Differenciálegyenletek a fizikában II

Lineáris dif. egyenlet

$$a_n(x)\, u^{(n)}(x) + \dots + a_0(x)\, u(x) = g(x)$$

u, v megoldások | $u \to g_1$ megoldás, $v \to g_2$ megoldás $\Big\} \Rightarrow u+v \to g_1+g_2$ is megoldás

$\lambda u + (1-\lambda) v$ is megoldás

kezdeti feltételek megadása

$y(x_0) = y_0$, $y'(x_0) = y'_0$ vagy $y(x_0) = y_0$, $y(x_1) = y_1$, $x_0 < x_1$

Egy pl: $a_2(x) y'' + a_1(x) y' + a_0(x) y = f(x) \xrightarrow{:a_2(x)} y'' + \frac{a_1(x)}{a_2(x)} y' + \frac{a_0(x)}{a_2(x)} y = \frac{f(x)}{a_2(x)}$

$\to (r(x) y')' + s(x) y = h(x)$ mert akkor

$$r(x) = \exp\left[\int^x \frac{a_1(\xi)}{a_2(\xi)}\, d\xi\right] = \exp\left[\int^x p(\xi)\, d\xi\right]$$

$r' = \frac{a_1}{a_2} r \qquad [r y']' + s y = h \to r' y' + r y'' + s y = h$

$\frac{a_1}{a_2} r y' + r y'' + s y = h$

$\frac{a_1}{a_2} y' + y'' + \frac{s}{r} y = \frac{h}{r}$

$y = w\, e^{-1/2 \int^x p(\xi)\, d\xi} = \frac{w}{\sqrt{r}} \qquad w = y\sqrt{r}$

$w' = 1/2 \frac{r'}{\sqrt{r}} y + \sqrt{r}\, y'$

$\sqrt{r}\, w' = 1/2\, r' y + r y'$

$\left(\sqrt{r}\, w' - 1/2 \frac{r' w}{\sqrt{r}}\right) + \frac{s}{\sqrt{r}} w = h$

$\boxed{w'' + t w = j} \qquad t = q - \frac{1}{2} p' - \frac{1}{4} p^2$

$j = g\, e^{\int 1/2\, p\, ds}$

példa: peremértéke feladat

$y'' + p y' + q y = g \qquad y(x_0) = a \qquad x_0 \le x \le x_1$

$\qquad\qquad\qquad\qquad y(x_1) = b$

$p(x) \le P \qquad P, Q, G$ a függvények maximumai

$q(x) \le Q$

$g(x) < G \qquad$ Van megoldás, ha $\frac{1}{8} Q (x_1 - x_0)^2 + \frac{1}{2} P (x_1 - x_0) < 1$

Lineáris függetlenség, Wronsky determináns

$y = A_1 z_1 + A_2 z_2 + y_1$

szabad paraméterek $\qquad$ homogén megoldás

$z'' - \frac{2}{x} z' + \left(4 + \frac{2}{x^2}\right) z = 0$

$z(x) = x \sin(2x) \qquad z_2(x) = x \sin x \cos x$

z_1 és z_2 függvények lineárisan összefüggő, ha $\exists$ B_1 és B_2 állandók, hogy $B_1 z_1 + B_2 z_2 = 0$

$\Downarrow$

$B_1 z_1' + B_2 z_2' = 0 \Big\} \to$

$\to$ difegyenlet rendszer.

$W = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} = 0$ akkor összefüggőek

biz: ha $0 = \frac{W^{=0}}{z_1^2} = \frac{z_1 z_2' - z_2 z_1'}{z_1^2} = \left(\frac{z_2}{z_1}\right)' = 0 \qquad \frac{z_1}{z_2} = C \qquad z_1 = C z_2$

z_1, z_2

$\left.\begin{aligned} [r z_1']' + s z_1 &= 0 \\ [r z_2']' + s z_2 &= 0 \end{aligned}\right\} \quad z_2 [r z_1']' - z_1 [r z_2']' = 0$

$[r (z_1 z_2' - z_2 z_1')]' = [r W]' = 0$

$r W = C$

$\boxed{W = \frac{C}{r}}$ Abel-azonosság

$$W(x) = \frac{C}{r(x)} = z_1^2(x)\left(\frac{z_2}{z_1}\right)'$$

Tfh egyik megoldást, z_1-et megkapjuk, keressük a másikat

$$r = e^{\int p(x)\,dx}$$

$$\left(\frac{z_2}{z_1}\right)' = \frac{C}{r(x) z_1^2(x)} \qquad \frac{z_2}{z_1} = \int \frac{C}{z_1^2 r}\,dx$$

ha megoldás $\longrightarrow$ $z_2 = C \cdot z_1 \int \frac{dx}{r z_1^2}$

akkor ez is $\longrightarrow$ $\bar{z}_2 = z_1 \int \frac{dx}{r z_1^2}$

$$z = A_1 z_1 + A_2 \bar{z}_2$$

z_1 és z_2 független megoldások, lin. kombinációjuk is nem lehetnek egyszerre 0-ák.

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pl: $x z'' + (2x-1) z' + (x-1) z = 0$

$$z'' + \underbrace{\left(2 - \frac{1}{x}\right)}_{p} z' + \left(1 - \frac{1}{x}\right) z = 0 \qquad r = e^{\int p(x)dx} \qquad \int\left(2 - \frac{1}{x}\right)dx = 2x - \ln x$$

$$r = e^{2x - \ln x} = \frac{e^{2x}}{x}$$

$$\left[\frac{e^{2x}}{x} z'\right]' + \frac{x-1}{x^2} e^{2x} z = 0$$

$$W = \frac{C}{r} = C x e^{-2x}$$

megsejtjük: $z_1 = e^{-x}$ megoldja az egyenletet

$$z_2 = e^{-x} \int x e^{-2x} e^{2x}\,dx = e^{-x} \int x\,dx = \frac{x^2}{2} e^{-x}$$

$z = A_1 e^{-x} + A_2 x^2 e^{-x}$ ennyit a homogén egyenletről

Együtthatók variálása

$y_1 = \alpha(t) z_1 + \beta(t) z_2$, továbbá legyen $\alpha'(t) z_1 + \beta'(t) z_2 = 0$

$y_1' = \alpha(t) z_1' + \underbrace{\alpha'(t) z_1 + \beta(t) z_2' + \beta'(t) z_2}_{=0}$

$y_1'' = \alpha' z_1' + \beta' z_2' + \alpha z_1'' + \beta z_2''$

$$\alpha \underbrace{\left(z_1'' + p z_1' + q z_1\right)}_{0} + \beta \underbrace{\left(z_2'' + p z_2' + q z_2\right)}_{0} + \alpha' z_1' + \beta' z_2' = g$$

$\alpha'(t) z_1 + \beta'(t) z_2 = 0$ $\qquad W = \begin{vmatrix} z_1 & z_2 \\ z_1' & z_2' \end{vmatrix} \neq 0$

$\alpha'(t) z_1' + \beta'(t) z_2' = g$

$\alpha' (z_1 z_2' - z_2 z_1') = - g z_2$

$\alpha' = -\dfrac{g z_2}{W} \qquad \alpha = -\int \dfrac{g z_2}{W} dx$

így megkapjuk y_1 partikuláris megoldást

$\beta' = \dfrac{g z_1}{W} \qquad \beta = \int \dfrac{g z_1}{W} dx$

$$y = A_1 z_1 + A_2 z_2 - z_1 \int \frac{g z_2}{W} dx + z_2 \int \frac{g z_1}{W} dx$$

$$y = A_1 z_1 + A_2 z_2 + \int^{x} \frac{z_1(\xi) z_2(x) - z_2(\xi) z_1(x)}{z_1(\xi) z_2'(\xi) - z_2(\xi) z_1'(\xi)} g(\xi)\, d\xi$$

pl: $(x+1) y'' + x y' - y = 2(x+1)^2 \quad y(0) = 0 \quad y(1) = 1$

$z_1 = x$ megoldása a homogénnek $\qquad \int \dfrac{x}{x+1} dx = \ln 1 + x$

$y'' + \dfrac{x}{x+1} y' - \dfrac{1}{x+1} y = 2(x+1) \qquad r = e^{\int p dx} = \dfrac{e^x}{x+1}$

$z_2 = Cx \int (x+1) e^{-x} \dfrac{1}{x^2} dx = C e^{-x}$

$z = A_1 x + A_2 e^{-x}$

$$\boxed{z'' + pz' + qz = g \iff [ry']' + \dots = g = h}$$ $\quad \leftarrow$ ez a forma könnyebb, $w = \frac{c}{r}$

$q = \frac{1}{r}$

$g = \frac{h}{r}$

$y_1 = \alpha(x) z_1 + \beta(x) z_2$

$\alpha'(x) z_1 + \beta'(x) z_2 = 0$

$\alpha'(x) z_1' + \beta'(x) z_2' = g$

$$\alpha(x) = -\int^x \frac{g(\xi) z_2(\xi)}{w(\xi)} d\xi \qquad \beta(x) \int \frac{g(\xi) z_1(\xi)}{w(\xi)} d\xi$$

$$y = A_1 z_1 + A_2 z_2 - z_1(x) \int^x \frac{g(\xi) z_2(\xi)}{w(\xi)} d\xi + z_2(x) \int^x \frac{g(\xi) z_1(\xi)}{w(\xi)} d\xi$$

$$y = A_1 z_1 + A_2 z_2 + \int^x \frac{z_1(\xi) z_2(x) - z_1(x) z_2(\xi)}{w(\xi) r(\xi)} h(\xi)\, d\xi$$

Green-függvény

peremfeltételek

$[r(x) y'(x)]' + s(x) y(x) = h(x) \qquad y(a) = y_a, \; y(b) = y_b$

$a \le x \le b$

$y(x) = u(x) + v(x)$ alakban írjuk fel,

hogy $v(a) = y_a$, $v(b) = y_b$ kielégítse $\Rightarrow u(x)$ ekkor homogén

Ekkor a difegyenlet $[r(x) u'(x)]' + s(x) u(x) = h(x) - [r(x) v'(x)]' - s(x) v(x) = k(x)$

Így $u_1(a) = 0$ és normálva, h. $u_1'(a) = 1$

$u_2(b) = 0$, $u_2'(b) = 1$

pl: $z_1 = e^{-x} \qquad z_2 = x^2 e^{-x}$ megoldásai, $[x^{-1} e^{2x} z'(x)]' + x^{-2}(x-1) e^{2x} z(x) = 0$

$$\begin{cases} u_1(x) = \frac{1}{2a}(x^2 - a^2) e^{-(x-a)} \\ u_2(x) = \frac{1}{2b}(x^2 - b^2) e^{-(x-b)} \end{cases}$$

