}_t \times \underline{H}_t^*\right)_z$$

TM módus:

$$\underline{E}_t = \frac{ih}{\gamma^2}\,\underline{\nabla}_t \psi$$

$$\underline{H}_t = \frac{i\omega\varepsilon}{\gamma^2}\,\underline{e}_z \times \underline{\nabla}_t \psi$$

$$S_z = \frac{h\omega\varepsilon}{2\gamma^4}\left(\underline{\nabla}_t\psi \times \left(\underline{e}_z \times \underline{\nabla}_t\psi^*\right)\right)_z = \frac{\varepsilon\omega h}{2\gamma^4}\,\underline{\nabla}_t\psi \cdot \underline{\nabla}_t\psi^*$$

$$\left(\underline{\nabla}_t\psi \cdot \underline{\nabla}_t\psi^*\right)\underline{e}_z - \underbrace{\left(\underline{e}_z \underline{\nabla}_t\psi\right)}_{0}\underline{\nabla}_t\psi^*$$

Teljesítményáram:

$$P = \int dF \cdot S_z = \frac{\varepsilon\omega h}{2\gamma^4}\int dF\, \underline{\nabla}_t\psi^* \cdot \underline{\nabla}_t\psi =$$

$$= \frac{\varepsilon\omega h}{2\gamma^4}\left[\underbrace{\int dF\, \underline{\nabla}_t\left(\psi^* \underline{\nabla}_t\psi\right)}_{\int_S d\underline{s}\, \underline{n}_t \psi^* \underline{\nabla}_t\psi = 0} - \int dF\, \psi^* \underbrace{\underline{\Delta}_t\psi}_{-\gamma^2\psi}\right]$$

$\underline{n}_t \psi^* \underline{\nabla}_t\psi = \psi^* \frac{\partial\psi}{\partial n_t}$, mert S-en $\psi = 0$

$$\boxed{P = \frac{\varepsilon\omega h}{2\gamma^2}\int |\psi|^2\, dF}$$

$$w = \frac{1}{4}\left(\varepsilon |\underline{E}|^2 + \mu |\underline{H}|^2\right) = \frac{1}{4}\left(\varepsilon |\underline{E}_t|^2 + \varepsilon |\underline{E}_z|^2 + \mu |\underline{H}_t|^2\right)$$

$$= \frac{1}{4}\left(\varepsilon |\Psi|^2 + \varepsilon \frac{k^2}{\gamma^4} |\nabla_t \Psi|^2 + \mu \frac{\omega^2 \varepsilon^2}{\gamma^4} \underbrace{|\underline{e}_z \times \nabla_t \Psi|^2}_{|\nabla_t \Psi|^2}\right)$$

Egységnyi hosszra eső energia

$$U = \int dF \cdot w = \frac{1}{4}\varepsilon\left(\int |\Psi|^2 dF + \frac{\varepsilon\mu\omega^2 + k^2}{\gamma^4} \underbrace{\int |\nabla_t \Psi|^2 dF}_{\gamma^2 \int |\Psi|^2 dF}\right)$$

$$U = \frac{1}{4}\varepsilon \underbrace{\left(1 + \frac{\varepsilon\mu\omega^2 + k^2}{\gamma^2}\right)}_{\frac{\gamma^2 + k^2 + \varepsilon\mu\omega^2}{\gamma^2} = \frac{2\varepsilon\mu\omega^2}{\gamma^2}} \int |\Psi|^2 dF$$

$$\boxed{U = \frac{\varepsilon^2\mu\omega^2}{2\gamma^2} \int |\Psi|^2 dF}$$

Energiaterjedés sebessége:

$$\boxed{\frac{P}{U} = \frac{\frac{\varepsilon\omega k}{2\gamma^2}}{\frac{\varepsilon^2\mu\omega^2}{2\gamma^2}} = \frac{1}{\varepsilon\mu} \cdot \underbrace{\frac{k}{\omega}}_{\frac{1}{v_f}} = v_\omega}$$

ahol $v_\omega = v_f = \frac{1}{\varepsilon\mu}$

TE modes:

$$\underline{E}_t = -\frac{i\omega}{\gamma^2}\,\underline{e}_z \times \underline{\nabla}_t \Psi$$

$$\underline{H}_t = \frac{ik}{\mu\gamma^2}\,\underline{\nabla}_t \Psi$$

$$S_z = \frac{1}{2}\left(\underline{E}_t \times \underline{H}_t^*\right)_z = -\frac{\omega k}{2\mu\gamma^4}\left(\left(\underline{e}_z \times \underline{\nabla}_t \Psi\right) \times \underline{\nabla}_t \Psi^*\right)_z =$$

$$= \frac{\omega k}{2\mu\gamma^4}\left(\underline{\nabla}_t \Psi^* \times \left(\underline{e}_z \times \underline{\nabla}_t \Psi\right)\right)_z = \frac{\omega k}{2\mu\gamma^4}\,\underline{\nabla}_t \Psi^* \cdot \underline{\nabla}_t \Psi$$

$$P = \int S_z \, dF = \frac{\omega k}{2\mu\gamma^4}\left(\underbrace{\int \underline{\nabla}_t\left(\Psi^* \underline{\nabla}_t \Psi\right) dF}_{\int_S d\gamma\; \Psi^* \frac{\partial \Psi}{\partial n_t} = 0} - \int \Psi^* \underbrace{\Delta_t \Psi}_{-\gamma^2 \Psi} dF\right)$$

$\frac{\partial \Psi}{\partial n_t} = 0$ ← S-en

$$\boxed{P = \frac{\omega k}{2\mu\gamma^2}\int |\Psi|^2 \, dF}$$

$$w = \frac{1}{4}\left(\varepsilon |\underline{E}_t|^2 + \mu |\underline{H}_t|^2 + \mu |\underline{H}_e|^2\right) =$$

$$= \frac{1}{4}\left(\varepsilon \cdot \frac{\omega^2}{\gamma^4} |\underline{\nabla}_t \Psi|^2 + \mu \cdot \frac{k^2}{\mu^2 \gamma^4} |\underline{\nabla}_t \Psi|^2 + \frac{1}{\mu} |\Psi|^2\right)$$

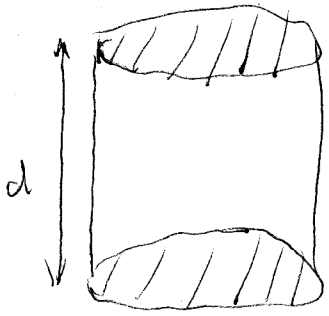
$$U = \int w \, dF = \frac{1}{4\mu}\left(\left(\frac{\varepsilon\mu\omega^2}{\gamma^4} + \frac{k^2}{\gamma^4}\right)\underbrace{\int |\underline{\nabla}_t \Psi|^2 dF}_{\gamma^2 \int |\Psi|^2 dF} + \int |\Psi|^2 dF\right)$$

$$u = \frac{1}{4\mu}\underbrace{\left(\frac{\varepsilon\mu\omega^2 + h^2}{\gamma^2} + 1\right)}_{\frac{2\varepsilon\mu\omega^2}{\gamma^2}} \int |\Psi|^2 \, dF$$

$$\boxed{u = \frac{\varepsilon\omega^2}{2\gamma^2} \int |\Psi|^2 \, dF}$$

$$\frac{P}{u} = \frac{\frac{\omega h}{2\mu\gamma^2}}{\frac{\varepsilon\omega^2}{2\gamma^2}} = \frac{1}{\varepsilon\mu} \cdot \frac{h}{\omega} = v_{cs}$$

Ürregenerátor



$$E_t(z=0,d) = 0$$

$$E_t(x,y,z) = E_t(x,y)\sin(hz)$$

$$h = \frac{\pi}{d} l \qquad l \in \mathbb{Z}$$

$$\varepsilon\mu\omega^2 = \gamma^2 + h^2 = \gamma^2 + \pi^2\frac{l^2}{d^2}$$

tételezt: $\gamma^2 = \pi^2\left(\frac{n^2}{a^2} + \frac{m^2}{b^2}\right)$

$$\omega^2 = \frac{\pi^2}{\varepsilon\mu}\left(\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{d^2}\right)$$

Fermat's principle

$\varepsilon(\underline{x})$ lassan változik $\quad \nabla \varepsilon(\underline{x}) \approx 0$

$$\Rightarrow \quad \Box \psi = -\varepsilon(\underline{x}) \mu_0 \omega^2 \cdot \psi - \Delta \psi = 0$$

$$v_f(\underline{x}) = \frac{1}{\sqrt{\varepsilon(\underline{x}) \mu_0}} = \frac{c}{n(\underline{x})} \quad \Rightarrow \quad \varepsilon(\underline{x}) \mu_0 = \frac{n^2(\underline{x})}{c^2}$$

$$\left[\Delta + \frac{\omega^2}{c^2} n^2(\underline{x}) \right] \psi = 0$$

$$\psi = e^{i \frac{\omega}{c} S(\underline{x})} = e^{i \frac{\omega}{c} S(\underline{x}_0) + i \underbrace{\frac{\omega}{c} \nabla S(\underline{x}_0)}_{\underline{k}(\underline{x}_0)} \cdot (\underline{x} - \underline{x}_0) + \dots}$$

$S(\underline{x})$: eikonál

hiba: lineáris ↑

$$\nabla \psi = i \frac{\omega}{c} \nabla S \cdot e^{i \frac{\omega}{c} S(\underline{x})}$$

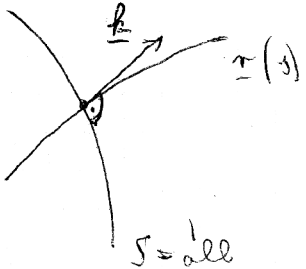
$$\Delta \psi = \left(i \frac{\omega}{c} \Delta S - \frac{\omega^2}{c^2} (\nabla S)^2 \right) e^{i \frac{\omega}{c} S(\underline{x})}$$

$$\frac{\omega^2}{c^2} \left(n^2 - (\nabla S)^2 \right) + i \underbrace{\frac{\omega}{c} \Delta S} = 0$$

$\Uparrow$ $\quad \frac{\omega^2}{c^2} (\nabla S)^2$

$\Downarrow$

$$\boxed{(\nabla S)^2 = n^2(\underline{x})}$$



$$\underline{e} = \frac{d\underline{r}}{ds} \qquad \nabla S \parallel \underline{k} \parallel \underline{e}$$

$$\nabla S = n(\underline{r}) \cdot \underline{e} \qquad / \cdot \underline{e}$$

$$\frac{d\underline{r}}{ds} \cdot \nabla S(\underline{r}) = n(\underline{r})$$

$$\Rightarrow \boxed{n(\underline{r}) = \frac{dS}{ds}(\underline{r}(s))}$$

$$\nabla S = n(\underline{r}) \cdot \frac{d\underline{r}}{ds} \xrightarrow{\frac{d}{ds}} \nabla \frac{dS}{ds} = \frac{d}{ds}\left(n(\underline{r}) \frac{d\underline{r}}{ds}\right)$$

<u>Alt. Snellius – Descartes</u>

$$\boxed{\frac{d}{ds}\left(n(\underline{r}) \frac{d\underline{r}}{ds}\right) = \nabla n(\underline{r})}$$

Liénard - Wiechert potenciál:

elnyírt pályán mozgó pontszerű test

$$\Box \phi = \frac{1}{\varepsilon_0} \cdot \varrho \qquad \varrho(\underline{x},t) = q\,\delta(\underline{x} - \underline{x}_0(t))$$

$$\Box \underline{A} = \mu_0 \cdot \underline{j} \qquad \underline{j}(\underline{x},t) = q\,\dot{\underline{x}}_0(t)\,\delta(\underline{x} - \underline{x}_0(t))$$