ált. megoldás

$$u(x) = A_1 u_1(x) + A_2 u_2(x) + \int_c^x \frac{u_1(\xi) u_2(x) - u_1(x) u_2(\xi)}{W(\xi) r(\xi)} k(\xi)\, d\xi \qquad u(a) = A_2 u_2 + \int_c^a \frac{u_1(\xi) u_2(a)}{w(\xi) r(\xi)} k(\xi) d\xi$$

$$A_2 = -\int_c^a \frac{u_1(\xi) k(\xi)}{w(\xi) r(\xi)} d\xi$$

$$u(b) = A_1 u_1(b) - \int_c^b \frac{u_1(b)\, u_2(\xi)}{W(\xi)\, r(\xi)}\, h(\xi)\, d\xi \qquad A_1 = \int_c^b \frac{u_2(\xi)\, h(\xi)}{W(\xi)\, r(\xi)}\, d\xi$$

$$u(x) = \int_c^b \frac{u_2(\xi)\, h(\xi)}{W(\xi)\, r(\xi)}\, d\xi\, u_1(x) - \int_c^a \frac{u_1(\xi)\, h(\xi)}{W(\xi)\, r(\xi)}\, d\xi\, u_2(x) + \int_c^x \frac{u_1(\xi) u_2(x) - u_1(x) u_2(\xi)}{W(\xi)\, r(\xi)}\, h(\xi)\, d\xi$$

$$u(x) = \int_a^x \frac{u_1(\xi)\, u_2(x)\, h(\xi)}{W(\xi)\, r(\xi)} + \int_x^b \frac{u_1(x)\, u_2(\xi)}{W(\xi)\, r(\xi)}\, h(\xi)\, d\xi$$

$$u(x) = \int_a^b g(x,\xi)\, h(\xi)\, d\xi \qquad g(x,\xi) = \begin{cases} \dfrac{u_1(x)\, u_2(\xi)}{r(\xi)\left[u_1(\xi) u_2'(\xi) - u_2(\xi) u_1'(\xi)\right]} & a \le x \le \xi \\[2ex] \dfrac{u_1(\xi)\, u_2(x)}{r(\xi)\, W(\xi)} & \xi \le x \le b \end{cases}$$

$$\frac{\partial g}{\partial x} \quad x \to \xi^+ \quad \frac{u_1(\xi)\, u_2'(\xi)}{r(\xi)\, W(\xi)}$$

Green-függvény folytonos ξ-ben

$$\frac{\partial g}{\partial x} \quad x \to \xi^- \quad \frac{u_1'(\xi)\, u_2(\xi)}{r(\xi)\, W(\xi)}$$

$$g_x'(\xi^+, \xi) - g_x'(\xi^-, \xi) = \frac{1}{r(\xi)}$$

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pl: $(x+1) y'' + x y' - y = 2(x+1) \qquad y(0) = y(1) = 0$

$$\left[(x+1)^{-1} e^x u'\right]' - (x+1)^{-2} e^x u = 2 e^x \qquad z_1 = x \quad z_2 = e^{-x}$$

$$u_1(x) = x \qquad u_2(x) = \tfrac{1}{2}\left(x - e^{1-x}\right)$$

$g(x,\xi)$ keressük $\quad 0 \le x \le \xi \quad u_1(x)$
$\quad \xi \le x \le 1 \quad u_2(x)$

$$g(x,\xi) = \begin{cases} A(\xi)\, x & 0 \le x \le \xi \\ B(\xi)\, \frac{1}{2}\left(x - e^{1-x}\right) & \xi \le x \le 1 \end{cases}$$

$$\left.\begin{aligned} 2A(\xi)\xi &= B(\xi)\left(\xi - e^{1-\xi}\right) \\ B(\xi)\, \tfrac{1}{2}\left(x - e^{1-x}\right) - A(\xi) &= (\xi + 1)\, e^{-\xi} \end{aligned}\right\} \quad \begin{aligned} B(\xi) &= -2 e^{-1} \xi \\ A(\xi) &= -e^{-1}\left(\xi - e^{1-\xi}\right) \end{aligned}$$

$$g(x,\xi) = \begin{cases} e^{-1}\left(\xi - e^{1-\xi}\right) x & 0 \le x \le \xi \\ e^{-1}\, \xi \left(x - e^{1-x}\right) & \xi \le x \le 1 \end{cases}$$

$u(x) = 2(x-1)^2 + 2e^{-1}x - 2e^{-x}$ $\frac{09.05}{3}$

$(r g')' + s g = \delta(x-\xi) \quad a \le x \le b$ ': x szerinti deriválás

$g(a,\xi) = g(b,\xi) = 0$

$\delta(x-\xi) = 0 \quad x \ne \xi$ és $\int_{-\infty}^{\infty} \delta(x-\xi)\, dx = 1$

$\delta_\varepsilon(x-\xi) = 0 \quad |x-\xi| > \varepsilon$

$\delta_\varepsilon(x-\xi) = \frac{1}{2\varepsilon} \quad |x-\xi| \le \varepsilon$

$\int_{\xi-\varepsilon}^{\xi+\varepsilon} (r g')' + s g = r(\xi+\varepsilon) \frac{dg}{dx}\Big|_{\xi+\varepsilon,\,\xi} - r(\xi-\varepsilon)\frac{dG}{dx}\Big|_{\xi-\varepsilon,\,\xi} +$

$+ \int_{\xi-\varepsilon}^{\xi+\varepsilon} s(x)\, g(x,\xi)\, dx = 1$

$\varepsilon \to 0$

pl: $\left((1-x^2)y'\right)' - (1-x^2)^{-1} y = (1+x^2)^{1/2} \quad y'(0) = y(1) = 0$

$$g(x,\xi) = \begin{cases} -(1+\xi)^{-1/2}(1-\xi)^{1/2}(1-x^2)^{-1/2} & 0 \le x \le \xi \\ -(1-\xi^2)^{-1/2}(1+x)^{-1/2}(1-x)^{-1/2} & \xi \le x \le 1 \end{cases}$$

$$y = \int_0^1 g(x,\xi)(1+\xi^2)^{1/2}\, d\xi$$

$$y(x) = \frac{4}{3}(1+x)^{-1/2}(1-x) - 2(1+x)^{-1/2}(1-x)^{1/2}$$

09.22.

példa: $x y'' + (2x-1) y' + (x-1) y = x^2 e^{-x} \quad \underline{y(a) = 0 \quad y(b) = 0}$

$z_1 = e^{-x}$

$y'' + \underbrace{\frac{2x-1}{x}}_{p} y' + \underbrace{\frac{x-1}{x}}_{q} y = \underbrace{x e^{x}}_{g}$

$r = e^{\int p\, dx}$

$\int p\, dx = \int \left(2 - \frac{1}{x}\right) dx = 2x - \ln x$

$r(x) = \frac{e^{2x}}{x}$

$\left[\frac{e^{2x}}{x} y'\right]' + \frac{(x-1)e^{2x}}{x^2} y = e^{x}$

$z_2(x) = z_1(x) \int^x \frac{d\xi}{z_1^2(\xi)\, r(\xi)}$

$z_2(x) = e^{-x} \int^x e^{2\xi} \xi e^{-2\xi} d\xi = e^{-x} \int^x \xi\, d\xi = \frac{x^2}{2} e^{-x}$

$z_2(x) \Leftarrow x^2 e^{-x}$

$u_1(a)=0 \quad u_1'(a)=1$

$u_2(b)=0 \quad u_2'(b)=1$

$u_1(x)=Ae^{-x}+Bx^2e^{-x}$

$u_1'(x)=-Ae^{-x}+2Bxe^{-x}-Bx^2e^{-x}$

$u_1(a)=Ae^{-a}+Ba^2e^{-a}:=0 \quad \frac{09.22}{2}$

$u_1'(a)=-Ae^{-a}+2Bae^{-a}-Ba^2e^{-a}=1$

$\Downarrow$

$2Bae^{-a}=1 \qquad Ae^{-a}=-Ba^2e^{-a}=\frac{-e^a}{2a}a^2e^{-a}$

$B=\frac{e^a}{2a} \qquad A=-\frac{a}{2}e^a$

$u_1(x)=\frac{-a}{2}e^ae^{-x}+\frac{e^a}{2a}x^2e^{-x}$

$\Leftarrow u_1(x)=\frac{x^2-a^2}{2a}e^{-(x-a)}$

$u_2(x)=\frac{x^2-b^2}{2b}e^{-(x-b)}$

$\Uparrow$

$u_2'(x)=\frac{x}{b}e^{-(x-b)}-\frac{x^2-b^2}{2b}e^{-(x-b)}$

$W=u_1u_2'-u_2u_1'=\frac{x^2-a^2}{2a}e^{-(x-a)}\cdot\left[\frac{x}{b}e^{-(x-b)}-\frac{x^2-b^2}{2b}e^{-(x-b)}\right]-$

$-\frac{x^2-b^2}{2b}e^{-(x-b)}\cdot\left[\frac{x}{a}e^{-(x-a)}-\frac{x^2-a^2}{2a}e^{-(x-a)}\right]=$

$=\frac{-a}{2b}xe^{-(2x-a-b)}+\frac{b}{2a}xe^{-(2x-a-b)}=\frac{b^2-a^2}{2ab}e^{a+b}\cdot xe^{-2x}=C\cdot\frac{1}{r}$

$$G(x,\xi)=\begin{cases}\frac{u_1(x)\,u_2(\xi)}{C} & a\le x\le\xi\\ \frac{u_2(x)\,u_1(\xi)}{C} & \xi\le x\le b\end{cases}$$

$y(x)=\int_a^b G(x,\xi)h(\xi)\,d\xi$

$y(x)=\int_a^x\frac{u_1(\xi)u_2(x)}{C}h(\xi)\,d\xi+\int_x^b\frac{u_2(\xi)u_1(x)}{C}h(\xi)\,d\xi$

$$y(x)=\int_a^x\frac{\frac{1}{2a}(\xi^2-a^2)e^{-(\xi-a)}\frac{1}{2b}(x^2-b^2)e^{-(x-b)}}{C}e^{\xi}\,d\xi+\int_x^b\frac{\frac{1}{2b}(\xi^2-b^2)e^{-(\xi-b)}\frac{1}{2a}(x^2-a^2)e^{-(x-a)}}{C}e^{\xi}\,d\xi=$$

$$=\frac{1}{2bC}(x^2-b^2)e^{-(x-b)}\int_a^x\frac{1}{2a}(\xi^2-a^2)e^a\,d\xi+\frac{1}{2aC}(x^2-a^2)e^{-(x-a)}\int_x^b\frac{\xi^2-b^2}{2b}e^b\,d\xi=$$

$$=\frac{1}{4abC}(x^2-b^2)e^{-(x-a-b)}\left(\frac{x^3}{3}-a^2x\right)\Bigg|_a^x+\frac{1}{4abC}(x^2-a^2)e^{-(x-a-b)}\left(\frac{x^3}{3}-b^2x\right)\Bigg|_x^b$$

$$y(x)=\frac{1}{4abC}(x^2-b^2)e^{-(x-a-b)}\left[\frac{x^3}{3}-a^2x-\frac{a^3}{3}+a^3\right]+\frac{1}{4abC}(x^2-a^2)e^{-(x-a-b)}\left[\frac{b^3}{3}-b^3-\frac{x^3}{3}+b^2x\right]$$

$$y(x)=\frac{1}{2(b^2-a^2)}e^{-x}\left[\frac{x^2-b^2}{3}\left(x^3-3a^2x+2a^3\right)-\frac{x^2-a^2}{3}\left(x^3-3b^2x+2b^3\right)\right]$$

ὑπόδ:

$(x+1)y''+xy'-y=2(x+1)^2 \quad y(0)=0 \quad y(1)=0$

$y''+\underbrace{\frac{x}{x+1}}_{p}y'-\underbrace{\frac{1}{x+1}}_{q}y=\underbrace{2(x+1)}_{g}$

$\int p\,dx=\int\frac{x}{x+1}dx=\int\left(1-\frac{1}{x+1}\right)dx=x-\ln(x+1)$

$r(x)=\frac{1}{x+1}e^x$

$\left[\frac{e^x}{x+1}y'\right]'-\frac{e^2}{(x+1)^2}y=\underbrace{2e^x}_{h}$

$z_1=x$

$\tilde{z}_2=x\int^x\frac{1}{\xi^2}(\xi+1)\,e^{-\xi}d\xi=x\int^x\left(\frac{1}{\xi}+\frac{1}{\xi^2}\right)e^{-\xi}d\xi$

$\int^x\frac{1}{\xi^2}e^{-\xi}d\xi=-\frac{1}{x}e^{-x}-\int^x\frac{1}{\xi}e^{-\xi}d\xi$

$\tilde{z}_2=x\left[-\frac{1}{x}e^{-x}+\int\left(\frac{1}{\xi}-\frac{1}{\xi}\right)e^{-\xi}d\xi\right]=-e^{-x} \quad z_2=e^{-x}$

$u_1(0)=0 \quad u_1'(0)=1$

$u_1(x)=x$

$u_2(x)=Ax+Be^{-x}$

$u_2'(x)=A-Be^{-x}$

$u_2(1)=A+Be^{-1}=0$

$u_2'(1)=A-Be^{-1}=1 \quad u_2(x)=\frac{x}{2}-\frac{1}{2}e^{1-x}$

$2A=1 \quad A=1/2$

$Be^{-1}=-A \quad B=-1/2\cdot e$

$W=u_1u_2'-u_2u_1'=x\left(\frac{1}{2}+\frac{1}{2}e^{1-x}\right)-\left(\frac{x}{2}-\frac{1}{2}e^{1-x}\right)\cdot 1=\frac{1}{2}e^{1-x}(x+1)=\frac{e}{2}e^{-x}(x+1)=\frac{e}{2}\cdot\frac{1}{r}=\frac{c}{r}$

$G(x,\xi)=\begin{cases}\frac{u_1(x)u_2(\xi)}{c} & 0\le x\le\xi\\ \frac{u_2(x)u_1(\xi)}{c} & \xi\le x\le 1\end{cases}$

$y(x)=\int_0^x\frac{u_1(\xi)u_2(x)}{c}h(\xi)\,d\xi+\int_x^1\frac{u_2(\xi)u_1(x)}{c}h(\xi)d\xi$

$\frac{u_2(x)}{c}\int_0^x\xi\cdot 2e^{\xi}d\xi=\frac{u_2(x)}{c}\left(2(x-1)e^x\right)\Big|_0^x=$

$=\frac{u_2(x)}{c}(2x-1)e^x+2$

$\frac{u_1(x)}{c}\int_x^1\frac{1}{2}\left(\xi-e^{1-\xi}\right)\cdot 2e^{\xi}d\xi=\frac{u_1(x)}{c}\int_x^1\left(\xi e^{\xi}-e\right)d\xi=$

$=\frac{u_1(x)}{c}\left((x-1)e^x-ex\right)\Big|_x^1=\frac{u_1(x)}{c}\left(-e^1-(x-1)e^x+ex\right)$

$y(x)=2\left[\left(x-e^{1-x}\right)\left(e^{x-1}(x-1)+e^{-1}\right)+x(x-1)-(x-1)e^{x-1}\right]$

$$\left[(1-x^2)y'\right]' - \frac{1}{1-x^2}\, y = (1+x)^{1/2}$$

$$\left[(1-x^2)z'\right]' - \frac{1}{1-x^2}\, z = 0 \qquad z_1 = \frac{1}{\sqrt{1-x^2}} \text{ megoldás}$$

$$z_1' = -\frac{1}{2}\,\frac{-2x}{(1-x^2)^{3/2}} = \frac{x}{(1-x^2)^{3/2}}$$

$$\left[\frac{x}{(1-x^2)^{1/2}}\right]' - \frac{1}{(1-x^2)^{3/2}} =$$

$$= \frac{1}{(1-x^2)^{1/2}} + \frac{x^2}{(1-x^2)^{3/2}} - \frac{1}{(1-x^2)^{3/2}} = 0$$

$$z_2 = \frac{1}{\sqrt{1-x^2}} \int (1-\xi^2)\frac{1}{1-\xi^2}\, d\xi = \frac{x}{\sqrt{1-x^2}}$$

$y'(0)=0 \quad y(1)=1$

$u_1'(0)=0 \quad u_1(0)=1$

$u_1(x) = z_1(x) = \frac{1}{\sqrt{1-x^2}}$

$u_2(1)=0$

$$\tilde{u}_2(x) = \frac{A+Bx}{\sqrt{1-x^2}} = A\,\frac{1-x}{\sqrt{1-x^2}} = A\sqrt{\frac{1-x}{1+x}} \qquad u_2 = \sqrt{\frac{1-x}{1+x}}$$

$$W = u_1 u_2' - u_2 u_1' = \frac{1}{\sqrt{1-x^2}}\left(-\frac{1}{2}\frac{1}{\sqrt{1-x^2}} - \frac{1}{2}\frac{\sqrt{1-x}}{(1+x)^{3/2}}\right) -$$

$$-\sqrt{\frac{1-x}{1+x}}\;\frac{x}{(1-x^2)^{3/2}} = -\frac{1}{(1-x^2)} = \frac{-1}{r} \Rightarrow C = 1$$

$1 - x^2 + x^4 - x^6 + \dots \qquad |x|<1$

$\frac{1}{1+x^2} \xrightarrow{\text{komplex}} \frac{1}{1+z^2}$ ennek szingularitása van $z = \pm i$ helyen

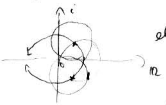
Az $f(z) = \frac{1}{(z-z_0)^n}$ típusú függvényeknek szingularitása van

$f(z) = \sum_{n=0}^{\infty} a_n z^n$

$z=0$ körül analitikusnak nevezzük, ha sorba fejthető

itt fejthető sorba, B-ben már R' lehet a konvergenciasugár

$\sqrt{z}$ függvény komplexben szinguláris lehet



elágazó szingularitás

z^N típusokban

$e^{-\frac{a}{z}}$ típusú függvényeknek lényeges szingularitása van

$w''(z) = p_1(z)\, w'(z) + p_0(z) w(z)$ $\quad w(0) = a$

$w'(0) = b$

$w(z) = w(0) + w'(0) z + \frac{1}{2!} w''(0) z^2 + \frac{1}{3!} w'''(0) z^3 \dots$

$w'' = p_1(0)\, b + p_0(0)\, a$

$w''' = p_1'(0)\, w'(0) + p_1(0) w''(0) + p_0'(0)\, w(0) + p_0(0)\, w'(0) = p_1'(0) b + p_1(0) \left(p_1(0)\, b + p_0(0) a\right) + p_0'(a) a + p_0(0) b$

$w(z) = a\, w_1(z) + b\, w_2(z)$

$w_1(z) = 1 + \frac{1}{2!} p_0(0) z^2 + \frac{1}{3!} \left[p_1(0) p_0(0) + p_0'(0)\right] z^3 + \dots$

$w_2(z) = z + \frac{1}{2!} p_1(0) z^2 + \frac{1}{3!} \left[p_1^2(0) + p_0(0) + p_1'(0)\right] z^3 + \dots$

Fuchs és Frobenius

$w''(z) = \frac{r(z)}{z} w'(z) + \frac{s(z)}{z^2} w(z)$ $\quad z = 0$ pontban reguláris szingularitás van

(pólus, z^d, $\ln z$ típusú szingularitás lehet)

$r(z) = \alpha$

$s(z) = \beta$

$z^2 w''(z) - \alpha z w'(z) - \beta w(z) = 0$

$w \sim z^d$

$d(d-1) z^d - \alpha d z^d - \beta z^d = 0$

$d(d-1) - \alpha d - \beta = 0 \quad d_1 \neq d_2$

$W(z) = A z^{d_1} + B z^{d_2}$

$d_1 = d_2 = u$

$w(z) = z^u (A + B \ln z)$

$w(z) = z^d (w_0 + w_1 z + w_2 z^2 + \dots) \quad w_0 \neq 0$

$\underset{\neq 0}{w_0} \underbrace{\left[d(d-1) - d r_0 - s\right]}_{0 \,\to\, d_1, d_2} = 0$

$$w_1\left[(d+1)d-(d+1)r_0-s_0\right]=(dr_1+s_1)w_0$$

$$w_2\left[(d+2)(d+1)-(d+r)r_0-s_0\right]=(dr_2+s_2)w_0+\left[(d+1)r_1+s_1\right]w_0$$

$$\dots$$

$$w_n\left[(d+n)(d+n-1)+(d+n)r_0-s_0\right]=(dr_n+s_n)w_0+\dots+\left[(d+n-1)r_1+s_1\right]w_{n-1}$$

r_n, s_n az r és s fv-ek hatványsorainak együtthatói

ha $d_1\neq d_2$ és $d_1\neq d_2+n$

$w_1=z^{d_1}(\dots)$

$w_2=z^{d_2}(\dots)$

miért vannak különválasztva $d_1=d_2$ és $d_1=d_2+n$

$$w_2(z)=C\,w_1(z)\int e^{\int\frac{-r(z)}{z}dz}\frac{1}{w_1(z)^2}\,dz$$

$$w_2(z)=w_1(z)\left[\ln z+h(z)\right]\qquad d_1=d_2$$

$$w_2(z)=A\left[\gamma\ln z+z^{d_2}h(z)\right].$$

fv viselkedése végtelenben

$w_1(z_1)=w(1/z_1)$

$z_1=1/z$

$$w_1''(z_1)=-\left[\frac{2}{z_1}+\frac{1}{z_1^2}r_1\left(\frac{1}{z_1}\right)\right]w_1'(z_1)+\frac{1}{z_1^4}r_0\left(\frac{1}{z_1}\right)w_1(z_1)$$

$z\to\dfrac{Az+B}{Cz+D}\qquad AD-BC\neq0\quad$ ezzel a szinguláris pontok a $0,1$ és ∞-be transzformálható

pl. $w''(z)=\dfrac{2}{z-z_0}w'(z)=0\qquad z=\infty$-ben

$w(z)=A_1+\dfrac{A_0}{z-z_0}\qquad w''(z)=0$

$w(z)=A+Bz$

$$w''(z) = (\alpha_1 + \alpha_2 - 1)\frac{1}{z} w'(z) - \alpha_1\alpha_2 z^{-2} w(z)$$

0, ∞ a szinguláris

$$w(z) = Az^{\alpha_1} + Bz^{\alpha_2} \quad \alpha_1 \neq \alpha_2$$
$$= z^{h}(A + B\ln z) \quad \alpha_1 = \alpha_2 = h$$

0, 1, ∞ szinguláris

$$z(1-z) w''(z) + [c - (a+b+1)z] w' - abw = 0$$

$z = 0 \to \quad \alpha_1 = 0 \quad \alpha_2 = 1-c$

$z = 1 \quad \alpha_1 = 0 \quad \alpha_2 = c-a-b$

$z = \infty \quad \alpha_1 = a \quad \alpha_2 = b$

---

konfluens hipergeometrikus, 0, ∞ szingularitással

$$zw'' + (c-z) w' - qw = 0$$

$\alpha_1 = 0$
$\alpha_2 = 1-c$

$z=0 \quad z=\infty$ szinguláris

$$w'' + (1-\alpha_1-\alpha_2)\frac{1}{z} w' + \left(-l^2 + \frac{2\alpha}{z} + \frac{\alpha_1\alpha_2}{z^2}\right) w = 0$$

$z = 0 \quad \alpha_1, \alpha_2$

$z = \infty \sim e^{lz}$

10.06

N-edrendű lin. egyenlet

$$b_n(x)\, y^{(n)}(x) + \ldots + b_1(x)\, y'(x) + b_0(x) y(x) = f(x)$$

$z(x) = c_1 z_1(x) + \ldots + c_n z_n(x)$ általános homogén megoldás

$(z_1, \ldots z_n)$ fundamentális halmazt alkotnak $\Leftrightarrow$ lin. függetlenek