(d'Alembert-tétel)

$$\phi(\underline{x},t) = \frac{1}{4\pi\varepsilon_0} \int d^3\underline{x}' \int dt' \; \frac{1}{|\underline{x} - \underline{x}'|} \cdot \varrho(\underline{x}',t') \cdot \delta\left(t - t' - \frac{|\underline{x} - \underline{x}'|}{c}\right)$$

$$= \frac{q}{4\pi\varepsilon_0} \int dt' \; \frac{1}{|\underline{x} - \underline{x}_0(t')|} \cdot \delta\Big(\underbrace{t - t' - \frac{|\underline{x} - \underline{x}_0(t')|}{c}}_{-t_1}\Big)$$

$$t_1(t') = t' + \frac{r(t')}{c} - t \qquad \underline{r}(t') = \underline{x} - \underline{x}_0(t')$$

$$\frac{dt_1}{dt'} = 1 + \frac{\underline{n}}{c} \cdot (-\dot{\underline{x}}_0(t')) \qquad r(t') = |\underline{x} - \underline{x}_0(t')|$$

$$dt' = \frac{dt_1}{1 - \frac{\underline{n}\,\dot{\underline{x}}_0(t')}{c}} \qquad \underline{n} = \frac{\underline{r}}{r}$$

$$\phi(\underline{x},t) = \frac{q}{4\pi\varepsilon_0}\int dt_r \cdot \frac{1}{1-\frac{1}{c}\underline{n}\cdot\dot{\underline{x}}_0} \cdot \frac{1}{r}\cdot \delta(t_r)$$

$$t_r(\tilde{t}) = 0 \rightarrow \text{megoldás: } \tilde{t} = t_{ret}$$

$$t_{ret} + \frac{r(t_{ret})}{c} - t = 0$$

$$\phi(\underline{x},t) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{1-\frac{1}{c}\underline{n}\cdot\dot{\underline{x}}_0(t_{ret})}\cdot\frac{1}{r(t_{ret})}$$

$$\underline{A}(\underline{x},t) = \frac{\mu_0}{4\pi}\, q\cdot\dot{\underline{x}}_0(t_{ret})\cdot\frac{1}{1-\frac{1}{c}\underline{n}\cdot\dot{\underline{x}}_0(t_{ret})}\cdot\frac{1}{r(t_{ret})}$$

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla_{\underline{x}}\phi = ? \qquad (\text{majd } \underline{B} = \operatorname{rot}\underline{A} = ?)$$

$$\underline{A} = \underbrace{\varepsilon_0\mu_0}_{\frac{1}{c^2}}\cdot\dot{\underline{x}}_0(t_{ret})\cdot\phi$$

$$\frac{\partial \underline{A}}{\partial t} = \frac{1}{c^2}\phi\cdot\ddot{\underline{x}}_0(t_{ret})\cdot\frac{\partial t_{ret}}{\partial t} + \frac{1}{c^2}\dot{\underline{x}}_0(t_{ret})\cdot\frac{\partial\phi}{\partial t_{ret}}\cdot\frac{\partial t_{ret}}{\partial t}$$

$$t_{ret}(\underline{x},t) \qquad \frac{\partial t_{ret}}{\partial t} = ? \qquad \nabla_{\underline{x}} t_{ret} = ?$$

$$r = |\underline{x} - \underline{x}_0(t_{ret})| = c(t - t_{ret})$$

$$\frac{\partial r}{\partial t} = \underline{n}\cdot(-\dot{\underline{x}}_0)\cdot\frac{\partial t_{ret}}{\partial t} = c\left(1-\frac{\partial t_{ret}}{\partial t}\right)$$

$$\Rightarrow \left(1-\frac{\underline{n}\cdot\dot{\underline{x}}_0}{c}\right)\cdot\frac{\partial t_{ret}}{\partial t} = 1$$

$$\frac{\partial t_{ret}}{\partial t} = \frac{1}{1 - \frac{1}{c}\,\underline{n}\cdot\dot{\underline{x}}_0(t_{ret})}$$

$$r = \left|\underline{x} - \underline{x}_0(t_{ret})\right| = c\,(t - t_{ret})$$

$$\nabla_{\underline{x}}\, r = \underline{n} + \underbrace{\frac{\partial r}{\partial t_{ret}}}_{-\underline{n}\cdot\dot{\underline{x}}_0} \cdot \nabla_{\underline{x}}\, t_{ret} = -c\,\nabla_{\underline{x}}\, t_{ret}$$

$$\left(1 - \frac{\underline{n}\cdot\dot{\underline{x}}_0}{c}\right)\nabla_{\underline{x}}\, t_{ret} = -\frac{\underline{n}}{c}$$

$$\nabla_{\underline{x}}\, t_{ret} = -\frac{\underline{n}}{c}\cdot\frac{1}{1 - \frac{1}{c}\,\underline{n}\cdot\dot{\underline{x}}_0(t_{ret})}$$

$$R(t_{ret}) = r(t_{ret})\cdot\left(1 - \frac{\underline{n}\cdot\dot{\underline{x}}_0(t_{ret})}{c}\right) = r(t_{ret}) - \frac{\underline{r}(t_{ret})\cdot\dot{\underline{x}}_0(t_{ret})}{c}$$

$$\phi = \frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{R} \qquad \frac{\partial\phi}{\partial t_{ret}} = -\frac{q}{4\pi\varepsilon_0 c}\cdot\frac{1}{R^2}\cdot\frac{\partial R}{\partial t_{ret}}$$

$$\frac{\partial R}{\partial t_{ret}} = \underbrace{\frac{\partial r}{\partial t_{ret}}}_{-\underline{n}\cdot\dot{\underline{x}}_0(t_{ret})} - \frac{1}{c}\,\ddot{\underline{x}}_0\cdot\underline{r} - \frac{1}{c}\,\dot{\underline{x}}_0\,\underbrace{\frac{\partial\underline{r}}{\partial t_{ret}}}_{-\dot{\underline{x}}_0(t_{ret})}$$

$$\frac{\partial R}{\partial t_{ret}} = -\underline{n}\cdot\dot{\underline{x}}_0 - \frac{1}{c}\,\ddot{\underline{x}}_0\cdot\underline{r} + \frac{1}{c}\,\dot{\underline{x}}_0^2$$

$$\nabla_{\underline{x}}\,\phi = -\frac{q}{4\pi\varepsilon_0}\cdot\frac{1}{R^2}\,\nabla_{\underline{x}}\, R$$

$$\nabla_x R = \underline{n} - \frac{\dot{\underline{x}}_0}{c} + \frac{\partial R}{\partial t_{ret}} \cdot \nabla_x t_{ret} =$$

$$= \underline{n} - \frac{\dot{\underline{x}}_0}{c} + \frac{1}{1 - \frac{1}{c}\underline{n}\dot{\underline{x}}_0} \cdot \frac{\underline{n}}{c}\left(\underline{n}\dot{\underline{x}}_0 + \frac{1}{c}\ddot{\underline{x}}_0 \underline{r} - \frac{1}{c}\dot{\underline{x}}_0^2\right)$$

$$\frac{\partial \underline{A}}{\partial t} = \frac{1}{c^2} \cdot \frac{q}{4\pi\varepsilon_0 R} \cdot \ddot{\underline{x}}_0 \cdot \frac{1}{1 - \frac{1}{c}\underline{n}\dot{\underline{x}}_0} - \frac{1}{c^2}\dot{\underline{x}}_0 \frac{q}{4\pi\varepsilon_0 R^2} \frac{\partial R}{\partial t_{ret}} \cdot \frac{1}{1 - \frac{1}{c}\underline{n}\dot{\underline{x}}_0}$$

$$= \frac{q}{4\pi\varepsilon_0 R} \cdot \frac{1}{1 - \frac{1}{c}\underline{n}\dot{\underline{x}}_0} \cdot \frac{1}{c^2}\left(\ddot{\underline{x}}_0 + \dot{\underline{x}}_0 \frac{1}{R}\left(\underline{n}\dot{\underline{x}}_0 + \frac{1}{c}\ddot{\underline{x}}_0 \underline{r} - \frac{1}{c}\dot{\underline{x}}_0^2\right)\right)$$

$$= \frac{q}{4\pi\varepsilon_0 R} \cdot \frac{\underline{r}}{R} \cdot \frac{1}{c^2}\left(\ddot{\underline{x}}_0 + \frac{1}{Rc}(\dot{\underline{x}}_0 \underline{r})\dot{\underline{x}}_0 + \frac{1}{R}(\underline{n}\dot{\underline{x}}_0)\dot{\underline{x}}_0 - \frac{1}{Rc}\dot{\underline{x}}_0^2 \cdot \dot{\underline{x}}_0\right)$$

$$\nabla_x \Phi = -\frac{q}{4\pi\varepsilon_0 R^2}\left(\underline{n} - \frac{\dot{\underline{x}}_0}{c} + \underbrace{\frac{1}{1 - \frac{1}{c}\underline{n}\dot{\underline{x}}_0}}_{\frac{r}{R}} \cdot \frac{\underline{n}}{c}\left(\underline{n}\dot{\underline{x}}_0 + \frac{1}{c}\ddot{\underline{x}}_0 \underline{r} - \frac{1}{c}\dot{\underline{x}}_0^2\right)\right)$$

$$\underline{E} = -\frac{\partial \underline{A}}{\partial t} - \nabla_x \Phi = \frac{q}{4\pi\varepsilon_0 R^2}\left[-\frac{\underline{r}}{c^2}\ddot{\underline{x}}_0 - \frac{\underline{r}}{Rc^3}(\dot{\underline{x}}_0 \underline{r})\dot{\underline{x}}_0 - \right.$$

$$- \frac{\underline{r}}{Rc^2}(\underline{n}\dot{\underline{x}}_0)\dot{\underline{x}}_0 + \frac{\underline{r}}{Rc^3}\dot{\underline{x}}_0^2 \cdot \dot{\underline{x}}_0 + \underline{n} - \frac{\dot{\underline{x}}_0}{c} + \frac{\underline{r}}{Rc}\underline{n}(\underline{n}\dot{\underline{x}}_0) +$$

$$\left. + \frac{\underline{r}}{Rc^2}\underline{n}(\dot{\underline{x}}_0 \underline{n}) - \frac{\underline{r}}{Rc^2}\underline{n}(\dot{\underline{x}}_0^2)\right]$$

Tagonkból $\underline{n} - \frac{\dot{\underline{x}}_0}{c}$ kiemelhető.

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 R^2}\left[-\frac{r}{c^2}\ddot{\underline{x}}_0 + \frac{r}{Rc^2}\left(\ddot{\underline{x}}_0 \underline{r}\right)\left(\underline{n} - \frac{\dot{\underline{x}}_0}{c}\right) + \frac{r}{Rc}\left(\underline{n}\,\dot{\underline{x}}_0\right)\left(\underline{n} - \frac{\dot{\underline{x}}_0}{c}\right)\right.$$

$$\left. - \frac{r}{Rc^2}\left(\dot{\underline{x}}_0^2\right)\cdot\left(\underline{n} - \frac{\dot{\underline{x}}_0}{c}\right) + \left(\underline{n} - \frac{\dot{\underline{x}}_0}{c}\right)\right]$$

$$\boxed{\underline{R} = r\left(\underline{n} - \frac{\dot{\underline{x}}_0}{c}\right)} \qquad \boxed{\underline{R}\,\underline{n} = r\left(1 - \frac{\underline{n}\,\dot{\underline{x}}_0}{c}\right) = R}$$

$$\underline{E} = \frac{q}{4\pi\varepsilon_0 R^2}\left[-\frac{r}{c^2}\ddot{\underline{x}}_0 + \frac{1}{Rc^2}\left(\ddot{\underline{x}}_0 \underline{r}\right)\underline{R} + \frac{1}{Rc}\left(\underline{n}\,\dot{\underline{x}}_0\right)\underline{R}\right.$$

$$\left. - \frac{1}{Rc^2}\left(\dot{\underline{x}}_0^2\right)\cdot\underline{R} + \frac{1}{r}\cdot\underline{R}\right] =$$

$$= \frac{q}{4\pi\varepsilon_0 R^2}\left[\frac{r}{Rc^2}\left(\underline{n}\,\ddot{\underline{x}}_0\right)\underline{R} - \frac{r}{Rc^2}R\cdot\ddot{\underline{x}}_0\right.$$

$$\left. - \frac{1}{c^2}\left(\dot{\underline{x}}_0^2\right)\frac{1}{R}\underline{R} + \frac{1}{r}\underline{R}\left(1 + \frac{r}{R}\cdot\frac{\underline{n}\,\dot{\underline{x}}_0}{c}\right)\right]$$

$$1 + \frac{1}{1 - \frac{1}{c}\underline{n}\,\dot{\underline{x}}_0}\cdot\frac{\underline{n}\,\dot{\underline{x}}_0}{c} = \frac{1}{1 - \frac{1}{c}\underline{n}\,\dot{\underline{x}}_0}\left(1 - \cancel{\frac{1}{c}\underline{n}\,\dot{\underline{x}}_0} + \cancel{\frac{1}{c}\underline{n}\,\dot{\underline{x}}_0}\right) = \frac{r}{R}$$

$$\underline{n}\times\left(\underline{R}\times\ddot{\underline{x}}_0\right) = \left(\underline{n}\,\ddot{\underline{x}}_0\right)\underline{R} - \underbrace{\left(\underline{n}\,\underline{R}\right)}_{R}\ddot{\underline{x}}_0$$