$g_1 \ldots g_N$ lin. összefüggő, ha $\exists A_i \quad A_1 g_1 + \ldots + A_N g_N = 0 \Rightarrow A_1 g_1' + A_2 g_2' + \ldots A_N g_N' = 0$

$\Downarrow \ldots$

$A_1 g_1^{(N-1)} + A_2 g_2^{(N-1)} + \ldots + A_N g_N^{(N-1)} = 0$

Ez N db egyenlet N db ismeretlenre

$$\underbrace{\begin{vmatrix} g_1 & g_2 & \cdots & g_N \\ g_1' & g_2' & \cdots & \\ g_1'' & & & \\ \vdots & & & \\ g^{(N-1)} & g_2^{(N-1)} & \cdots & g_N^{(N-1)} \end{vmatrix}}_{= W} = 0$$ akkor g_i-k összefüggők

$\mathcal{L} = \mathcal{L}_N \circ \mathcal{L}_{N-1} \circ \dots \circ \mathcal{L}_1$ ha ilyen alakban felírható, akkor jó

elsőrendű difoperátorok

↑ N-edrendű dif. operátor

pl: $(x+2)z''' + (2x+3)z'' + xz' - z = 0 = \mathcal{L}(z) = \alpha_3 \alpha_2 \alpha_1 (z)$

Ekkor $\alpha_1 = \frac{d}{dx} + 1$ $\quad \alpha_2 = (x+1)\frac{d}{dx} - 1$ és $\alpha_3 = (x+2)\frac{d}{dx} + (x+1)$,

a független megoldások:

$z_1 = w_1$, $z_2 = w_1 \int^x w_2(\xi)\,d\xi$ és $z_3 = w_1 \int^x w_2(\xi)\,d\xi \int^\xi w_3(\eta)\,d\eta$

$\alpha_1 z_1 := 0 \quad \alpha_2 \alpha_1 z_2 := 0 \quad \alpha_3 \alpha_2 \alpha_1 := 0$

$\left(\frac{d}{dx} + 1\right) w_1 = 0 \qquad w_1 = e^{-x}$

$z_2 = e^{-x} \int^x w_2(\xi)\,d\xi \qquad w_2 = (x+1)e^x \qquad z_2 = e^{-x}\int (\xi+1)e^\xi\,d\xi = e^{-x}(xe^x) = x$

$(x+1)w_2' - (x+2)w_2 = 0$

$(x+1)(x+2)w_3' + (x^2+4x+5)w_3 = 0$

$z_3 = e^{-x}\int^x (\xi+1)e^\xi \int^\xi w_3(\eta)\,d\xi\,d\eta$

$w_3 = \frac{x+2}{(x+1)^2}e^{-x}$

$z_3 = e^{-x}\int^x (\xi+1)e^\xi \int^\xi \frac{\eta+2}{(\xi+1)^2}e^{-\xi}\,d\xi\,d\eta = xe^{-x}$ $\qquad \int^\xi w_3\,d\eta = -\frac{1}{(\xi+1)}e^{-\xi}$

$y_1 = \alpha_1(x)z_1(x) + \dots + \alpha_N(x)z_N(x)$ inhomogén partikuláris megoldásra

$\alpha_1'(x)z_1(x) + \dots + \alpha_N'(x)z_N(x) = 0$

$\alpha_1'(x)z_1'(x) + \dots + \alpha_N'(x)z_N'(x) = 0$

$\vdots$

$\alpha_1'(x)z_1^{(N-2)}(x) + \dots + \alpha_N'(x)z_N^{(N-2)}(x) = 0$

$\alpha_1'(x)z_1^{(N-1)}(x) + \dots + \alpha_N'(x)z_N^{(N-1)}(x) = f(x)$

$$\begin{vmatrix} z_1 & \dots & z_N \\ z_1' & \dots & z_N' \\ & \vdots & \\ z_1^{(N-1)} & \dots & z_N^{(N-1)} \end{vmatrix} \neq 0$$

KEZDETI FELTÉTELEK

$y(0) = a_0 \quad y'(0) = a_1 \quad \dots \quad y^{(N-1)}(0) = a_{N-1}$

Sajátérték probléma

$\mathcal{L}\,\phi(t)=f(x) \qquad [p(t)\cdot y'(t)]' + q(t)\,y(t)=f(x)$

Spektrál elmélet

olyanok nem-triviális megoldások, amelyek eleget tesznek a peremfeltételeknek

$[p(t)\,y'(t)]' + [q(t)+\lambda\,r(t)]\,y(t)=0 \qquad a\le x\le b$

$y(a)=0\,,\quad y(b)=0$

paraméter   súlyfüggvény

lehet, hogy $\exists\lambda$ hogy meg tudom oldani a feladatot

$\mathcal{L}\,y(x)=\lambda\,r(x)\,y(x)$

sajátfüggvénye $\mathcal{L}$-nek

$\lambda_1 \dots \lambda_N \dots$
$y_1 \dots y_N \dots$ megoldások, diszkrét spektrum

$A\le\lambda\le B$ folytonos spektrum

függvények skalárszorzata

$\langle g,h\rangle=\int_a^b w(t)\,h(t)\,g(t)\,dx \qquad w(t), a, b$ még definiálható

$\langle g,g\rangle=N(g)$ ha $N(g)=1$, akkor normált a tér

$\int_a^b w(x)\,y_i\,y_k\,dx=\delta_{ik}$ akkor ortonormáltak

ha $\phi(x)=\sum_{n=1}^{\infty} c_n\,y_n(t)$

$c_m=\langle\phi, y_n\rangle=\int_a^b w(x)\,\phi(x)\,y_n(t)\,dx$

$N(\phi-\phi_m)\to 0$

$\phi_m=\sum_{n=1}^{m} c_n\,y_n$

Sturm-Liouville elmélet

p, p', q, r valós folyt fv. $p(x), r(x) > 0$ $\quad a \le x \le b$

$(pu')' + (q + \lambda r)u = 0 \quad \exists\, u$ és λ $\qquad$ u és v mikor lennének ortogonálisak?

$(pv')' + (q + \mu r)v = 0 \quad \exists\, v$ és $\mu \neq \lambda$

$\cdot v,\ \cdot u:\quad v(pu')' - u(pv')' + (\lambda - \mu) r u v = 0$

$\Downarrow \int$

$$(\lambda-\mu)\int_a^b u v\, dx = -\int_a^b (vpu' - upv')'\, dx$$

$$\underbrace{(\lambda-\mu)\int_a^b r u v\, dx}_{0} = \underbrace{p(b)\, w(b) - p(a)\, w(a)}_{0} \Rightarrow \quad p(b)\, w(b) - p(a)\, w(a) = 0$$

spec eset

$p(a)=0 \quad w(b)=0$ VAGY

$p(b)=0 \quad w(a)=0$ VAGY

$p(a)=0 \quad p(b)=0$

spec eset: $w(a) = w(b) = 0$

$\alpha y(a) + \beta y'(a) = 0$ $\quad$ Reguláris SL

$\gamma y(b) + \delta y'(b) = 0$

spec eset $\quad p(a) = p(b)$ VAGY $\quad$ Szinguláris SL

$y(a) = y(b)$ VAGY

$y'(a) = y'(b)$

periodikus SL

10.13.

$[p(x)\, y'(x)]' + [q(x) + \lambda r(x)]\, y(x) = 0 \qquad a \le x \le b \qquad \exists\lambda$, hogy megoldható

$$\underbrace{\int_a^b w(x) \cdot y_n(x)\, y_m(x)}_{\langle y_n | y_m \rangle \text{ skaláris szorzás}} = \delta_{nm} \qquad y_n(t) \text{ és } y_m(t) \text{ ortogonálisak}$$

$y(x) = \sum_{n=1}^{\infty} c_n y_n(x) \qquad c_n = \langle y | y_n \rangle$

Sturm-Liouville:

p, p', q, r valós, $p, r > 0$

$a \le x \le b$

$(pu')' + (q + \lambda r)u = 0$ $\quad$ megoldás u valamilyen λ-val $\quad$ legyenek merőlegesek, ortonormáltak a súlyfüggvénnyel

$(pv')' + (q + \mu r)v = 0$ $\quad$ megoldás v valamilyen μ-vel

$$(\lambda-\mu)\int_a^b r(x)\, u(x)\, v(x)\, dx = p(a)\, W(a) - p(b)\, W(b)$$

$$w(u,v) = uv' - vu'$$

reguláris SL
$u(a)=0$
$u(b)$

$\alpha y(a)+\beta y'(a)=0$
$\gamma y(b)+\delta y'(b)=0$

reguláris SL
$p(a)u(a)=p(b)\ u(b)$
$p(a)=p(b)$
$y(a)=y(b)$
$y'(a)=y'(b)$

sing
$p(a)=0$
$u(b)=0$

$p(b)=0$
$u(a)=0$

$p(a)=0$
$p(b)=0$

---

pl: $\frac{d^2u}{dx^2}+\left(s(x)+\lambda\right)u=0$ ha $\lambda>0$, $s(x)>0$

gömbölék

oszcillációs tétel

végtelen sok λ-ra van megoldás $\lambda_1<\lambda_2\dots<\lambda_n<\dots$ $n\to\infty$, $\lambda_n\to\infty$

$y_n(x,\lambda_n)$ függvénynek $n-1$ egyszerű nullája van (az intervallumban)

y_n-nek nullája x_1 és x_2-ben, y_{n+1}-nek lesz x_1 és x_2 közti nullája

nézzük $u_n(0)=u_n(\ell)=0$ peremfeltételeket $(x\in[0,\ell])$

$\int_0^\ell u^2(x)\,dx=1$ $\quad |u_n(x,\lambda_n)|$ korlátosak

ha $n\to\infty$ $\quad u_n=\sqrt{\frac{2}{\ell}}\sin\sqrt{\lambda_n}\cdot x+\frac{1}{\sqrt{\lambda_n}}O(1)$

$\lambda_n=\frac{n^2\pi^2}{\ell^2}+O(1)$

---

pl: $y''+\lambda y=0$
$p=r=1$ $\quad q=0$
$y(-\pi)=:0:=y(\pi)$

$\frac{1}{\sqrt{\pi}}\sin(\sqrt{\lambda}x)=y_n$
$\lambda_n=n^2$ $\quad n=1,2\dots$

$y''+\lambda y=0$
$p=r=1$ $\quad q=0$
$y'(-\pi)=:0:=y'(\pi)$
$y_n=\frac{1}{\sqrt{\pi}}\cos(\lambda_n x)$ $\quad \lambda_n=n^2$ $\quad n=1,2\dots$
$y_0=1/\sqrt{2\pi}$ $\quad \lambda_0=0$

---

periodikus határfeltétel
$y(-\pi)=y(\pi)$ $\quad y'(\pi)=y'(-\pi)$

$\lambda_n=n^2$ $\quad n=0,1\dots$

$f(x)=\frac{1}{2}a_0+\sum_{n=1}^{\infty}\left[a_n\cos(nx)+b_n\sin(nx)\right]$

$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos(nx)\,dx$ $\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin(nx)\,dx$

$\langle u, Lu\rangle = \int_a^b r(t)\, u(x)\, Lu(x)\, dt = \lambda \int_a^b r(t)^2 u(t)^2 dt = \lambda \langle u, r u\rangle$

$Lu = \lambda u$ $\qquad$ L. poz. definit

$\uparrow$ dif. operátor $\qquad$ $\langle u, Lu\rangle > 0,\ r > 0 \to \lambda \geq 0$