$$\boxed{\underline{E}(x,t) = \frac{q}{4\pi\varepsilon_0 R^2}\left\{\frac{r}{Rc^2}\,\underline{n}\times\left(\underline{R}\times\ddot{\underline{x}}_0\right) + \left(1 - \frac{\dot{\underline{x}}_0^2}{c^2}\right)\frac{\underline{R}}{R}\right]}$$

regolari ter. $\sim \frac{1}{r}$ — Coulomb-ter. $\sim \frac{1}{r^2}$

$$\underline{B} = \mathrm{rot}\,\underline{A} = \mathrm{rot}\left(\frac{1}{c^2}\,\dot{\underline{x}}_0\,\Phi\right)$$

$$B_i = \frac{1}{c^2}\,\varepsilon_{ijk}\,\partial_j\left(\dot{x}_{0k}\,\Phi\right) = \frac{1}{c^2}\,\varepsilon_{ijk}\,\ddot{x}_{0k}\,\Phi\cdot\partial_j t_{ret} +$$

$$+\frac{1}{c^2}\,\varepsilon_{ijk}\,\dot{x}_{0k}\cdot\partial_j\Phi$$

$$\underline{B} = \frac{1}{c^2}\,\Phi\left(\nabla_{\underline{x}}\,t_{ret}\right)\times\ddot{\underline{x}}_0 + \frac{1}{c^2}\,\nabla_{\underline{x}}\Phi\times\dot{\underline{x}}_0 = \dots$$

$$\underline{E}_{fern} = \frac{q}{4\pi\varepsilon_0 c^2}\cdot\frac{r}{R^3}\,\underline{n}\times\left(\underline{R}\times\ddot{\underline{x}}_0\right)$$

$$\underline{B}_{fern} = \frac{1}{c}\left(\underline{n}\times\underline{E}_{fern}\right)$$

$$\underline{S} = \frac{1}{\mu_0}\left(\underline{E}_{fern}\times\underline{B}_{fern}\right) = \frac{1}{\mu_0 c}\left[\underline{E}_{fern}\times\left(\underline{n}\times\underline{E}_{fern}\right)\right] =$$

$$\underline{E}_{fern}^2\,\underline{n} - \underbrace{\left(\underline{E}_{fern}\cdot\underline{n}\right)}_{0}\underline{E}_{fern}$$

$$= \underline{n}\cdot\frac{1}{\mu_0 c}\cdot\frac{q^2}{16\pi^2\varepsilon_0^2 c^4}\cdot\frac{r^2}{R^6}\left(\underline{n}\times\left(\underline{R}\times\ddot{\underline{x}}_0\right)\right)^2$$

$$\underline{S}(\underline{x},t) = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\cdot\frac{r^2}{R^6}\left(\underline{n}\times\left(\underline{R}\times\ddot{\underline{x}}_0\right)\right)^2\cdot\underline{n}$$

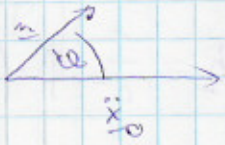
Nichtrelativistisches Retardat: $\dot{\underline{x}}_0^2 \ll c^2$

$$R = r\left(1-\frac{\underline{n}\,\dot{\underline{x}}_0}{c}\right)\approx r \qquad \underline{R} = \underline{r} - r\,\frac{\dot{\underline{x}}_0}{c}\approx\underline{r}$$

$$\underline{S}(\underline{x},t) = \frac{q^2}{16\pi^2\varepsilon_0 c^3}\cdot\frac{1}{r^2}\cdot\left[\underline{n}\times(\underline{n}\times\ddot{\underline{x}}_0)\right]^2\cdot\underline{n}$$

$$\underbrace{\underline{n}^2}_{1}\cdot(\underline{n}\times\ddot{\underline{x}}_0)^2\cdot\underbrace{\sin^2\frac{\pi}{2}}_{1} = \ddot{\underline{x}}_0^2\cdot\sin^2\vartheta$$

ϑ: $\underline{n}$ és $\ddot{\underline{x}}_0$ által bezárt szög



$$P(t) = \int \underline{S}\,d\underline{F} = \int (\underline{S}\,\underline{n})\,r^2\,d\Omega =$$

$$= \frac{q^2}{16\pi^2\varepsilon_0 c^3}\,\ddot{\underline{x}}_0^2\cdot\underbrace{\int_0^{2\pi}d\varphi\int_{-1}^{+1}d(\cos\vartheta)\,\sin^2\vartheta}_{\xi=\cos\vartheta}$$

$$2\pi\int_{-1}^{+1}d\xi\,(1-\xi^2) = 2\pi\underbrace{\left[\xi-\frac{1}{3}\xi^3\right]_{-1}^{+1}}_{\frac{4}{3}}$$

$$\boxed{P(t) = \frac{q^2}{6\pi\varepsilon_0 c^3}\,\ddot{\underline{x}}_0^2(t_{ret})}$$ Larmor-képlet

Relativisztikus mozgás

$\beta = \frac{\dot{x}_0}{c} \lesssim 1$ $\quad t_{ret}\in(T_1,T_2)$ időintervallum

$\underline{R} = r(\underline{n}-\beta)$ $\quad R = (1-\underline{n}\beta)\,r$

$$\frac{\partial t_{ret}}{\partial t} = \frac{1}{1-\underline{n}\beta}$$

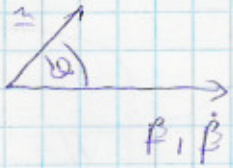
$$\underline{S} = \frac{q^2}{16\pi^2 \varepsilon_0 c} \cdot \frac{1}{r^2} \cdot \frac{1}{(1-\vec{n}\vec{\beta})^5} \cdot \left(\vec{n} \times \left((\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right)\right)^2$$

$$\frac{dW}{d\Omega} = \int_{T_1 + \frac{1}{c} r(T_1)}^{T_2 + \frac{1}{c} r(T_2)} dt \; r^2 \underline{S}(t_{ret}) = \int_{T_1}^{T_2} dt_{ret} \cdot \underbrace{\frac{dt}{dt_{ret}}}_{1-\vec{n}\vec{\beta}} \cdot r^2 \underline{S}(t_{ret})$$

$$\frac{dW}{d\Omega} = \int_{T_1}^{T_2} dt_{ret} \; \frac{q^2}{16\pi^2 \varepsilon_0 c} \cdot \frac{1}{(1-\vec{n}\vec{\beta})^5} \left(\vec{n} \times \left((\vec{n}-\vec{\beta}) \times \dot{\vec{\beta}}\right)\right)^2$$

Lineáris gyorsítás: $\vec{\beta} \parallel \dot{\vec{\beta}}$ $\qquad \vec{\beta} \times \dot{\vec{\beta}} = 0$

$$\left|\vec{n} \times (\vec{n} \times \dot{\vec{\beta}})\right|^2 = \left|\vec{n} \times \dot{\vec{\beta}}\right|^2 = \dot{\beta}^2 \sin^2\vartheta$$



$$\frac{dW}{d\Omega} = \int_{T_1}^{T_2} dt_{ret} \; \frac{q^2 \dot{\beta}^2(t_{ret})}{16\pi^2 \varepsilon_0 c} \cdot \frac{\sin^2\vartheta}{(1-\beta\cos\vartheta)^5}$$

$\vartheta_{max} = ?$ $\qquad \xi = \cos\vartheta$

$$\frac{1-\xi^2}{(1-\beta\cdot\xi)^5} = \max \rightarrow \ln(1-\xi^2) - 5\ln(1-\beta\cdot\xi) = \max$$

$$-\frac{2\xi}{1-\xi^2} + \frac{5\beta}{1-\beta\cdot\xi} = 0$$

$$-2\xi\cdot(1-\beta\cdot\xi) + 5\beta\cdot(1-\xi^2) = 0$$

$$-3\beta\cdot\xi^2 - 2\xi + 5\beta = 0$$

$$3\beta\cdot\xi^2 + 2\xi - 5\beta = 0 \qquad \xi \in [-1,+1]$$

$$\xi = \frac{-1+\sqrt{1+15\beta^2}}{3\beta}$$

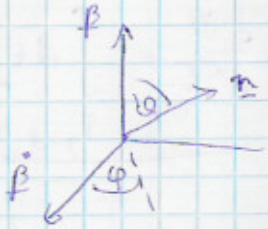
$$\vartheta_{max} = \arccos\left[\frac{1}{3\beta}\left(\sqrt{1+15\beta^2}-1\right)\right]$$

$\beta \to 0 \Rightarrow \vartheta_{max} \to \frac{\pi}{2}$ $\qquad$ $\beta \to 1 \Rightarrow \vartheta_{max} \to 0$

$$W = \int_{T_1}^{T_2} dt_{ret}\cdot\frac{q^2\cdot\dot\beta^2}{16\pi^2\varepsilon_0 c}\cdot\underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\int_{-1}^{+1} d(\cos\vartheta)\,\frac{\sin^2\vartheta}{(1-\beta\cos\vartheta)^5}$$

$$\cdot\int_{-1}^{+1}\frac{1-\xi^2}{(1-\beta\cdot\xi)^5}\,d\xi = \frac{4}{3}\cdot\frac{1}{(1-\beta^2)^3}$$

$$W = \int_{T_1}^{T_2} dt_{ret}\,\frac{q^2}{6\pi\varepsilon_0 c}\,\frac{\dot\beta^2}{(1-\beta^2)^3}$$



Synchrotron (Kämpfer): $\dot\beta \perp \beta$

$$\beta = \begin{pmatrix}0\\0\\\beta_z\end{pmatrix} \qquad \dot\beta = \begin{pmatrix}|\dot\beta|\\0\\0\end{pmatrix} \qquad n = \begin{pmatrix}\sin\vartheta\cos\varphi\\ \sin\vartheta\sin\varphi\\ \cos\vartheta\end{pmatrix}$$

$$n\times(n\times\dot\beta) - n\times(\beta\times\dot\beta) = (n\dot\beta)\,n - (nn)\,\dot\beta -$$

$$-(n\dot\beta)\,\beta + (n\beta)\,\dot\beta = (n\dot\beta)\cdot(n-\beta) - (1-n\beta)\,\dot\beta =$$

$$= |\dot\beta|\sin\vartheta\cos\varphi\cdot(n-\beta) - (1-\beta\cos\vartheta)\,\dot\beta$$

$$\left|\vec{n}\times\left(\left(\vec{n}-\vec{\beta}\right)\times\dot{\vec{\beta}}\right)\right|^2 = \dot{\beta}^2 \sin^2\vartheta\cos^2\varphi\cdot\left(\vec{n}-\vec{\beta}\right)^2 -$$

$$-2\left|\dot{\vec{\beta}}\right|\sin\vartheta\cos\varphi\cdot\left(1-\beta\cos\vartheta\right)\left(\vec{n}-\vec{\beta}\right)\dot{\vec{\beta}} + \left(1-\beta\cos\vartheta\right)^2\cdot\dot{\beta}^2 =$$

$$= \dot{\beta}^2\sin^2\vartheta\cos^2\varphi\cdot\left(1+\beta^2-2\beta\cos\vartheta\right) -$$

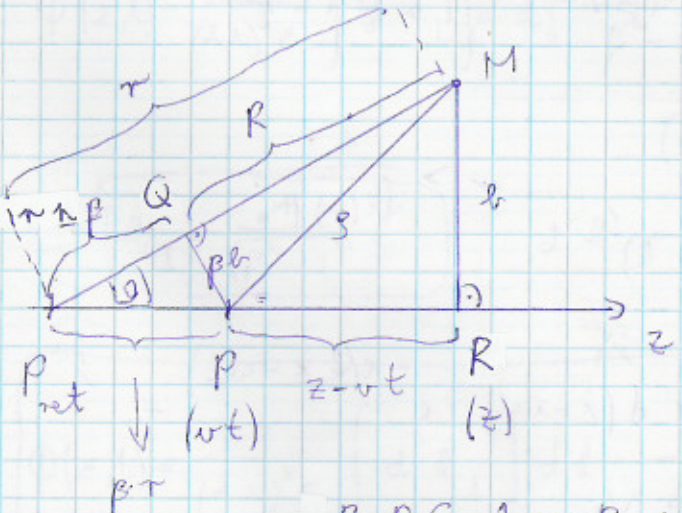
$$-2\dot{\beta}^2\sin^2\vartheta\cos^2\varphi\left(1-\beta\cdot\cos\vartheta\right) + \left(1-\beta\cos\vartheta\right)^2\cdot\dot{\beta}^2 =$$

$$= \dot{\beta}^2\left[\left(1-\beta\cos\vartheta\right)^2 - \left(1-\beta^2\right)\sin^2\vartheta\cos^2\varphi\right]$$

$$\frac{dW}{d\Omega} = \int_{T_1}^{T_2} dt_{ret}\,\frac{q^2\dot{\beta}^2}{16\pi^2\varepsilon_0 c}\left[\frac{1}{\left(1-\beta\cos\vartheta\right)^3} - \frac{\left(1-\beta^2\right)\sin^2\vartheta\cos^2\varphi}{\left(1-\beta\cos\vartheta\right)^5}\right]$$

$$W = \int_{T_1}^{T_2} dt_{ret}\,\frac{q^2}{6\pi\varepsilon_0 c}\cdot\frac{\dot{\beta}^2}{\left(1-\beta^2\right)^2}$$

Elvipte'isli felületek dist'ációja $v = áll$



$\overline{P_{ret}M} = r \qquad \overline{PM} = \rho$

$\overline{P_{ret}P} = \beta r \qquad \overline{MR} = \ell$

$\overline{P_{ret}Q} = \beta r\cos\vartheta = r\cdot\vec{n}\vec{\beta}$

$\overline{QM} = r\left(1-\vec{n}\vec{\beta}\right) = R$

$P_{ret}PQ\,\Delta \sim P_{ret}MR\,\Delta \Rightarrow \overline{PQ} = \beta\cdot\ell$

$MPQ\,\Delta$ Pitagorasz-tétel $\Rightarrow \rho^2 = R^2 + \beta^2\ell^2$

$MPR\,\Delta$ ——— " ——— $\Rightarrow \rho^2 = (z-vt)^2 + \ell^2$

$$\rho^2 = R^2 + \beta^2 \varrho^2 = (z - vt)^2 + \varrho^2 \quad \Rightarrow \quad \boxed{R^2 = (z - vt)^2 + (1 - \beta^2)\,\varrho^2}$$

$$\phi = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{R} = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{(z - vt)^2 + (1 - \beta^2) \cdot \varrho^2}}$$

$$\boxed{\phi = \frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{\sqrt{1 - \beta^2}} \cdot \frac{1}{\sqrt{\dfrac{(z - vt)^2}{1 - \beta^2} + \varrho^2}}}$$

ekvipot. felület: $\left(\dfrac{z - vt}{\sqrt{1 - \beta^2}}\right)^2 + \varrho^2 = \text{áll}$

Lorentz-kontrakció: kör $\longrightarrow$ ellipszis

Csernikov-sugárzás

Fourier-transzformált kifejezések:

$$\left(-\varepsilon(\omega)\,\mu_0\,\omega^2 + k^2\right)\,\Phi(\underline{k}, \omega) = \frac{1}{\varepsilon(\omega)} \cdot \varrho(\underline{k}, \omega)$$

$$\left(-\varepsilon(\omega) \cdot \mu_0\,\omega^2 + k^2\right)\,\underline{A}(\underline{k}, \omega) = \mu_0 \cdot \underline{j}(\underline{k}, \omega)$$

$$\varrho(\underline{x}, t) = q\,\delta(\underline{x} - \underline{v}\,t) \qquad v > \frac{1}{\sqrt{\varepsilon(\omega)\,\mu_0}}$$

$$\underline{j}(\underline{x}, t) = q\,\underline{v}\,\delta(\underline{x} - \underline{v}\,t)$$

$$\varrho(\underline{k}, \omega) = \int d^3x \int dt \cdot q\,\delta(\underline{x} - \underline{v}\,t) \cdot e^{-i(\underline{k}\,\underline{x} - \omega t)} =$$

$$= q \cdot \int dt\; e^{-i(\underline{k}\,\underline{v} - \omega)\,t} = 2\pi q \cdot \delta(\omega - \underline{k}\,\underline{v})$$

$$\phi(\underline{k},\omega) = 2\pi \cdot \frac{q}{\varepsilon(\omega)} \cdot \frac{\delta(\omega - \underline{k}\underline{v})}{k^2 - \varepsilon(\omega)\mu_0\omega^2}$$

$$\underline{A}(\underline{k},\omega) = \varepsilon(\omega)\mu_0 \underline{v}\, \phi(\underline{k},\omega)$$

$$\phi(x,t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \cdot 2\pi \cdot \frac{q}{\varepsilon(\omega)} \cdot \frac{\delta(\omega - \underline{k}\underline{v})}{k^2 - \varepsilon(\omega)\mu_0\omega^2} e^{i(\underline{k}\underline{x} - \omega t)} =$$

$$= \int \frac{d^3k}{(2\pi)^3} \cdot \frac{q}{\varepsilon(\underline{k}\underline{v})} \cdot \frac{1}{k^2 - \varepsilon(\underline{k}\underline{v})\mu_0(\underline{k}\underline{v})^2} e^{i\underline{k}(\underline{x} - \underline{v}t)}$$

$d^3k = d^2k_\perp \cdot dk_\parallel$ + filtering: $\varepsilon(\underline{k}\underline{v}) = \varepsilon = \text{const}$

$$\phi(\underline{x},t) = \frac{q}{(2\pi)^3\varepsilon} \int d^2k_\perp \int dk_\parallel \cdot \frac{1}{k_\perp^2 - (\varepsilon\mu_0 v^2 - 1)k_\parallel^2} e^{i(\underline{k}_\perp \underline{x}_\perp + k_\parallel(x_\parallel - vt))}$$

$$\boxed{\beta = \frac{v}{v_f} = \sqrt{\varepsilon\mu_0} \cdot v} \qquad \boxed{\xi = x_\parallel - v \cdot t}$$

$$\phi(\underline{x},t) = \frac{q}{(2\pi)^3\varepsilon} \int d^2k_\perp \int dk_\parallel \frac{1}{\beta^2 - 1} \frac{e^{i(\underline{k}_\perp \underline{x}_\perp + k_\parallel \xi)}}{\left(\frac{k_\perp}{\sqrt{\beta^2-1}}\right)^2 - k_\parallel^2}$$

$$\boxed{\underline{k}'_\perp = \frac{1}{\sqrt{\beta^2-1}}\underline{k}_\perp} \qquad d^2k'_\perp = \frac{1}{\beta^2-1} d^2k_\perp$$

$$\boxed{\phi(\underline{x},t) = \frac{q}{(2\pi)^3\varepsilon} \int d^2k'_\perp \int dk_\parallel \frac{1}{k'^2_\perp - k_\parallel^2} e^{i(\underline{k}_\perp \underline{x}_\perp + k_\parallel \xi)}}$$

$\sqrt{\beta^2-1}\cdot\underline{k}'_\perp$

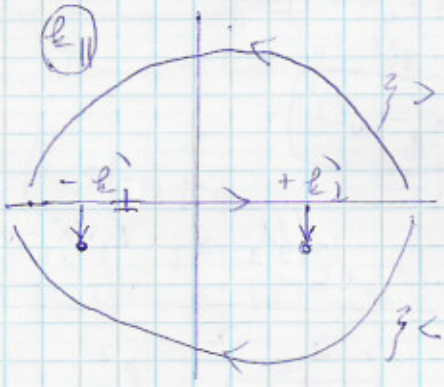
$$\frac{1}{k_\perp'^2 - k_\parallel^2} = -\frac{1}{(k_\parallel + k_\perp')(k_\parallel - k_\perp')} = \frac{a}{k_\parallel - k_\perp'} + \frac{b}{k_\parallel + k_\perp'}$$

$$-1 = a(k_\parallel + k_\perp') + b(k_\parallel - k_\perp')$$

$$\left.\begin{aligned} a + b &= 0 \\ (a - b)\, k_\perp' &= -1 \end{aligned}\right\} \Rightarrow \quad a = -\frac{1}{2k_\perp'} \qquad b = \frac{1}{2k_\perp'}$$

$$\phi(x,t) = \frac{q}{(2\pi)^3 \varepsilon} \int d^2 k_\perp' \int dk_\parallel \left(\frac{1}{k_\parallel + k_\perp'} - \frac{1}{k_\parallel - k_\perp'}\right) \frac{e^{i(k_\perp' x_\perp + k_\parallel \zeta)}}{2k_\perp'}$$

$$= \frac{q}{(2\pi)^3 \varepsilon} \int d^2 k_\perp' \, \frac{e^{i k_\perp' x_\perp}}{2k_\perp'} \int dk_\parallel \left(\frac{1}{k_\parallel + k_\perp'} - \frac{1}{k_\parallel - k_\perp'}\right) e^{i k_\parallel \zeta}$$



$$e^{i k_\parallel \zeta} = e^{i(\mathrm{Re}\, k_\parallel)\zeta} \cdot e^{-(\mathrm{Im}\, k_\parallel)\cdot \zeta}$$

$$(\mathrm{Im}\, k_\parallel)\cdot \zeta > 0$$

$$\zeta > 0 \Rightarrow \quad \phi(x,t) = 0$$

$\underline{\zeta < 0}$

$$\int dk_\parallel \left(\frac{e^{i k_\parallel \zeta}}{k_\parallel + k_\perp'} - \frac{e^{i k_\parallel \zeta}}{k_\parallel - k_\perp'}\right) =$$

$$= -2\pi i \cdot \left(e^{-i k_\perp' \cdot \zeta} - e^{i k_\perp' \cdot \zeta}\right)$$

$$\phi(x_\perp, \zeta<0) = \frac{qi}{(2\pi)^2\varepsilon}\int d^2\hat{k}_\perp \; \frac{e^{i\sqrt{\beta^2-1}\,\hat{k}_\perp \cdot x_\perp}}{2\hat{k}_\perp}\cdot\left(e^{i\hat{k}_\perp\zeta} - e^{-i\hat{k}_\perp\zeta}\right)$$

$\hat{k}_\perp$ sulla $k_\perp$, ψ coordinate polari

$$= \frac{qi}{(2\pi)^2\varepsilon}\int_{-\pi}^{+\pi} d\psi \int_0^\infty d\hat{k}_\perp \cdot \hat{k}_\perp \cdot \frac{e^{i\sqrt{\beta^2-1}\,\hat{k}_\perp |x_\perp| \cos\psi}}{2\hat{k}_\perp}\left(e^{i\hat{k}_\perp\zeta} - e^{-i\hat{k}_\perp\zeta}\right)$$

$$= \frac{qi}{8\pi^2\varepsilon}\int_{-\pi}^{+\pi} d\psi \int_0^\infty d\hat{k}_\perp \left[e^{i\left(\sqrt{\beta^2-1}\,|x_\perp|\cos\psi + \zeta\right)\hat{k}_\perp} - \right.$$

$$\left. - e^{i\left(\sqrt{\beta^2-1}\,|x_\perp|\cos\psi - \zeta\right)\hat{k}_\perp}\right] = \leftarrow$$

regolarizzazione:
$$\int_0^\infty e^{iax}\,dx = -\frac{1}{ia}$$

$$= \frac{q}{8\pi^2\varepsilon}\int_{-\pi}^{+\pi} d\psi \left[-\frac{1}{\sqrt{\beta^2-1}\,|x_\perp|\cos\psi + \zeta} + \frac{1}{\sqrt{\beta^2-1}\,|x_\perp|\cos\psi - \zeta}\right] =$$

$$= \frac{q}{8\pi^2\varepsilon}\int_{-\pi}^{+\pi} d\psi \; \frac{-\left(\sqrt{\beta^2-1}\,|x_\perp|\cos\psi - \zeta\right) + \left(\sqrt{\beta^2-1}\,|x_\perp|\cos\psi + \zeta\right)}{(\beta^2-1)\,x_\perp^2\cos^2\psi - \zeta^2}$$

$$\boxed{\phi(x_\perp, \zeta<0) = \frac{q}{4\pi^2\varepsilon}\int_{-\pi}^{+\pi} d\psi \; \frac{\zeta}{(\beta^2-1)\,x_\perp^2\cos^2\psi - \zeta^2}}$$

$$\int_{-\pi}^{+\pi} d\psi = 2\cdot\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\psi$$

$$\phi(\underline{x}_\perp | \xi<0) = \frac{q}{2\pi^2\varepsilon}\cdot\xi\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} d\psi \; \frac{1}{(\beta^2-1)\underline{x}_\perp^2\cos^2\psi - \xi^2}$$