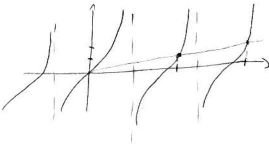
$y'' + \lambda y = 0$ $\quad [0,\pi]$ intervallumban

$y(0) = 0 \quad y'(\pi) = y(\pi)$ határfeltételek

$\sin(\sqrt{\lambda}\, x)$ függvényekkel jöh-e? $\to \sqrt{\lambda} \cos(\sqrt{\lambda}\, \pi) = \sin(\sqrt{\lambda}\, \pi)$

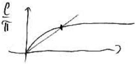
$\sqrt{\lambda} = \operatorname{tg} \sqrt{\lambda}\, \pi \qquad \lambda_n = \frac{k^2}{\pi^2}$ ← legyen ilyen alakú

$\operatorname{tg} k = \frac{k}{\pi}$



$\lambda = -\frac{\ell^2}{\pi^2}$

$\operatorname{th} \ell = \frac{\ell}{\pi}$



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pl: $\left[(1-x^2)\, y'\right]' + \lambda y = 0 \qquad -1 \leq x \leq 1 \qquad L = (1-x^2)\frac{d}{dt} + \lambda$

$p(x) = 1 - x^2 \quad q = 0 \quad r = 1$

$\langle u, Lu\rangle = -\int_{-1}^{1} u(x)\left[(1-x^2)(u'(x))\right]' dx =$

$y = $ áll $\quad \lambda_0 = 0$

$= -\left[(1-x^2)\, u u'\right]_{-1}^{1} + \int_{-1}^{1} (1-x^2)\, u'^2(x)\, dx > 0$

$(1-x^2)y'' - 2xy' + \lambda y = 0$

$u(x) = (x^2-1)^n \qquad u'(x) = n\, 2x (x^2-1)^{n-1}$

$(1-x^2)u' + 2nxu = 0$

$(1-x^2)u'' + 2(n-1)xu' + 2nu = 0$

$(1-x^2)u''' + 2(n-2)xu'' + 2(2n-1)u' = 0$

$\vdots$

$(1-x^2)u^{(k+2)}(x) + 2(n-k-1)x u^{(k+1)} + (2n-k)(k+1)u^{(k)} = 0$

$$(1-x^2)\,u^{(k+1)}(x) + 2(n-k)\,x\,u^{(k)} + (2n-k+1)\,k\,u^{(k-1)} = 0 \qquad \Big\rfloor\, d/dx$$

$$(1-x^2)\,u^{(k+2)}(x) + 2(n-k-1)\,u^{(k+1)} + (2n-k)(k+1)\,u^{(k)} = 0 \qquad \text{TIND} \qquad k=n$$

visszaan eredeti egyenletbe

$$\lambda_n = n(n+1)$$

$$y_n = c\left[(x^2-1)^n\right]^{(n)} = P_n(x)$$

Legendre-polinomok

$$\int_{-1}^{1} P_n(x)\,P_m(x)\,dx = 0 \qquad n \neq m$$

$$f(x) = \sum_{n=0}^{\infty} c_n P_n(x)$$

$\mathcal{L}\varphi = \lambda\varphi$ + határfeltételek

$\mathcal{L}\psi = g$ érhomogén lineáris probléma, ugyanazon határfeltételekkel

$\mathcal{L}G = \delta$

Green-függvény

$$\mathcal{L}\varphi_k = \lambda_k \varphi_k$$

$$\psi = \sum_k c_k \varphi_k$$

$$g = \sum_k g_k \varphi_k \qquad \sum_k c_k \mathcal{L}\varphi_k = \sum_k g_k \varphi_k$$

$$\Downarrow$$

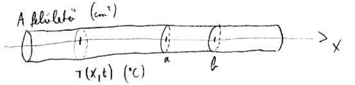
$$\sum_k c_k \lambda_k \varphi_k = \sum_k g_k \varphi_k \qquad c_k = \frac{g_k}{\lambda_k}$$

$$G(x,x') = \sum_k \frac{\varphi_k(x)\,\varphi_k(x')}{\lambda_k}$$

$$\int_a^b G(x,x')\,g(x')\,dx' = \psi(x') \qquad \sum_k \frac{g_k}{\lambda_k}\varphi_k(x) = \psi(x)$$

Parciális diff. egyenletek

Hővezetés problémája



$\rho\left(\frac{g}{cm^3}\right)$ $\quad c\left(\frac{cal}{g\,°C}\right)$ $\quad$ hővezetési együttható: $k\left(\frac{cal}{s\cdot cm\cdot °C}\right)$

$\Delta E = c\cdot m\cdot \Delta T$

$J_{\text{hőáramsűrűség}} = k\frac{\Delta T}{\Delta x}$

$$\frac{\partial}{\partial t}\int_a^b c\,T\rho A\,dx = \int_a^b c\cdot\frac{\partial T}{\partial t}\rho A\,dx = A\left[k(b)T_x(b,t) - k(a)T_x(a,t)\right] = A\int_a^b (kT_x)_x\,dx$$

$\frac{\partial T}{\partial t} \equiv T_t \qquad \frac{\partial T}{\partial x} = T_x$

kontinuitás miatt

$$\int_a^b\left[c\cdot\rho T_t - (kT_x)_x\right]dx = 0$$

másik lehetőség

$$\int_a^b c(x)\rho(x)AT_t(x,t)\,dx \overset{\exists\xi \text{ belső pont}}{=} A\,c(\xi)\rho(\xi)T_t(\xi,t)(b-a) = A\left[k(b)T_x(b,t) - k(a)T_x(a,t)\right]$$

$b \to a$, $\xi \to a$ $\qquad c(a)\rho(a)T_t(a,t) = (kT_x)_x\big|_{x=a}$

másik lehetőség

vizsgáljuk $a = x$ és $b = x+\Delta x$ esetre $\quad c\cdot\rho\cdot A\,\Delta x\,T_t = (kAT_x)\big|_{x+\Delta x} - (kAT_x)\big|_x$

Difegyenletek osztályozása

$$\left(A\frac{\partial^2}{\partial x^2} + 2B\frac{\partial^2}{\partial x\partial y} + C\frac{\partial^2}{\partial y^2} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial y} + F\right)\varphi = G$$

új koordinátákkal, $\xi = \xi(x,y)$, $\eta = \eta(x,y)$

I. eset

$$\frac{\partial}{\partial x} = \frac{\partial\xi}{\partial x}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial x}\frac{\partial}{\partial\eta}$$

$$\frac{\partial}{\partial y} = \frac{\partial\xi}{\partial y}\frac{\partial}{\partial\xi} + \frac{\partial\eta}{\partial y}\frac{\partial}{\partial\eta} \qquad \frac{\partial^2}{\partial x\partial y} = \left(\xi_x\frac{\partial}{\partial\xi} + \eta_x\frac{\partial}{\partial\eta}\right)\left(\xi_y\frac{\partial}{\partial\xi} + \eta_y\frac{\partial}{\partial\eta}\right)$$

$$\frac{\partial^2}{\partial x^2} = \left(\xi_x\frac{\partial}{\partial\xi} + \eta_x\frac{\partial}{\partial\eta}\right)\left(\xi_x\frac{\partial}{\partial\xi} + \eta_x\frac{\partial}{\partial\eta}\right)$$

$$\frac{\partial^2}{\partial y^2} = \left(\xi_y\frac{\partial}{\partial\xi} + \eta_y\frac{\partial}{\partial\eta}\right)\left(\xi_y\frac{\partial}{\partial\xi} + \eta_y\frac{\partial}{\partial\eta}\right)$$

$$\left(A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2\right)\frac{\partial}{\partial\xi^2} + \left(2A\eta_x\xi_x + 2\left(\eta_x\xi_y + \eta_y\xi_x\right)B + 2C\eta_y\xi_y\right)\frac{\partial^2}{\partial\xi\partial\eta} +$$

$$+\left(A\eta_y^2 + 2B\eta_x\eta_y + C\eta_y^2\right)\frac{\partial}{\partial\eta^2} + \dots \quad \circ\,\varphi = G$$

mikor lesz $A\xi_x^2 + 2B\xi_x\xi_y + C\xi^2 = 0$

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + 2B\frac{\xi_x}{\xi_y} + C = 0$$

$$A\left(\frac{dy}{dx}\right)^2 + 2B\left(\frac{dy}{dx}\right) + C = 0$$

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{B^2 - AC}}{A} \quad \text{ha } B^2 \geq AC$$

$$\frac{dy}{dx} = \frac{-B + \sqrt{B^2 - AC}}{A} \quad v. \quad \frac{dy}{dx} = \frac{-B \pm \sqrt{B^2 - AC}}{A}$$

C_+ $\quad$ C_-

paraméterek maradnak lent

$\xi(x,y) = C_+$ és $\eta(x,y) = C_-$

hiperbolikus alak

$$\varphi_{\xi\eta} + D_1\varphi_\xi + E_1\varphi_\eta + F_1\varphi = G_1$$

$\xi := \alpha + \beta \quad \eta = \alpha - \beta$

$$\varphi_{\alpha\alpha} - \varphi_{\beta\beta} + D_2\varphi_\alpha + E_2\varphi_\beta + F_2\varphi = G_2$$

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példa:

$$(1+x^2)\varphi_{xx} + 2\varphi_{xy} + 3\varphi = 0$$

$A = 1+x^2 \quad B^2 - AC = 1 > 0$

$B = 1$

$C = 0$

$\frac{dy}{dx} = \frac{1+\sqrt{1}}{1+x^2} = \frac{2}{1+x^2} \quad C_+$

$\frac{dy}{dx} = \frac{1-\sqrt{1}}{1+x^2} = 0 \quad C_-$

$\xi = y - 2\,\mathrm{arctg}\,x$

$\eta = y$

$$\to \varphi_{\xi\eta} - \left[\mathrm{tg}\left(\frac{\eta-\xi}{2}\right)\right]\varphi_\xi - \frac{3}{4}\left[\frac{1}{\cos^2\frac{\eta-\xi}{2}}\right]\varphi = 0$$

$B^2 - AC = 0 \quad \underline{\text{II}}$ eset

$\frac{dy}{dx} = \frac{B}{A} \to \xi(x,y)$

$\eta(x,y) \xrightarrow{!} \eta_x\xi_y - \eta_y\xi_x \neq 0$

$$\varphi_{\eta\eta} + D_2\varphi_\xi + E_2\varphi_\eta + F_2\varphi = G \quad \text{parabolikus dif.egyenlet}$$

pl: $x^2\varphi_{xx} + 2x\varphi_{xy} + \varphi_{yy} + y\varphi_x - \varphi = 0$

$A = x^2 \quad B^2 - AC = 0 \quad \frac{dy}{dx} = \frac{1}{x} \Rightarrow \left.\begin{array}{l}\xi = y - \ln x \\ \eta = y\end{array}\right\}$ teljesítik

$B = x$

$C = 1$

$$\varphi_{\eta\eta} + \varphi_\xi\left(1 - e^{\xi-\eta}\right) - \varphi = 0$$

III. eset

$B^2 - AC < 0$ elliptikus

$$\varphi_{\xi\xi} + \varphi_{\eta\eta} + D_3\varphi_\xi + E_3\,\varphi_\eta + F_3\varphi = G_3$$

$$A\left(\xi_x + i\,\eta_x\right)^2 + 2B\left(\xi_x + i\eta_x\right)\left(\xi_y + i\eta_y\right) + C_3\left(\xi_y + i\eta_y\right)^2 = 0$$

$$\frac{\xi_x + i\eta_x}{\xi_y + i\eta_y} = \frac{B + i\sqrt{AC - B^2}}{A} \qquad \varphi(x,y) = \xi(x,y) + i\eta(x,y)$$

példa: $x^2\varphi_{xx} + 2\varphi_{xy} + \varphi_{yy} = 0 \qquad x > 1$

$$\frac{dy}{dx} = \frac{1 + i\sqrt{x^2-1}}{x^2} \qquad \varphi(x,y) = \overbrace{y + \frac{1}{x}}^{\xi} + i\overbrace{\left[\frac{\sqrt{x^2-1}}{x} - \ln\left(x + \sqrt{x^2-1}\right)\right]}^{\eta} = \text{áll}$$

$$\varphi_{\xi\xi} + \varphi_{\eta\eta} + \frac{2x}{x^2-1}\varphi_\xi - \frac{x(x^2-2)}{(x^2-1)^{3/2}}\varphi_\eta = 0$$

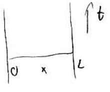
$$\eta = \frac{\sqrt{x^2-1}}{x} - \ln\left(x + \sqrt{x^2-1}\right) \qquad \xi = y + \frac{1}{x}$$

peremfeltételek

1. eset: hiperbolikus egyenlet $\quad \varphi_{tt} - \varphi_{xx} = 0$

kezdeti érték $\quad \varphi(x,0)$ megadandó

$\varphi_t(x,0) \qquad \varphi(0,t)$, $\varphi(L,t)$ peremfeltétel



2. eset: parabolikus $\quad \varphi_t = \varphi_{xx}$

$\varphi(x,0) =$ megadandó

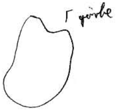
$\varphi(0,t)$ és $\varphi(L,t)$ (peremfeltétel)

3. eset elliptikus

$\varphi_{xx} + \varphi_{yy} = 0$

$\varphi(\Gamma)$ VAGY

$\frac{\partial\varphi}{\partial n}(\Gamma)$



3. kiméretek van
- megoldás létezik
- megoldás egyértelmű
- folytonosan függ a határadatoktól

10.20
4

$u_{xx}+u_{yy}=0 \quad y>0$ ha direkt béna vagyok, nem feltételeket adok meg

$u(x,0)=0 \quad u_y(x,0)=f(x)$ ha $f(x)$ nem analitikus, az nem

$f(x)=\frac{\sin Nx}{N^2} \quad N\in\mathbb{Z}$

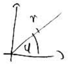
ekkor $u=\operatorname{sh}(Ny)\cdot\sin(Nx)/N^3$

ha $N\to\infty$, $f(x)\to 0$, viszont $u\to\infty$

$\varphi(x,0)=0 \quad x>0$

$\varphi(0,y)=1 \quad y>0$

$\varphi_{xx}+\varphi_{yy}=0 \hookrightarrow \varphi=\frac{2}{\pi}\operatorname{arctg}\frac{y}{x}$



$\frac{1}{r^{2n}}\sin n\varphi$

11.07.

Hiperbolikus egyenlet

$$\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u}{\partial x^2}=0 \qquad u(x,0)=F(x)$$

$$\left(\frac{\partial u}{\partial t}\right)_{x,0}=G(x)$$

d'Alambert megoldása:

$$u(x,t)=f_1(x-t)+f_2(x+t)$$

$$f_1(x)+f_2(x)=F(x)$$

$$-f_1'(x)+f_2'(x)=G(x) \Rightarrow f_1(x)-f_2(x)=-\int_{x_1}^{x}G(x)dx$$

$$f_1(x)=\frac{1}{2}\left[F(x)-\int_{x_0}^{x}G(x)dx\right] \qquad f_2(x)=\frac{1}{2}\left[F(x)+\int_{x_0}^{x}G(x)dx\right]$$

$$u(x,t)=\frac{1}{2}\left[F(x-t)+F(x+t)+\int_{x-t}^{x+t}G(x)dx\right]$$

most legyen $u(0,t)=u(l,t)=0$

határfeltételek figyelembe vétele

$$\boxed{\begin{aligned}&f_1(-t)+f_2(t)=0\\&f_1(l-t)+f_2(l+t)=0\end{aligned}}$$

$\Downarrow$

$$\boxed{\begin{aligned}&f_1(-x)+f_2(x)=0\\&f_1(l-x)+f_2(l+x)=0\end{aligned}}$$

$\Downarrow$

$$\begin{aligned}&f_1(-x)+f_2(x)=0\\&f_1(-x)+f_2(2l+x)=0\end{aligned} \Rightarrow \begin{aligned}&f_2(x)=f_2(2l+x)\\&f_1(x)=f_1(2l+x)\end{aligned}$$

$0\le x\le l$

$$f_1(x)=\frac{1}{2}\left[F(x)-\int_{x_0}^{x}G(x)dx\right]$$

$$f_2(x)=\frac{1}{2}\left[F(x)+\int_{x_0}^{x}G(x)dx\right]$$

Mi van, ha a határfeltétel megváltoztatom, és

$u(l,t) = \cos \Omega t$

$u(0,t) = 0$

keresem a megoldást $u(x,t) = \eta(x,t) + v(x,t)$ alakban, ahol

$$0 = \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{x}{l}\Omega^2 \cos(\Omega t) \quad \left\{ \begin{array}{ll} \eta(l,t) = 0 & v(0,t) = 0 \\ \eta(0,t) = 0 & v(l,t) = \cos \Omega t \end{array} \right\} \; v(x,t) = \frac{x}{l}\cos \Omega t$$

$$u(x,t) = \eta(x,t) + \frac{x}{l}\cos(\Omega t)$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0$$

Mivel $\sqrt{\frac{2}{l}}\sin\frac{n\pi x}{l}$ ortonormált bázis

$$\eta(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} V_n(t)$$

$$\sum_{n=1}^{\infty}\left(\sin\frac{n\pi x}{l} V_n''(t) + \left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi x}{l} V_n(t)\right) = \frac{x}{l}\Omega^2\cos(\Omega t) \quad \Big\downarrow \cdot \sin\frac{n\pi x}{l},\ \int_0^l dx$$

$$V_n''(t) + \left(\frac{n\pi}{l}\right)^2 V_n(t) = \frac{2}{l}\int_0^l \frac{x}{l}\Omega^2 \cos(\Omega t)\sin\frac{n\pi x}{l}\,dx$$

$$\int_0^l x \sin\frac{n\pi x}{l}\,dx = -\frac{l}{n\pi}\cos\frac{n\pi x}{l}\,x\Big|_0^l + \int_0^l \frac{l}{n\pi}\cos\frac{n\pi x}{l}\,dx = -\frac{l^2}{n\pi}$$

$$V_n''(t) + \left(\frac{n\pi}{l}\right)^2 V(t) = -\frac{2}{n\pi}\Omega^2\cos(\Omega t)$$

$$a_n \cos\frac{n\pi t}{l} + b_n \sin\frac{n\pi t}{l} + \text{inh.} = u(x,t)$$

Parabolikus egyenlet

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$u(x,0) = G(x)$

$u(0,t) = 0$

$u(l,t) = 0$

Megoldás Fourier módszerrel

$u(x,t) = U(x)\cdot V(t)$

$V'U - U''V = 0 \Rightarrow \frac{V'}{V} = \frac{U''}{U} := -k^2$

$U'' = -k^2 U \Rightarrow U = A_u\cos(kx) + B_u\sin(kx)$

peremfeltételek

$U_n = \sin\left(\frac{n\pi}{l}x\right)$

$V' = -k^2 V$

$\Downarrow$

$A_V \cdot e^{-k^2 t}$

peremfeltételek

$V_n = e^{-\left(\frac{n\pi}{l}\right)^2 t}$

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l} e^{-\left(\frac{n\pi}{l}\right)^2 t}$$

$$u(x,0) = G(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{l}x$$

$$a_n = \frac{2}{l}\int_0^l G(x)\sin\frac{n\pi}{l}x\,dx$$

$-l \le x \le 0$

$f_1(x) = f_2(x)$

$f_2(x) = -f_1(x)$

$F(-x) = -F(x) \quad G(-x) = -G(x)$ esetén

$f_1(-x) + f_2(x) = 0$

$f_1(l-x) + f_2(l+x) = 0$

Fourier, Bernoulli módszer

keressük a megoldást $u(x,t) = \sum_n \left(u_n(t) v_n(x) + u_n(x) w_n(t) \right)$ v_n és w_n saját függvények

ha $u(x,t) = U(x) V(t) \Rightarrow V'' U = V U'' \Leftrightarrow \frac{V''}{V} = \frac{U''}{U} := -k^2$

x-től, t-től függ

határfeltétel:
$U(0) = 0$
$U(l) = 0$

$V'' = -k^2 V \Rightarrow V = A_v \cos(kt) + B_v \sin(kt) \rightarrow V =$

$U'' = -k^2 U \Rightarrow U = A_u \cos(kt) + B_u \sin(kt) \xrightarrow{\text{határfeltételek}} U = B_u \sin\frac{n\pi x}{l}$

$$u(x,t) = \sum_{n=1}^{\infty} \left(a_n \sin\left(\frac{n\pi}{l} x\right) \cos\frac{n\pi t}{l} + b_n \sin\left(\frac{n\pi}{l} x\right) \sin\frac{n\pi t}{l} \right)$$

ekkor $F(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{l}$

$G(x) = \sum_{n=1}^{\infty} b_n \frac{n\pi}{l} \sin\frac{n\pi x}{l}$

$a_m = \frac{2}{l} \int_0^l F(x) \sin\frac{m\pi x}{l} dx$

$\frac{n\pi}{l} b_n = \frac{2}{l} \int_0^l G(x) \sin\left(\frac{n\pi}{l} x\right) dx$

$$\int_0^l \underbrace{\sin\frac{m\pi x}{l} \sin\frac{n\pi x}{l}}_{I} dx = \begin{cases} 0 & m \ne n \\ l/2 & m = n \end{cases}$$

$\frac{\pi x}{l} := \varphi$

$I = \sin(m\varphi)\sin(n\varphi) = \frac{1}{2}\cos[(m-n)\varphi] - \frac{1}{2}\cos[(m+n)\varphi]$

$$\int_0^{\pi} I = \begin{cases} 0 & \text{ha } m \ne n \\ l/2 & \text{ha } m = n \end{cases}$$

$\sqrt{\frac{2}{l}} \sin\left(\frac{n\pi}{l} x\right)$ ortonormált bázis

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{l} \sin\left(\frac{n\pi t}{l} + \gamma_n\right) =$$

$$= \sum_{n=1}^{\infty} \frac{c_n}{2} \left[\cos\left[\frac{n\pi}{l}(x-t) + \gamma_n\right] - \cos\left[\frac{n\pi}{l}(x+t) + \gamma_n\right] \right]$$

$T_{,t} = \frac{1}{c\rho}\left(k T_{,x}\right)_{,x}$ ha k állandó

$T_{,t} = \frac{k}{c\rho} T_{,xx}$

$\downarrow T = u, \ \frac{k}{c\rho} = a^2$

$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$

kezdeti feltétel $u(x,0) = f(x)$

$\tau := a^2 t$

$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \tau}\frac{\partial \tau}{\partial t} = a^2 \frac{\partial u}{\partial \tau}$

$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$

$u(x, \tau = 0) = f(x)$ kezdeti feltétel len. keresem a $u(x,\tau) = X(x) T(\tau)$ alakú megoldásokat.