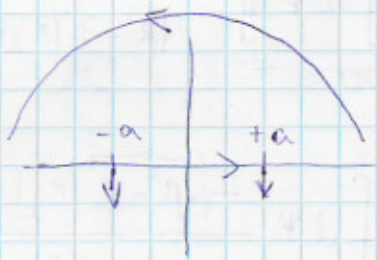
$$u = \operatorname{tg}\psi \qquad \cos^2\psi = \frac{1}{1+u^2} \qquad d\psi = \frac{du}{1+u^2}$$

$$\phi(\underline{x}_\perp | \xi<0) = \frac{q}{2\pi^2\varepsilon}\cdot\xi\int_{-\infty}^{+\infty} du \; \frac{1}{(\beta^2-1)\underline{x}_\perp^2 - \xi^2(1+u^2)}$$

$$\hat{u} = |\xi|\, u \qquad \xi<0$$

$$\boxed{\phi(\underline{x}_\perp | \xi<0) = -\frac{q}{2\pi^2\varepsilon}\int_{-\infty}^{+\infty} d\hat{u}\cdot\frac{1}{(\beta^2-1)\underline{x}_\perp^2 - \xi^2 - \hat{u}^2}}$$

$$(\beta^2-1)\underline{x}_\perp^2 - \xi^2 = a^2 > 0$$



$$\int_{-\infty}^{+\infty}\frac{1}{a^2-x^2}\,dx = 0$$

$$\Rightarrow \quad \boxed{\phi(\underline{x}_\perp | \xi<0) = 0}$$

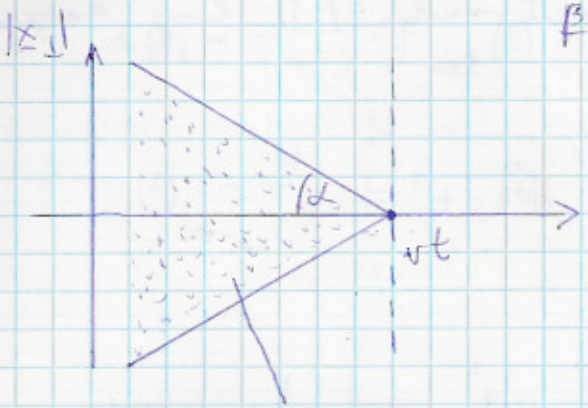
$$(\beta^2-1)\underline{x}_\perp^2 - \xi^2 = -a^2 < 0$$

$$\int_{-\infty}^{+\infty}\frac{1}{-a^2-x^2}\,dx = -\int_{-\infty}^{+\infty}\frac{1}{a^2+x^2}\,dx = -\frac{1}{a}\underbrace{\left[\operatorname{arctg}\left(\frac{x}{a}\right)\right]_{-\infty}^{+\infty}}_{\pi}$$

$$\varphi(\underline{x}_\perp, \xi < 0) = \frac{q}{2\pi\varepsilon} \cdot \frac{1}{\sqrt{\xi^2 - (\beta^2 - 1)\,\underline{x}_\perp^2}} \qquad \xi = x_\parallel - v t$$

$$\underline{A}(\underline{x}_\perp, \xi) = \varepsilon \cdot \mu_0 \cdot \underline{v} \cdot \varphi(\underline{x}_\perp, \xi) = \sqrt{\varepsilon \cdot \mu_0} \cdot \underline{\beta} \cdot \varphi(\underline{x}_\perp, \xi)$$

$$\underline{\beta} = \frac{\underline{v}}{v_\varphi} = \sqrt{\varepsilon \mu_0} \cdot \underline{v}$$



$$\xi^2 \geq (\beta^2 - 1)\,\underline{x}_\perp^2 \quad \Leftrightarrow \quad \xi < 0$$

$$\operatorname{tg}\alpha = \frac{1}{\sqrt{\beta^2 - 1}} \qquad \frac{\sin^2\alpha}{1 - \sin^2\alpha} = \frac{1}{\beta^2 - 1}$$

$$(\beta^2 - 1)\sin^2\alpha = 1 - \sin^2\alpha \quad \Rightarrow \quad \boxed{\sin\alpha = \frac{1}{\beta}}$$

Sugárzási visszahatás a töltött részecske mozgására

$$P_{sug}(t) = \frac{q^2}{6\pi\varepsilon_0 c^3}\cdot \dot{\underline{v}}^2$$

$m\,\dot{\underline{v}} = \underline{F}_{ext} + \underline{F}_{sug}$ → sugárzási visszahatási erő

$[0,T]$ int.-lan $\underline{F}_{sug}$ által végzett munka $= (-1)\cdot$ kisug. energia:

$$\int_0^T \underline{F}_{sug}\cdot \underline{v}\,dt = -\int_0^T \frac{q^2}{6\pi\varepsilon_0 c^3}\cdot \dot{\underline{v}}^2\,dt =$$

$$= -\frac{q^2}{6\pi\varepsilon_0 c^3}\underbrace{\left[\underline{v}\cdot\dot{\underline{v}}\right]_0^T}_{\to 0} + \int_0^T \underbrace{\frac{q^2}{6\pi\varepsilon_0 c^3}\ddot{\underline{v}}}_{\underline{F}_{sug}}\cdot\underline{v}\,dt$$

T szerint periodikus mozgásra

$$\underline{F}_{sug} = m\cdot\tau\cdot\dot{\underline{v}} \qquad \tau = \frac{q^2}{6\pi\varepsilon_0 c^3 m} = 6{,}3\cdot 10^{-24}\,\mathrm{s}$$

↑ e⁻-ra

$$\boxed{m\left(\dot{\underline{v}} - \tau\ddot{\underline{v}}\right) = \underline{F}_{ext}}$$

$$\underline{F}_{ext} = 0 \Rightarrow \ddot{\underline{v}} = \frac{1}{\tau}\dot{\underline{v}} \Rightarrow \dot{\underline{v}}(t) = \underline{a}\cdot e^{\frac{t}{\tau}}$$

öngyorsító elektron | csak $\underline{a}=0$-ra jó megoldás

állandó variálása: $\underline{v}(t) = \underline{a}(t) \cdot e^{\frac{t}{T}}$

$$\dot{\underline{v}}(t) = \left(\dot{\underline{a}}(t) + \frac{1}{T}\underline{a}(t)\right) e^{\frac{t}{T}}$$

$$m\left(\cancel{\underline{a}(t)} - T\cdot\dot{\underline{a}}(t) - \cancel{\underline{a}(t)}\right) e^{\frac{t}{T}} = \underline{F}_{ext}(t)$$

$$\dot{\underline{a}}(t) = -\frac{1}{mT}\cdot\underline{F}_{ext}(t)\cdot e^{-\frac{t}{T}}$$

$$\underline{a}(t) = -\frac{1}{mT}\int_{t_0}^{t}\underline{F}_{ext}(t')\cdot e^{-\frac{t'}{T}}\,dt' + \underline{a}_0 \quad /\cdot e^{\frac{t}{T}}$$

$$\underline{v}(t) = -\frac{1}{mT}\int_{t_0}^{t}\underline{F}_{ext}(t')\, e^{\frac{t-t'}{T}}\,dt' + \underbrace{\underline{a}_0 \cdot e^{\frac{t}{T}}}_{\text{megoldás}}$$

megoldás

$\underline{a}_0 := 0$

$$\tau = \frac{t'-t}{T} \qquad \tau_0 = \frac{t_0-t}{T}$$

$$\underline{v}(t) = -\frac{1}{m}\int_{\tau_0}^{0}\underline{F}_{ext}(t+T\cdot\tau)\cdot e^{-\tau}\,d\tau$$

$$\underline{v}(t) = \frac{1}{m}\int_{0}^{\tau_0}\underline{F}_{ext}(t+T\cdot\tau)\, e^{-\tau}\,d\tau$$

$\tau_0 = ?$ $\quad T \to 0$-ra $\quad \underline{v}(t) = \frac{\underline{F}_{ext}(t)}{m}\int_0^{\tau_0} e^{-\tau}\,d\tau = \frac{\underline{F}_{ext}(t)}{m}\left(1 - e^{-\tau_0}\right)$

$\tau_0 \to \infty$ – ez hogyan viszi a Newton-törvényt

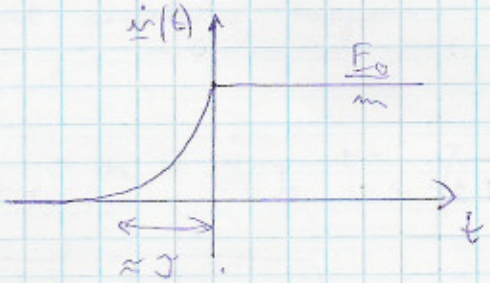
$$\underline{v}(t) = \frac{1}{m}\int_0^{+\infty} \underline{F}_{ext}(t+\tau\cdot s)\, e^{-s}\, ds$$ nem használjuk!

$$\underline{F}_{ext}(t) = \begin{cases} 0 & t<0 \\ \underline{F}_0 & t>0 \end{cases}$$

$$\underline{v}(t<0) = \frac{1}{m}\int_{-\frac{t}{\tau}}^{+\infty} \underline{F}_0 \cdot e^{-s}\, ds = \frac{\underline{F}_0}{m}\cdot e^{\frac{t}{\tau}}$$

$t+\tau\cdot s \geq 0 \Rightarrow s \geq -\frac{t}{\tau}$ - ra $\underline{F}_{ext}(t+\tau\cdot s) = \underline{F}_0$

$$\underline{v}(t>0) = \frac{1}{m}\int_0^{+\infty} \underline{F}_0 \cdot e^{-s}\, ds = \frac{\underline{F}_0}{m}$$



Ø sebesség perturbációsor: $\underline{F}_{ext}(t+\tau\cdot s) = \sum_{n=0}^{\infty} \frac{(\tau\cdot s)^n}{n!} \frac{\partial^n}{\partial t^n} \underline{F}_{ext}(t)$

$$m\,\underline{v}(t) = \sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{d^n}{dt^n} \underline{F}_{ext}(t) \cdot \underbrace{\int_0^{+\infty} s^n \cdot e^{-s}\, ds}_{\Gamma(n+1) = n!}$$

$$m\ddot{\underline{x}}(t) = \sum_{n=0}^{\infty} \tau^n \cdot \frac{d^n}{dt^n} \underline{F}_{ext}(t) = \underline{F}_{ext}(t) + \tau \cdot \frac{d\underline{F}_{ext}}{dt} + \dots$$

1d harmonikus rezgőmozgás

$$F_{ext} = -m\omega_0^2 \cdot x$$

$$\ddot{x}(t) = -\omega_0^2 \int_0^{+\infty} x(t+\tau\cdot\vartheta)\, e^{-\vartheta}\, d\vartheta$$

próbafv.: $x(t) = x_0 \cdot e^{-\lambda t}$

$$\lambda^2 \cancel{x_0 e^{-\lambda t}} = -\omega_0^2 \cancel{x_0} \int_0^{+\infty} e^{-\lambda(t+\tau\cdot\vartheta)} \cdot e^{-\vartheta}\, d\vartheta$$

$$\lambda^2 = -\omega_0^2 \int_0^{+\infty} e^{-(\lambda\cdot\tau+1)\vartheta}\, d\vartheta = -\omega_0^2 \cdot \frac{1}{\lambda\cdot\tau+1}$$

$$(\mathrm{Re}\,\lambda)\cdot\tau + 1 > 0$$

$\lambda^3\cdot\tau + \lambda^2 + \omega_0^2 = 0$ perturbáció alkalmazása

$\tau = 0$ -ra $\lambda = \pm i\omega_0$

$\lambda = i\omega_0 + \lambda_1\cdot\tau + \lambda_2\cdot\tau^2 + \dots$ τ^2 rendig

$$(i\omega_0 + \lambda_1\tau)^3\cdot\tau + (i\omega_0 + \lambda_1\tau + \lambda_2\tau^2)^2 + \omega_0^2 = 0$$

$$(i\omega_0)^3\tau + 3(i\omega_0)^2\lambda_1\tau^2 + \cancel{(i\omega_0)^2} + 2i\omega_0\cdot\lambda_1\tau + \lambda_1^2\tau^2 +$$

$$+ 2i\omega_0\cdot\lambda_2\tau^2 + \cancel{\omega_0^2} = 0$$

$$\underbrace{\left(-i\omega_0^3 + 2i\omega_0\alpha_1\right)}_{0}\cdot\gamma + \underbrace{\left(-3\omega_0^2\alpha_1 + \alpha_1^2 + 2i\omega_0\cdot\alpha_2\right)}_{0}\gamma^2 = 0$$

$$\alpha_1 = \frac{1}{2}\omega_0^2$$

$$\alpha_2 = \frac{3\omega_0^2\left(\frac{1}{2}\omega_0^2\right) - \left(\frac{1}{2}\omega_0^2\right)^2}{2i\omega_0}$$

$$\alpha_2 = -i\omega_0^3\left(\frac{3}{4} - \frac{1}{8}\right) = -\frac{5}{8}i\omega_0^3$$

$$\alpha = i\omega_0\left(1 - \frac{5}{8}\omega_0^2\gamma^2\right) + \frac{1}{2}\omega_0^2\gamma$$

$$\alpha = i\left(\omega_0 + \Delta\omega\right) + \frac{1}{2}\Gamma$$

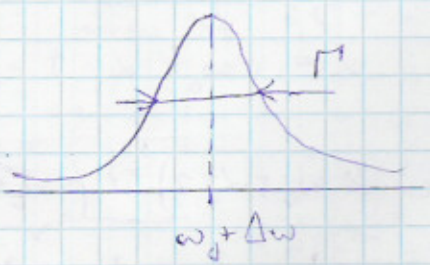
$$\Delta\omega = -\frac{5}{8}\omega_0^3\gamma^2 \qquad \Gamma = \omega_0^2\gamma$$

sajátfrekvencia eltolódása — csillapítási tényező

$$\tilde{x}(\omega) = \int_{-\infty}^{+\infty} dt\, x(t)\, e^{i\omega t} = x_0 \int_0^{+\infty} dt\, e^{(i\omega - \alpha)t} = x_0 \cdot \frac{1}{\alpha - i\omega}$$

$$\tilde{x}(\omega) = x_0 \cdot \frac{1}{\frac{1}{2}\Gamma + i\left(\omega_0 + \Delta\omega - \omega\right)}$$

$$\left|\tilde{x}(\omega)\right|^2 = x_0^2 \cdot \frac{1}{\left(\frac{1}{2}\Gamma\right)^2 + \left(\omega_0 + \Delta\omega - \omega\right)^2}$$



Lorentz-görbe

Γ: tenzsetes vonalszélesség

3d-s mozgás $V(r)$ gömbszimmetrikus potenciálban

E, L lassan változik → hogyan? (T-ben elsőrendig)

$$\underline{F}_{ext} = -\operatorname{grad} V = -\frac{dV}{dr}\cdot\underline{n} \qquad \underline{n} = \frac{\underline{r}}{r}$$

$$\frac{dE}{dt} = -P_{sug} = -m\cdot T\cdot \dot{\underline{v}}^2 \approx -\frac{T}{m}\cdot\left(\underline{F}_{ext}\right)^2 = -\frac{T}{m}\cdot\left(\frac{dV}{dr}\right)^2$$

$$\dot{\underline{v}} \approx \frac{\underline{F}_{ext}}{m}$$

egy periódusra átlagoljuk

$$\boxed{\frac{dE}{dt} = -\frac{T}{m}\cdot\overline{\left(\frac{dV}{dr}\right)^2}^{T}}$$

$$m\dot{\underline{v}} = \underline{F}_{ext} + T\cdot\frac{d\underline{F}_{ext}}{dt} = -\frac{dV}{dr}\cdot\underline{n} - T\cdot\frac{d}{dt}\left(\frac{1}{r}\frac{dV}{dr}\cdot\underline{r}\right) =$$

$$= -\left(\frac{dV}{dr} + T\cdot\frac{d}{dt}\left(\frac{1}{r}\frac{dV}{dr}\right)\cdot r\right)\underline{n} - T\cdot\frac{1}{r}\frac{dV}{dr}\cdot\underline{v}$$

$$\underline{L} = m\left(\underline{r}\times\underline{v}\right)$$

$$\frac{d\underline{L}}{dt} = m\left(\underline{r}\times\dot{\underline{v}}\right) = -T\cdot\frac{1}{r}\frac{dV}{dr}\cdot\underline{r}\times\underline{v} = -\frac{T}{m}\left(\frac{1}{r}\frac{dV}{dr}\right)\cdot\underline{L}$$

egy periódusra átlagoljuk

$$\boxed{\frac{d\underline{L}}{dt} = -\frac{T}{m}\cdot\overline{\left(\frac{1}{r}\frac{dV}{dr}\right)}^{T}\cdot\underline{L}}$$

spec.: $\boxed{V(r) = \frac{1}{2}m\omega_0^2\cdot r^2}$ $\Rightarrow$ $\frac{1}{r}\frac{dV}{dr} = m\omega_0^2 = \text{áll.}$

$$\boxed{\frac{d\underline{L}}{dt} = -\omega_0^2\cdot T\cdot\underline{L}} \qquad \Gamma = \omega_0^2 T \quad \text{csillapítási tényező}$$

Lorentz - dorentz elektron modell

e^- impulzusa a saját EM terétől származik

$$\underline{E} = \underline{E}_s + \underline{E}_{ext} \qquad \underline{B} = \underline{B}_s + \underline{B}_{ext}$$

impulzustétel:

$$\frac{\partial \underline{G}}{\partial t} + \operatorname{div} \underline{\underline{\Pi}} = -\underline{f} = -(\rho \underline{E} + \underline{j} \times \underline{B})$$

$$\underline{G} = \int \underline{g} \, d^3x = 0$$

teljes (saját + külső) EM tér összimpulzusa

$$\frac{d\underline{G}}{dt} = -\int d^3x (\rho \underline{E} + \underline{j} \times \underline{B}) =$$

$$= \underbrace{-\int d^3x (\rho \underline{E}_s + \underline{j} \times \underline{B}_s)}_{\frac{d\underline{f}}{dt}} \underbrace{- \int d^3x (\rho \underline{E}_{ext} + \underline{j} \times \underline{B}_{ext})}_{\underline{F}_{ext}} = 0$$

von. rendszer választás: Vizsgált t időpillanatban az e^- áll: $\underline{j}(x,t) = 0$

(de a retardált időpillanatban nem!)

$$\boxed{\frac{d\underline{f}}{dt} = -\int d^3x \, \rho \underline{E}_s = \int d^3x \, \rho \left(\nabla \phi_s + \frac{\partial \underline{A}_s}{\partial t} \right)}$$

$$\phi_g(\underline{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \frac{\rho\left(\underline{x}', t-\frac{|\underline{x}-\underline{x}'|}{c}\right)}{|\underline{x}-\underline{x}'|}$$

$$\underline{A}_g(\underline{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \frac{\underline{j}\left(\underline{x}', t-\frac{|\underline{x}-\underline{x}'|}{c}\right)}{|\underline{x}-\underline{x}'|}$$

$$\mu_0 = \frac{1}{\varepsilon_0 c^2}$$

$$\underline{R} = \underline{x} - \underline{x}' \qquad R = |\underline{R}| \qquad \hat{\underline{R}} = \frac{\underline{R}}{R}$$

$\underline{x}$; $\underline{x}'$ az e^- helyében $\Rightarrow \frac{R}{c}$ szerint sorfejthetünk:

$$\rho\left(\underline{x}', t-\frac{R}{c}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{\partial^n \rho}{\partial t^n}(\underline{x}', t)\cdot\left(\frac{R}{c}\right)^n$$

$$\phi_g(\underline{x},t) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^n}\int d^3x' \left(\frac{\partial}{\partial t}\right)^n \rho(\underline{x}',t)\cdot R^{n-1}$$

$$\underline{A}_g(\underline{x},t) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,c^{n+2}}\int d^3x' \left(\frac{\partial}{\partial t}\right)^n \underline{j}(\underline{x}',t)\cdot R^{n-1}$$

ϕ_g sorában $\frac{1}{c^n}$, $\underline{A}_g$ sorában $\frac{1}{c^{n+2}}$ van

$\Rightarrow$ Vizsgáljuk ϕ_g $n=0,1$ tagjainak járulékait, majd tegyük el 2-vel az összegzőindexet

$\underline{n=0}$: e^- által önmagára gyakorolt elektrosztatikus erő

$$\frac{1}{4\pi\varepsilon_0}\int d^3x \int d^3x' \underbrace{\rho(\underline{x},t)\,\rho(\underline{x}',t)}_{\downarrow}\underbrace{\nabla_x\left(\frac{1}{R}\right)}_{-\frac{\underline{R}}{R^3}} = 0$$

$(\underline{x},\underline{x}')$-ben szimmetrikus

$\rightarrow (\underline{x},\underline{x}')$-ben antiszimmetrikus

$n=1$: $\nabla_x \left(R^{1-1}\right) = 0$

Második sorban $n \to n+2$ helyettesítés:

$$\nabla_x \phi_1(x,t) = \ldots + \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(n+2)!\, c^{n+2}} \int d^3x' \left(\frac{\partial}{\partial t}\right)^{n+2} \varrho(x',t)\, \nabla_x R^{n+1}$$

$$\frac{\partial A_1}{\partial t}(x,t) = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\, c^{n+2}} \int d^3x' \left(\frac{\partial}{\partial t}\right)^{n+1} j(x',t) \cdot R^{n-1}$$

$$\frac{d\underline{\mathcal{F}}}{dt} = \frac{1}{4\pi\varepsilon_0} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!\, c^{n+2}} \int d^3x \int d^3x'\, \varrho(x,t) \cdot R^{n-1} \cdot \left(\frac{\partial}{\partial t}\right)^{n+1} \cdot$$

$$\cdot \left[j(x',t) + \frac{1}{(n+1)(n+2)} \cdot \frac{\partial \varrho}{\partial t}(x',t) \cdot \frac{\nabla_x R^{n+1}}{R^{n-1}} \right]$$

$[\ldots]$ előtti operátort jelölje $\S$

$$\frac{d\underline{\mathcal{F}}}{dt} = \S \left[j(x',t) + \frac{1}{(n+1)(n+2)} \cdot \underbrace{\frac{\partial \varrho}{\partial t}(x',t)}_{-\nabla_{x'} j(x',t)} \cdot \underbrace{\frac{\nabla_x R^{n+1}}{R^{n-1}}}_{(n+1) R \cdot \underbrace{\nabla_x R}_{\frac{R}{R}}} \right]$$

$$\frac{d\underline{\mathcal{F}}}{dt} = \S \left[j(x',t) - \frac{1}{n+2} R \left(\nabla_{x'} j(x',t) \right) \right]$$

2. tagot parc. integrálással alakítsuk át

($\S$-ben lévő R^{n-1} faktor!)

$$\int d^3\grave{x}\; R^{n-1}\cdot \underline{R}_\ell\, \partial_\ell\, j_\ell(\underline{\grave{x}},t) = -\int d^3\grave{x}\; \partial_\ell\left(R^{n-1}\cdot \underline{R}_\ell\right)\cdot j_\ell(\underline{\grave{x}},t)$$

$$= -\int d^3\grave{x}\left((n-1)\,R^{n-2}\cdot \underbrace{\partial_\ell R}_{-\frac{\underline{R}_\ell}{R}}\cdot \underline{R}_k + R^{n-1}\cdot \underbrace{\partial_\ell \underline{R}_k}_{-\delta_{k\ell}}\right) j_\ell(\underline{\grave{x}},t)$$

$$= \int d^3\grave{x}\; R^{n-1}\left(j_k(\underline{\grave{x}},t) + (n-1)\,\frac{\underline{R}_k \underline{R}_\ell}{R^2}\, j_\ell(\underline{\grave{x}},t)\right)$$

$$\underline{R}\;\underline{\nabla}_{\underline{\grave{x}}}\, j(\underline{\grave{x}},t) \to j(\underline{\grave{x}},t) + (n-1)\,\frac{1}{R^2}\,\underline{R}\left(\underline{R}\, j(\underline{\grave{x}},t)\right)$$

$$\boxed{\frac{d\underline{\dot{r}}}{dt} = \S\left[\frac{n+1}{n+2}\, j(\underline{\grave{x}},t) - \frac{n-1}{n+2}\,\frac{1}{R^2}\,\underline{R}\left(\underline{R}\, j(\underline{\grave{x}},t)\right)\right]}$$

<u>merev testlét nyom e~</u>: $j(\underline{\grave{x}},t) = \varrho(\underline{\grave{x}},t)\cdot \underline{v}(t)$

ϱ gömbszimmetrikus $\Rightarrow$ $[\ldots]$-nak csak a $\underline{v}$-vel párhuzamos komponense marad meg az integrálások után

$\hat{\underline{v}} = \frac{\underline{v}}{v}$ $\qquad \S \to \S\cdot\left(\hat{\underline{v}}\circ\hat{\underline{v}}\right)$ helyettesíthető