$\frac{T'(\tau)}{T(\tau)} = \frac{X''(x)}{X(x)} = C \Rightarrow \begin{array}{l} T'(\tau) = C\, T(\tau) \Rightarrow T(\tau) = C_1 \cdot e^{-\lambda^2 \tau} \\ X''(x) = C \cdot X(x) \overset{C = -\lambda^2}{\Rightarrow} X(x) = A\cos(\lambda x) + B\sin(\lambda x) \end{array} \Bigg\} u(x,\tau) = \left(\overset{\alpha}{\widetilde{AC}}\cos\lambda x + \overset{\beta}{\widetilde{BC}}\sin\lambda x\right) \cdot e^{-\lambda^2 t}$

partic. $\Rightarrow u_\lambda(x,\tau) = \left(\alpha(\lambda)\cos(\lambda x) + \beta(\lambda)\sin(\lambda x)\right) e^{-\lambda^2 \tau}$

ált. $\Rightarrow u(x,\tau) = \int_{-\infty}^{\infty} u_\lambda(x,\tau)\, d\lambda = \int_{-\infty}^{\infty} \left(\alpha(\lambda)\cos(\lambda x) + \beta(\lambda)\sin(\lambda x)\right) e^{-\lambda^2 \tau}\, d\lambda$

$u(x,0) = \int_{-\infty}^{\infty} \left[\alpha(\lambda)\cos(\lambda x) + \beta(\lambda)\sin(\lambda x)\right] d\lambda \overset{!}{=} f(x)$ Követelmény a kezdeti feltételre: $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

ha $f(x)$ Fourier-integrálja létezik, beszorozva elhinni $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi)\cos\left[\lambda(\xi - x)\right] d\xi$

$f(x) = \int_{-\infty}^{\infty} \left\{\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\lambda\xi)\, d\xi\right)\cos(\lambda x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\lambda\xi) d\xi \cdot \sin\lambda x\right\} d\lambda$ $\boxed{\cos\left[\lambda(\xi - x)\right] = \cos(\lambda\xi)\cos(\lambda x) + \sin\lambda\xi\sin\lambda x}$

$\alpha(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\cos(\lambda\xi)\, d\xi \quad |\alpha(\lambda)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\xi)|\, d\xi \le \infty$

$\beta(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\sin(\xi\lambda)\, d\xi \quad |\beta(\lambda)| \le \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(\xi)|\, d\xi \le \infty$

$$u(x,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi)\left[\cos(\lambda x)\cos(\lambda \xi)+\sin(\lambda x)\sin(\lambda \xi)\right] e^{-\lambda^2 \tau}\, d\xi$$

$$u(x,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} f(\xi)\cos\left[\lambda(x-\xi)\right] e^{-\lambda^2\tau}\, d\xi$$

$$u(x,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\left[\int_{-\infty}^{\infty} e^{-\lambda^2\tau}\cos\left(\lambda(x-\xi)\right) d\lambda\right] d\xi \qquad \lambda=\frac{\sigma}{\sqrt{\tau}} \qquad \frac{x-\xi}{\sqrt{\tau}}=\omega \qquad d\lambda=\frac{d\sigma}{\sqrt{\tau}}$$

$$\int_{-\infty}^{\infty} e^{-\lambda^2\tau}\cos\left[\lambda(x-\xi)\right] d\lambda=\frac{1}{\sqrt{\tau}}\int_{-\infty}^{\infty} e^{-\sigma^2}\cos(\sigma\omega)\, d\sigma=\frac{1}{\sqrt{\tau}} I(\omega)$$

$$I(\omega)=\int_{-\infty}^{\infty} e^{-\sigma^2}\cos(\sigma\omega)\, d\sigma \qquad I(0)=\int_{-\infty}^{\infty} e^{-\sigma^2}\, d\sigma=\sqrt{\pi}$$

$$I'(\omega)=-\int_{-\infty}^{\infty} e^{-\sigma^2}\sigma\cdot\sin(\sigma\cdot\omega)\, d\sigma=\underbrace{\frac{1}{2}e^{-\sigma^2}\sin(\sigma\omega)\Big|_{-\infty}^{\infty}}_{0}-\frac{\omega}{2}\underbrace{\int_{-\infty}^{\infty} e^{-\sigma^2}\cos(\sigma\omega)\, d\sigma}_{I(\omega)}$$

$$I'(\omega)=-\frac{\omega}{2}I(\omega)$$

$$I(\omega)=C_2\cdot e^{-\frac{\omega^2}{4}} \qquad I(0)=\sqrt{\pi} \Rightarrow C_2=\sqrt{\pi}$$

$$\int_{-\infty}^{\infty} e^{-\lambda^2\tau}\cos\left[\lambda(x-\xi)\right] d\lambda=\frac{1}{\sqrt{\tau}}I(\omega)=\sqrt{\frac{\pi}{\tau}}\, e^{-\frac{(x-\xi)^2}{4\tau}}$$

$$u(x,\tau)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\xi)\left[\int_{-\infty}^{\infty} e^{-\lambda^2\tau}\cos\left[\lambda(x-\xi)\right] d\lambda\right] d\xi=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4\tau}}\, d\xi$$

$$u(x,\tau)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x-2\omega\sqrt{\tau})\, e^{-\omega^2}\, d\omega$$

$$u(x,0)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-\omega^2}\, d\omega=f(x)\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-\omega^2}\, d\omega=f(x)$$

$$\varphi_\xi(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\, e^{-\frac{(x-\xi)^2}{4\tau}}$$ a megoldja a $\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2}$ egyenletet

$$\frac{\partial\varphi_\xi}{\partial\tau}=\left[-\frac{1}{4\tau\sqrt{\pi\tau}}+\frac{(x-\xi)^2}{8\tau^2\sqrt{\pi\tau}}\right]e^{-\frac{(x-\xi)^2}{4\tau}}$$

$$\frac{\partial^2\varphi_\xi}{\partial x^2}=\left[-\frac{1}{4\tau\sqrt{\pi\tau}}+\frac{(x-\xi)^2}{8\tau^2\sqrt{\pi\tau}}\right]e^{-\frac{(x-\xi)^2}{4\tau}}$$

$$u(x,t)=\frac{1}{2a\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\xi)\, e^{-\frac{(x-\xi)^2}{4a^2t}}\, d\xi$$

$$\varphi_\xi=\frac{1}{2a\sqrt{\pi t}}\, e^{-\frac{(x-\xi)^2}{4a^2t}}$$

pl: most legyen

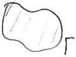
$$f(x) = \delta(x - x_0)$$

$$u_{x_0}(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_{-\infty}^{\infty} \delta(\xi - x_0)\, e^{-\frac{(x-\xi)^2}{4a^2 t}}\, d\xi = \frac{1}{2a\sqrt{\pi t}}\, e^{-\frac{(x-x_0)^2}{4a^2 t}}$$

Elliptikus egyenlet

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 = \Delta u \quad (2D\text{-ben})$$

$$u(x,y) = G(\Gamma)$$



pl: legyen ez egy téglalap

$u(x,0) = 0$

$u(x,L_2) = 0$

$u(0,y) = Ky/L_2$

$u(L_1,y) = Ky/L_2$

$z = x + iy \quad \bar{z} = x - iy$

$$2\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$$

$$2\frac{\partial}{\partial z} = \frac{\partial}{\partial x} - i\frac{\partial}{\partial y}$$

$$4\frac{\partial^2}{\partial \bar{z}\,\partial z} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = 0 \qquad u = f_1(z) + f_2(\bar{z})$$

$$u = f_1(z) + \overline{f_1(z)}$$

$$f_1(z) = \varphi(x,y) + i\psi(x,y)$$

$$\frac{\partial}{\partial \bar{z}} f_1(z) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\varphi(x,y) + i\psi(x,y)\right) =$$

$$= \frac{\partial \varphi}{\partial x} + i\frac{\partial \varphi}{\partial y} + i\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial y} = 0$$

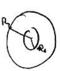
$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\partial^2 \psi}{\partial x \partial y} = -\frac{\partial^2 \varphi}{\partial y^2}$$

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$f_1(z) = -i\frac{K}{L_2} z = i\frac{K}{L_2}(x + iy) = \frac{K}{L_2} y - i\frac{K}{L_2} x$$

$$u(x,y) = \frac{K}{L_2} y$$



$u(r_1) = 0$

$u(r_2) = B$

$$f_1(z) = C \ln\left(\frac{z}{R_1}\right) \quad C \in \mathbb{R}$$

$$z = r e^{i\varphi}$$

$$f_1(z) = C \cdot (\ln z - \ln R_1) = C\left(\ln(re^{i\varphi}) - \ln R_1\right) = C(\ln r + i\varphi - \ln R_1)$$

$$u(r,\varphi) = C \ln \frac{r}{R_1}$$

$$u(R_2) = C \ln \frac{R_2}{R_1} = B \qquad C = \frac{B}{\ln R_2/R_1} \qquad u(r,\varphi) = B\frac{\ln(r/R_1)}{\ln(R_2/R_1)} \qquad \left.\frac{\partial u}{\partial r}\right|_{r=R} = 0 \qquad \left.\frac{\partial u}{\partial x}\right|_{x,y=a} = 0$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$f(z) = v_0\left(z + \frac{R^2}{z}\right) = v_0\left(x + iy + \frac{R^2}{x+iy}\right) = v_0\left(x + iy + \frac{R^2(x-iy)}{x^2+y^2}\right)$$

$$u(x,y) = v_0 x\left(1 + \frac{R^2}{x^2+y^2}\right) = v_0 r\cos\varphi\left(1 + \frac{R^2}{r^2}\right)$$

$$\left.\frac{\partial u}{\partial r}\right|_R = v_0\cos\varphi - v_0\cos\varphi\left.\frac{R^2}{r^2}\right|_R = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{x,y\to\infty} = v_0$$

---

11. 24

3 változós eset

$\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0$ elliptikus

peremérték-megadás

- Megadunk egy 3D-s felületet, melyen megadjuk a fv értékét : $\varphi(\Gamma)$
- megadjuk a felületre merőleges egységvektort $\left.\frac{\partial \varphi}{\partial n}\right|_\Gamma$

$\varphi_{tt} - \varphi_{xx} - \varphi_{yy}$ hiperbolikus

$\varphi(x,y,0)$ vagy $\left.\frac{\partial \varphi}{\partial t}\right|_{x,y,0}$

$\varphi_t - \varphi_{xx} - \varphi_{yy}$ parabolikus

$\varphi(x,y,0)$

Mégtöbb változós eset

$$\sum_i d_i \frac{\partial^2 \varphi}{\partial y_i^2} + \sum_k \alpha_k \frac{\partial \varphi}{\partial y_k} = 0$$

$d_i \in \{-1, 0, 1\}$

ha $d_1 = -1$, $d_{i+1} = 1$ hiperbolikus

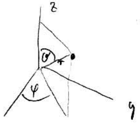
$d_i = 1$ elliptikus

$d_1 = 0$ $d_{i+1} = 1$ parabolikus

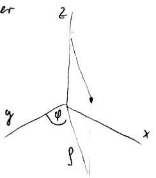
$d_i = -1$ $i = 1 \ldots k$ $d_j = 1$ $j = k+1 \ldots$ ultraparabolikus

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

polár



henger



gömbi: $x = r \sin\vartheta \cos\varphi$

$y = r \sin\vartheta \sin\varphi$

$z = r\cos\vartheta$

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial u}{\partial \vartheta}\right) + \frac{1}{r^2 \sin\vartheta}\frac{\partial^2 u}{\partial \varphi^2} = 0$$

henger $x = \rho\cos\varphi$

$y = \rho \sin\varphi$ itt $\Delta = \dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \varphi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$