$\underline{v}$ irányra vetítő projektor

$$\left(\hat{\underline{v}}\circ\hat{\underline{v}}\right)\underline{v} = \underline{v}$$

$$\left(\hat{\underline{v}}\circ\hat{\underline{v}}\right)\underline{R}\left(\hat{\underline{R}}\,\underline{v}\right) = \hat{\underline{v}}\left(\hat{\underline{v}}\,\underline{R}\right)\left(\hat{\underline{R}}\,\underline{v}\right)\cdot v = \left(\hat{\underline{R}}\cdot\hat{\underline{v}}\right)^2\cdot\underline{v}$$

$$\boxed{\frac{d\underline{\dot{r}}}{dt} = \S\left[\frac{n+1}{n+2} - \frac{n-1}{n+2}\left(\hat{\underline{R}}\cdot\hat{\underline{v}}\right)^2\right]\cdot\varrho(\underline{\grave{x}},t)\cdot\underline{v}(t)}$$

$\S$ gömbi koordinátákra $\Rightarrow$ az integrálban $\left(\hat{\underline{R}}\cdot\hat{\underline{v}}\right)^2$ helyettesítheto az egységgömb felületére vett átlagával:

$$\overline{\left(\hat{\underline{R}}\,\hat{\underline{v}}\right)^2} = \frac{1}{4\pi}\int\left(\hat{\underline{R}}\cdot\hat{\underline{v}}\right)^2 d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\vartheta)\cdot\cos^2\vartheta =$$

$$\hat{\underline{R}}\cdot\hat{\underline{v}} = \cos\vartheta = \zeta$$

$$= \frac{1}{2}\int_{+1}^{+1}\zeta^2\cdot d\zeta = \frac{1}{6}\cdot\left[\zeta^3\right]_{-1}^{+1} = \frac{1}{3}$$

$$\boxed{\frac{d\underline{P}}{dt} = \S\left[\frac{n+1}{n+2} - \frac{n-1}{n+2}\cdot\frac{1}{3}\right]\rho(\underline{x},t)\cdot\underline{v}(t) = \S\left[\frac{2}{3}\rho(\underline{x},t)\,\underline{v}(t)\right]}$$

$$\frac{3(n+1)-(n-1)}{3(n+2)} = \frac{2n+4}{3(n+2)} = \frac{2}{3}$$

$$\frac{d\underline{P}}{dt} = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\,c^{n+2}}\int d^3x\int d^3\grave{x}\;\rho(\underline{x},t)\,R^{n-1}\left(\frac{\partial}{\partial t}\right)^{n+1}\left[\frac{2}{3}\rho(\underline{\grave{x}},t)\,\underline{v}(t)\right]$$

$\rho(\underline{\grave{x}},t)$ deriváltjaira a kontinuitási egyenlet használható

v-ben nemlineáris tagok jelennek meg $\Rightarrow$ elhanyagoljuk

$$\boxed{\frac{d\underline{P}}{dt} = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n!\,c^{n+2}}\int d^3x\int d^3\grave{x}\;\rho(\underline{x},t)\,\rho(\underline{\grave{x}},t)\,R^{n-1}\cdot\frac{2}{3}\cdot\frac{d^{n+1}\underline{v}}{dt^{n+1}}(t)}$$

fontosan lineáris $\Rightarrow$ R^{n-1} nincs az $n\geq 2$ tagok 0-ra történnek

$n = 0$:

$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{c^2} \cdot \int d^3x \int d^3x' \, \frac{\rho(\underline{x},t)\,\rho(\underline{x}',t)}{R} \cdot \frac{2}{3} \frac{d\underline{v}}{dt}(t)$$

Elektrosztatikus energia változásához:

$$W_{elsztat} = \frac{1}{4\pi\varepsilon_0} \cdot \sum_{i<j} \frac{q_i q_j}{|\underline{x}_i - \underline{x}_j|} = \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0} \sum_{i \neq j} \frac{q_i q_j}{|\underline{x}_i - \underline{x}_j|}$$

$\Rightarrow$ ugyanez részletes töltéseloszlásra:

$$W_{elsztat} = \frac{1}{2} \cdot \frac{1}{4\pi\varepsilon_0} \int d^3x \int d^3x' \, \frac{\rho(\underline{x},t)\,\rho(\underline{x}',t)}{|\underline{x} - \underline{x}'|}$$

$\Rightarrow$ **nulladrendben:**

$$\boxed{\frac{d\underline{F}}{dt} = \underbrace{\frac{4}{3} \frac{W_{elsztat}}{c^2}}_{m_e} \cdot \frac{d\underline{v}}{dt}}$$

$\rightarrow$ problém: relativisztikusnál $\frac{4}{3}$ nem jó

$n = 1$:

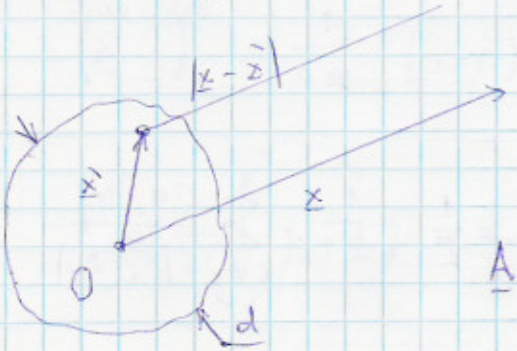
$$-\frac{1}{4\pi\varepsilon_0} \cdot \frac{1}{c^3} \cdot \underbrace{\int d^3x \, \rho(\underline{x},t)}_{-e} \underbrace{\int d^3x' \, \rho(\underline{x}',t)}_{-e} \cdot \frac{2}{3} \ddot{\underline{v}}(t) =$$

$$= -\frac{e^2}{6\pi\varepsilon_0 c^3} \ddot{\underline{v}} = -m_e \gamma \cdot \ddot{\underline{v}} \qquad \gamma = \frac{e^2}{6\pi\varepsilon_0 c^3 \cdot m_e}$$

$$\Rightarrow \boxed{\frac{d\underline{F}}{dt} = m_e \left(\dot{\underline{v}} - \gamma \ddot{\underline{v}} \right) = \underline{F}_{ext}}$$

Konzultáció

Multipólussorfejtés



$$j(\underline{x}', t') = j(\underline{x}')\, e^{-i\omega t'}$$

$$\underline{A}(\underline{x}, t) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{j\left(\underline{x}', t - \frac{|\underline{x} - \underline{x}'|}{c}\right)}{|\underline{x} - \underline{x}'|} =$$

$$= \frac{\mu_0}{4\pi} \int d^3x' \, \frac{j(\underline{x}')\, e^{-i\omega\left(t - \frac{|\underline{x} - \underline{x}'|}{c}\right)}}{|\underline{x} - \underline{x}'|} = \underline{A}(\underline{x}) \cdot e^{-i\omega t}$$

$$\boxed{\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \, \frac{j(\underline{x}')}{|\underline{x} - \underline{x}'|}\, e^{ik|\underline{x} - \underline{x}'|}} \qquad k = \frac{\omega}{c}$$

Lehetséges skálák: d, λ, $r = |\underline{x}|$

$d \ll \lambda$ mellett három zóna:

1) statikus zóna: $\lambda \gg r$ $\quad 1 \gg kr$

$e^{ik|\underline{x} - \underline{x}'|} \approx 1 \rightarrow$ statikus megoldás

2) sugárzási zóna: $r \gg \lambda$ $\quad kr \gg 1 \gg kd$

$\rightarrow$ reverőlen: $|\underline{x} - \underline{x}'| \approx r$

$\rightarrow$ hitevőlen: $|\underline{x} - \underline{x}'| = r - \underline{n} \cdot \underline{x}'$ $\qquad \underline{n} = \frac{\underline{x}}{r}$

$$\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\underline{j}(\underline{x}')}{r} \; e^{-ik(r - \underline{n}\,\underline{x}')}$$

$$\boxed{\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \cdot \frac{e^{ikr}}{r} \int d^3x' \; \underline{j}(\underline{x}') \; e^{-ik\,\underline{n}\,\underline{x}'}}$$

multipól sorfejtés: $kd << 1$

$$e^{-ik\,\underline{n}\,\underline{x}'} = \sum_{n=0}^{+\infty} \frac{(-ik)^n}{n!} \cdot (\underline{n}\,\underline{x}')^n$$

$$\boxed{\underline{A}(\underline{x}) = \frac{\mu_0}{4\pi} \cdot \frac{e^{ikr}}{r} \sum_{n=0}^{+\infty} \frac{(-ik)^n}{n!} \int d^3x' \; \underline{j}(\underline{x}') \cdot (\underline{n}\,\underline{x}')^n}$$

$$\underline{A}^{(0)}(\underline{x}) = \frac{\mu_0}{4\pi} \cdot \frac{e^{ikr}}{r} \cdot \int d^3x' \; \underline{j}(\underline{x}')$$

$$0 = \int d^3x' \; \partial'_\ell \left(x'_k \; j_\ell(\underline{x}') \right) = \int d^3x' \; j_k(\underline{x}') + \int d^3x' \; x'_k \; \partial'_\ell \; j_\ell(\underline{x}')$$

$$\Rightarrow \int d^3x' \; \underline{j}(\underline{x}') = -\int d^3x' \cdot \underline{x}' \cdot \underbrace{\operatorname{div} \underline{j}(\underline{x}')}_{i\omega \rho(\underline{x}')}$$

$$\underline{A}^{(1)}(\underline{x}) = -\frac{i\omega\mu_0}{4\pi} \cdot \frac{e^{ikr}}{r} \cdot \underbrace{\int d^3x' \; \rho(\underline{x}') \cdot \underline{x}'}_{\uparrow}$$

el. dipólmomentum

$$\underline{A}^{(rad)}(\underline{x}) = -\frac{i\omega\mu_0}{4\pi}\cdot\frac{e^{ikr}}{r}\cdot\underline{f} \qquad \underline{B} = \nabla\times\underline{A} = ? \qquad \underline{E} = ?$$

$$\nabla\left(e^{ikr}\right) = ik\,e^{ikr}\cdot\nabla r = ik\,\underline{n}\cdot e^{ikr}$$

$$\boxed{\nabla \to ik\,\underline{n}} \quad\Rightarrow\quad \boxed{\underline{B} = ik\,\underline{n}\times\underline{A}}$$

$$\nabla\times\underline{B} = \underbrace{\mu_0\varepsilon_0}_{\frac{1}{c^2}}\frac{\partial\underline{E}}{\partial t} \quad\Rightarrow\quad ik\,\underline{n}\times\underline{B} = -\frac{i\omega}{c^2}\underline{E}$$

$$\boxed{\underline{E} = -c\,\underline{n}\times\underline{B}}$$

$$\nabla\times\underline{E} = -\frac{\partial\underline{B}}{\partial t} \quad\Rightarrow\quad k\,\underline{n}\times\underline{E} = \omega\cdot\underline{B}$$

$$\boxed{\underline{B} = \frac{1}{c}\,\underline{n}\times\underline{E}}$$

$$\boxed{\underline{B}^{(rad)} = \frac{\mu_0\omega^2}{4\pi c}\cdot\frac{e^{ikr}}{r}\cdot\underline{n}\times\underline{f}}$$

$$\boxed{\underline{E}^{(rad)} = -\frac{\mu_0\omega^2}{4\pi}\cdot\frac{e^{ikr}}{r}\,\underline{n}\times(\underline{n}\times\underline{f})}$$

$$\boxed{\underline{S} = \frac{1}{2\mu_0}\,\underline{E}\times\underline{B}^* = -\frac{c}{2\mu_0}(\underline{n}\times\underline{B})\times\underline{B}^* =}$$

$$= \frac{c}{2\mu_0}\underbrace{\underline{B}^*\times(\underline{n}\times\underline{B})}_{(\underline{B}^*\cdot\underline{B})\underline{n} - (\underline{B}^*\underline{n})\underline{B}} = \boxed{\frac{c}{2\mu_0}|\underline{B}|^2\cdot\underline{n}}$$

$$\underbrace{(\underline{B}^*\cdot\underline{B})}_{|\underline{B}|^2}\underline{n} - \underbrace{(\underline{B}^*\underline{n})}_{0}\underline{B}$$

$$\underline{S}^{(\mathrm{ed})} = \frac{c}{2\mu_0} \cdot \frac{\mu_0^2 \omega^4}{16\pi^2 c^2} \cdot \frac{1}{r^2} |\underline{n} \times \underline{p}|^2 = \frac{\mu_0 \omega^4}{32\pi^2 c} \cdot \frac{1}{r^2} |\underline{n} \times \underline{p}|^2$$

$\underline{p}$ valós $\Rightarrow$ $|\underline{n} \times \underline{p}|^2 = p^2 \cdot \sin^2\vartheta$

$$P^{(\mathrm{ed})} = \int \underline{S}^{(\mathrm{ed})} \cdot \underline{n} \, dF = \frac{\mu_0 \omega^4}{32\pi^2 c} \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\vartheta) \cdot p^2 \sin^2\vartheta =$$

$dF = r^2 \, d\Omega = r^2 \, d(\cos\vartheta) \, d\varphi$

$\zeta = \cos\vartheta$

$$= \frac{\mu_0 \omega^4}{16\pi c} p^2 \int_{-1}^{+1} d\zeta \, (1 - \zeta^2) = \frac{\mu_0 \omega^4}{12\pi c} \cdot p^2$$

$$\left[\zeta - \frac{1}{3}\zeta^3\right]_{-1}^{+1} = \frac{4}{3}$$

$$\underline{A}^{(1)}(\underline{x}) = -\frac{ik\mu_0}{4\pi} \cdot \frac{e^{ikr}}{r} \cdot \int d^3x' \, \underline{j}(\underline{x}') \, (\underline{n} \cdot \underline{x}')$$

$$\underline{j}(\underline{x}') \cdot (\underline{n} \cdot \underline{x}') = \frac{1}{2}\left(\underline{j}(\underline{n} \cdot \underline{x}') + \underline{x}'(\underline{n} \cdot \underline{j})\right) + \frac{1}{2}\left(\underline{j}(\underline{n} \cdot \underline{x}') - \underline{x}'(\underline{n} \cdot \underline{j})\right)$$

$\underline{n} \times (\underline{j} \times \underline{x}')$

$\downarrow$

$\underline{x}', \underline{j}$ - ben szimmetrikus

elektromos kvadrupól

$\underline{x}', \underline{j}$ - ben antiszimmetrikus

mágneses dipól

$$\underline{A}^{(md)}(\underline{x}) = -\frac{ikr_0}{4\pi}\cdot\frac{e^{ikr}}{r}\cdot\int d^3\grave{x}\left(\frac{1}{2}\,\underline{n}\times\left(\underline{j}(\underline{\grave{x}})\times\underline{\grave{x}}\right)\right) =$$

$$= \frac{i\omega r_0}{4\pi c}\cdot\frac{e^{ikr}}{r}\cdot\underline{n}\times\underbrace{\frac{1}{2}\int d^3\grave{x}\left(\underline{\grave{x}}\times\underline{j}(\underline{\grave{x}})\right)}_{\text{magnetic dipole moment}}$$

$$\boxed{\underline{A}^{(md)} = \frac{i\omega r_0}{4\pi c}\cdot\frac{e^{ikr}}{r}\;\underline{n}\times\underline{m}}$$

$$\underline{A}^{(ed)}(\underline{x}) = -\frac{ikr_0}{4\pi}\cdot\frac{e^{ikr}}{r}\int d^3\grave{x}\cdot\frac{1}{2}\left[\underline{j}(\underline{n}\,\underline{\grave{x}}) + \underline{\grave{x}}(\underline{n}\,\underline{j})\right]$$

$$A^{(ed)}_{\ell}(\underline{x}) = -\frac{i\omega r_0}{8\pi c}\cdot\frac{e^{ikr}}{r}\underbrace{\int d^3\grave{x}\left[j_k(n_\ell\,\grave{x}_\ell) + \grave{x}_k(n_\ell\,j_\ell)\right]}$$

$$n_\ell\cdot\int d^3\grave{x}\left(j_k\,\grave{x}_\ell + \grave{x}_k\,j_\ell\right)$$

$$0 = \int d^3\grave{x}\;\partial_i\left(\grave{x}_k\,\grave{x}_\ell\cdot j_i\right) = \int d^3\grave{x}\left(\grave{x}_\ell\cdot j_k + \grave{x}_k\cdot j_\ell\right) +$$

$$+\int d^3\grave{x}\;\grave{x}_k\,\grave{x}_\ell\,\underbrace{\partial_i\,j_i(\underline{x})}_{\operatorname{div}\underline{j}(\underline{x}) = i\omega\rho(\underline{x})}$$

$$\Rightarrow \int d^3\grave{x}\left(\grave{x}_\ell\cdot j_k + \grave{x}_k\cdot j_\ell\right) = -i\omega\int d^3\grave{x}\cdot\rho(\underline{\grave{x}})\cdot\grave{x}_k\,\grave{x}_\ell$$

$$A_\ell^{(el)}(\underline{x}) = -\frac{\omega^2 \mu_0}{8\pi c} \cdot \frac{e^{ikr}}{r} \cdot n_e \int d^3x' \; \rho(\underline{x}') \, x'_\ell \cdot x'_\ell$$

$$Q_{\ell\ell} = \int d^3x' \; \rho(\underline{x}') \left[3 x'_\ell x'_\ell - \delta_{\ell\ell} \cdot r'^2 \right]$$ elektromos kvadrupólusmomentum

$$\Rightarrow \int d^3x' \; \rho(\underline{x}') \, x'_k x'_\ell = \frac{1}{3} \left[Q_{k\ell} + \delta_{k\ell} \underbrace{\int d^3x' \; \rho(\underline{x}') \cdot r'^2}_{\sim} \right]$$

$$\underline{A}^{(el)}(\underline{x}) = -\frac{\omega^2 \mu_0}{24\pi c} \cdot \frac{e^{ikr}}{r} \cdot \left(\underline{\underline{Q}} \, \underline{n} + \sim \cdot \underline{n} \right)$$

$$\underline{B}^{(el)}(\underline{x}) = ik \, \underline{n} \times \underline{A}^{(el)} = -\frac{i\omega^3 \mu_0}{24\pi c^2} \cdot \frac{e^{ikr}}{r} \cdot \underline{n} \times \left(\underline{\underline{Q}} \, \underline{n} \right)$$

Thomson-szórás

szabad elektron, nemrelativisztikus

$$m \dot{\underline{v}} = -e \cdot \underline{E}_0 \cdot e^{-i\omega t} \qquad \underline{E}_0 = \underline{\varepsilon}^{(be)} \cdot E_0$$

$$\underline{E}^{(szórt)} = -\frac{e}{4\pi\varepsilon_0 c^2} \cdot \frac{1}{r} \underbrace{\underline{n} \times \left(\underline{n} \times \dot{\underline{v}} \right)}$$

$$\left(\underline{n} \, \dot{\underline{v}} \right) \underline{n} - \underbrace{\left(\underline{n} \, \underline{n} \right)}_{1} \dot{\underline{v}}$$

két hullám polarizációjának leírása:

$\underline{\varepsilon}_1^{(szórt)}$, $\underline{\varepsilon}_2^{(szórt)}$, $\underline{n}$ ortonormált bázis

$$\underline{E}^{(\text{szórt})} = \frac{e}{4\pi\varepsilon_0 c^2} \cdot \frac{1}{r} \cdot \left[\underline{\dot{v}} - \underline{n}\left(\underline{n}\,\underline{\dot{v}}\right)\right]$$

kifejtési együtthatók:

$$\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{E}^{(\text{szórt})} = \frac{e}{4\pi\varepsilon_0 c^2} \cdot \frac{1}{r} \cdot \left(\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{\dot{v}}\right) =$$

$$= \frac{e}{4\pi\varepsilon_0 c^2} \cdot \frac{1}{r} \left(-\frac{e}{m} \cdot E_0\right) \cdot \left(\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{\varepsilon}^{(be)}\right) e^{-i\omega t_{ret}}$$

$$= -\underbrace{\frac{e^2}{4\pi\varepsilon_0 m c^2}}_{r_e} \cdot \frac{1}{r} \cdot E_0 \cdot \left(\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{\varepsilon}^{(be)}\right) \cdot e^{-i\omega t_{ret}}$$

$r_e \rightarrow$ klasszikus elektronsugár

$$t_{ret} = t - \frac{r}{c}$$

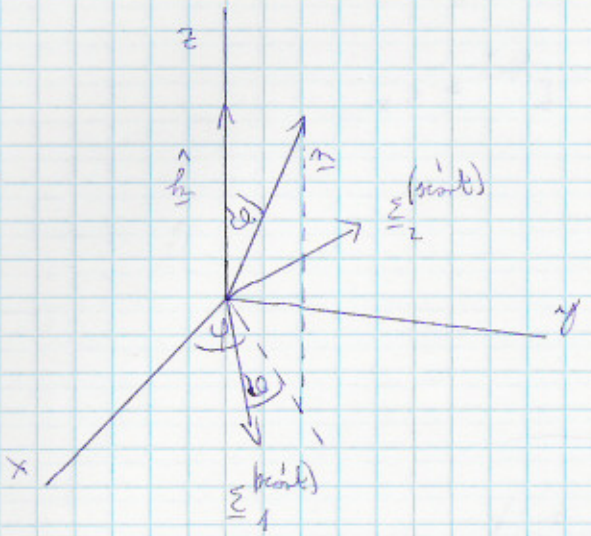
$$\frac{1}{4\pi\varepsilon_0} \cdot \frac{e^2}{r_e} = m c^2 \qquad r_e = 2{,}8 \cdot 10^{-15}\ \text{m}$$

$$\boxed{\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{E}^{(\text{szórt})} = -\frac{r_e}{r} \cdot E_0 \cdot \left(\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{\varepsilon}^{(be)}\right) \cdot e^{i(kr - \omega t)}}$$

<u>i. polarizációra vett hatáskeresztmetszet:</u>

$$\left(\frac{d\sigma}{d\Omega}\right)_i = \frac{\left(\frac{dP}{d\Omega}\right)_i^{(\text{szórt})}}{\left(\frac{dP}{dF}\right)^{(be)}} = \frac{r^2 \cdot \left|\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{E}^{\text{szórt}}\right|^2}{|E_0|^2}$$

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_i = r_e^2 \cdot \left|\underline{\varepsilon}_i^{(\text{szórt})^*} \cdot \underline{\varepsilon}^{(be)}\right|^2}$$



$$\hat{\underline{k}} = \frac{\underline{k}}{k} = \underline{e}_z$$

$$\underline{n} = \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix}$$

$$\underline{\xi}_2^{(\text{scorr})} = \frac{\hat{\underline{k}} \times \underline{n}}{|\hat{\underline{k}} \times \underline{n}|} = \frac{1}{\sin\vartheta} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} =$$

$$= \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \underline{\xi}_2^{(\text{scorr})}$$

$$\underline{\xi}_1^{(\text{scorr})} = \underline{\xi}_2^{(\text{scorr})} \times \underline{n} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \times \begin{pmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\vartheta\cdot\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{pmatrix} = \underline{\xi}_1^{(\text{scorr})}$$

x irányban polarizált bejövő: $\xi^{(be)} = \underline{e}_x$

$$\left(\frac{d\sigma}{d\Omega}\right)_1 = r_e^2 \cdot \cos^2\vartheta \cos^2\varphi$$

$$+ \left(\frac{d\sigma}{d\Omega}\right)_2 = r_e^2 \cdot \sin^2\varphi$$

$$\frac{d\sigma}{d\Omega} = r_e^2 \left(\cos^2\vartheta \cos^2\varphi + \sin^2\varphi\right)$$

y irányban polarizált bejövő: $\xi^{(be)} = \underline{e}_y$

$$\left(\frac{d\sigma}{d\Omega}\right)_1 = r_e^2 \cdot \cos^2\vartheta \sin^2\varphi$$

$$+ \left(\frac{d\sigma}{d\Omega}\right)_2 = r_e^2 \cdot \cos^2\varphi$$

$$\frac{d\sigma}{d\Omega} = r_e^2 \left(\cos^2\vartheta \cdot \sin^2\varphi + \cos^2\varphi\right)$$

polarizálatlan bejövő $\Rightarrow$ fenti kettő összegeként

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \left(\cos^2\vartheta + 1\right)}$$

$\xi = \cos\vartheta$

$$\boxed{\sigma_T = \frac{1}{2} r_e^2 \int_0^{2\pi} d\varphi \int_{-1}^{+1} d(\cos\vartheta) \left(\cos^2\vartheta + 1\right)} = \ldots$$

$$= r_e^2 \pi \int_{-1}^{+1} d\xi \left(\xi^2 + 1\right) = r_e^2 \pi \underbrace{\left[\frac{1}{3}\xi^3 + \xi\right]_{-1}^{+1}}_{\frac{8}{3}} = \boxed{\frac{8\pi}{3} \cdot r_e^2}$$