$z = z$

$u(r, \vartheta, \varphi)$

ez van megadva kezdeti értéknek

$r = 1$

$$u(1, \vartheta, \varphi) = f(\vartheta) \to u(r, \vartheta) \Rightarrow \frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\vartheta}\frac{\partial}{\partial \vartheta}\left(\sin\vartheta \frac{\partial u}{\partial \vartheta}\right) = 0$$

keresem a megoldást $u(r, \vartheta) = R(r)\,\Theta(\vartheta)$ alakban, ekkor

$$\Theta \frac{d}{dr}\left(r^2 \frac{dR}{dr}\right) + \frac{R}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta \frac{d\Theta}{d\vartheta}\right) = 0 \qquad / \, R \cdot \Theta$$

$$\underbrace{\frac{1}{R}\left(r^2 R'' + 2rR'\right)}_{r\text{-től}} = \underbrace{-\frac{1}{\Theta}\left(\Theta'' + \operatorname{ctg}\vartheta\, \Theta'\right)}_{\vartheta\text{-tól függ}} = \gamma(\gamma+1)$$

$$r^2 R'' + 2rR' - \gamma(\gamma+1) R = 0 \qquad R_1 = r^\gamma \quad R_2 = r^{-(\gamma+1)} \quad -\gamma \neq 1/2$$

$$\Theta'' + \operatorname{ctg}\vartheta\,\Theta' + \gamma(\gamma+1)\Theta = 0$$

$x = \cos\vartheta$

$$\frac{d\Theta}{d\vartheta} = -\sin\vartheta \frac{d\Theta}{dx} \qquad \operatorname{ctg}\frac{d\Theta}{d\vartheta} = -x\frac{d\Theta}{dx}$$

$$\frac{d^2\Theta}{d\vartheta^2} = (1-x^2)\frac{d^2\Theta}{dx^2} - x\frac{d\Theta}{dx} \qquad \Theta(\vartheta) = w(\cos\vartheta) = w(x)$$

$-1 \le x \le 1$

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \gamma(\gamma+1)\Theta = 0$$

$$(1-x^2)\frac{d^2 w}{dx^2} - 2x\frac{dw}{dx} + \gamma(\gamma+1) w = 0$$

$$\frac{d}{dx}\left[(1-x^2)\frac{dw}{dx}\right] + \gamma(\gamma+1) = 0 \qquad \gamma = n \text{ egész}$$

$P_n(t) \sim \frac{d^n}{dt^n}(t^2-1)^n$ Legendre-polinomok

$P_n(t) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

$\int_{-1}^{1} P_n(t) P_m(t)\,dt = 0 \quad n \neq m$

$\int_{-1}^{1} P_n(t)^2\,dt = \frac{2}{2n+1}$

$P_0 = 1$

$P_1 = x$

$P_2 = \frac{3}{2}x^2 - \frac{1}{2}$

$u_n(r,\vartheta) = r^n P_n(\cos\vartheta)$

$u(r,\vartheta) = \sum_{n=0}^{\infty} a_n r^n P_n(\cos\vartheta)$

$u(1,\vartheta) = f(\vartheta) = \sum_{n=0}^{\infty} a_n P_n(\cos\vartheta)$

$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \quad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(t) P_n(t)\,dx$

Rekurziós formulák Legendre-polinomokra

- $(n+1) P_{n+1}(x) = (2n+1)x \cdot P_n(x) - n P_{n-1}(x)$
- $(1-x^2) P_n'(x) = -n x P_n(x) + n P_{n-1}(x)$

generátorfüggvény $g(z;x) = \sum_{n=0}^{\infty} f_n(x) z^n$

Mi a Legendre-polinomok generátorfüggvénye? $(f_n \to P_n)$

$P_0(x) = 1$

$\sum_{n=0}^{\infty}(n+1)P_{n+1}(x) z^n = \sum_{n=0}^{\infty}\frac{d}{dz}\left[P_{n+1}(x) z^{n+1}\right] = \frac{d}{dz}\left[\sum_{n=0}^{\infty} P_n(x) z^n - P_0(x)\right] = \frac{dy}{dz} \quad y(z,x)$

$\sum_{n=0}^{\infty}(2n+1) x P_n(x) z^n = 2xz\frac{dy}{dz} + xy$ (honnan?)

$\sum_{n=0}^{\infty} n P_{n-1}(x) z^n = z^2\frac{dy}{dz} + yz$

$(1-2xz+z^2)\frac{dy}{dz} + (z-x)y = 0$

$y = (1-2zx+z^2)^{-1/2} \Rightarrow P_n(x) = \frac{1}{n!}\frac{d^n}{dz^n}(1-2zx+z^2)^{-1/2}$

Hol jön be a Legendre-polinom?

$$\frac{1}{|x-y|} = \frac{1}{\sqrt{|x|^2+|y|^2-2|x||y|\cos\vartheta}} = \frac{1}{|y|}\frac{1}{\sqrt{1+\frac{|x|^2}{|y|^2}-2\frac{|x|}{|y|}\cos\vartheta}} = \frac{1}{|y|}\sum_{n=0}^{\infty} P_n(\cos\vartheta)\left(\frac{|x|}{|y|}\right)^n$$

Csebisev polinomok

$$(1-x^2)\frac{d^2T_n}{dx^2} - x\frac{dT_n}{dx} + n^2 T_n = 0 \qquad -1\le x\le 1 \text{ ért. SL probléma}$$

$$\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dT_n}{dx}\right) + \frac{n^2}{\sqrt{1-x^2}}T_n = 0$$

$$T_n(\cos\vartheta) = \cos(n\vartheta)$$

$$T_n(x) = \frac{(-1)^n\sqrt{\pi}}{2^{n+1}\Gamma(n+1/2)}\sqrt{1-x^2}\frac{d^n}{dx^n}\left[(1-x^2)^{n-1/2}\right]$$

$$\int_{-1}^{1}\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \frac{\pi}{2}\delta_{mn} \qquad \int_{-1}^{1}\frac{|f|^2}{\sqrt{1-x^2}}dx < \infty \qquad \to \quad T_{n+1} = 2xT_n - T_{n-1}$$

$$\int_{-1}^{1}\frac{T_n(x)T_0(x)}{\sqrt{1-x^2}}dx = \pi\delta_{n0} \qquad f(x) = \sum_{n=0}^{\infty} a_n T_n(x) \qquad (1-x)^2\frac{dT_n}{dx} = -nxT_n + nT_{n-1}$$

generátorfüggvény $g(z,x) = \dfrac{1-xz}{1-2xz+z^2}$

Jacobi polinomok mint egy SL probléma megoldásai

$$P_n^{(\alpha,\beta)} \qquad \alpha > -1 \qquad \beta > -1 \qquad -1\le x\le 1$$

$$\frac{d}{dx}\left((1-x)^{\alpha+1}(1+x)^{\beta+1}\frac{dP_n^{(\alpha,\beta)}}{dx}\right) + n(n+\alpha+\beta+1)(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)} = 0$$

$$P_n^{(\alpha,\beta)} = \frac{(-1)^n}{2^n n!}\frac{1}{(1-x)^\alpha(1+x)^\beta}\frac{d^n}{dx^n}\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right]$$

$$\int_{-1}^{1}(1-x)^\alpha(1+x)^\beta P_m^{(\alpha,\beta)}(x)P_n^{(\alpha,\beta)}(x)dx = C(n,\alpha,\beta)\,\delta_{n,m}$$

Laguerre polinomok $L_n^{(\alpha)}(x)$ $\quad 0 \le x \le \infty$

$$\frac{d}{dx}\left(x^{\alpha+1} e^{-x} \frac{dL_n^{(\alpha)}}{dx}\right) + n x^{\alpha} e^{-x} L_n^{(\alpha)} = 0$$

$$L_n^{(\alpha)} = \frac{e^{x} x^{-\alpha}}{n!} \frac{d^n}{dx^n}\left(x^{n+\alpha} e^{-x}\right)$$

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}$$

Hermite polinomok

$-\infty \le x \le \infty \quad H_n(x)$

$$\frac{d^2 H_n}{dx^2} - 2x \frac{dH_n}{dx} + 2n H_n = 0$$

$$\frac{d}{dx}\left(e^{-x^2} \frac{dH_n}{dx}\right) + 2n e^{-x^2} H_n = 0$$

$$u_n = e^{-\frac{x^2}{2}} H_n \qquad \boxed{\frac{d^2 u_n}{dx^2} + (2n+1-x^2) u_n = 0}$$

Rodriges-formulája $\quad H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left(e^{-x^2}\right)$

$$H_{n+1} = 2x H_n - 2n H_{n-1} \qquad \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n n!\, 2\pi\, \delta_{nm}$$

$$\frac{dH_n}{dx} = 2n H_{n-1}$$

$$e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

12. 01.

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt \qquad Re(z)$$

$$\int_0^\infty e^{-t} t^{z-1}\, dt = -e^{-t} t^{z-1}\Big|_0^\infty + \int_0^\infty e^{-t} \frac{t^z}{z}\, dt = \frac{1}{z}\int_0^\infty e^{-t} t^z\, dt \quad \text{tehát} \quad \Gamma(z) = \frac{1}{z}\Gamma(z+1) \ \text{ill} \ \Gamma(z+1) = z \cdot \Gamma(z)$$

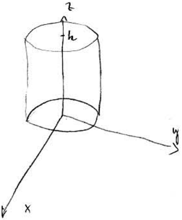
$$\Gamma(z+n) = z(z+1)(z+2)\dots(z+(n-1))\,\Gamma(z)$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

$$\Gamma(n+1) = n! \quad n \in \mathbb{N}$$

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left[\left(1+\frac{z}{n}\right) e^{-\frac{z}{n}} \right] \qquad \lim_{n\to\infty} 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n} - \ln(n) = \gamma \approx 0,5$$

Laplace-egyenlet hengerkoordinátákban



$x = \rho \cos\varphi$

$y = \rho \sin\varphi$

$z = z$

$$\Delta u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$u(\rho, h, \varphi) = 0 \qquad u(1, z, \varphi) = 0$

$u(\rho, 0, \varphi) = f(\rho) \qquad \Rightarrow \quad \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial \varphi^2} = 0$

$u(1, z) = 0 \quad u(\rho, h) = 0 \quad u(\rho, 0) = f(\rho)$

Keressük a megoldást $u(\rho, z) = R(\rho) Z(z) \Rightarrow \left(\frac{d^2 R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\varphi}\right) Z + R \frac{d^2 Z}{dz^2} = 0$

$$\frac{R'' + \frac{1}{\rho} R'}{R} = -\frac{Z''}{Z} =: -\lambda^2$$

ha $R = Y(\lambda \rho)$ $\quad \leftarrow \quad R'' + \frac{1}{\rho} R' + \lambda^2 R = 0$

$\lambda^2 Y'' + \frac{\lambda}{\rho} Y' + \lambda^2 Y = 0$ $\qquad Z'' = \lambda^2 Z$

$\lambda \cdot \rho = t$ $\qquad \Downarrow$ határfeltételek

$Y'' + \frac{1}{t} Y' + Y = 0$ $\qquad Z = \operatorname{sh}\left[\lambda (h - z)\right]$

$$Y(t) = \sum_{n=0}^{\infty} c_n t^n \qquad \sum_{n=2}^{\infty} n(n-1) c_n t^{n-2} + \sum_{n=1}^{\infty} n c_n t^{n-2} + \sum_{n=0}^{\infty} c_n t^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} t + c t^{-1} + \sum_{n=0}^{\infty} (n+2) c_n t^n + \sum_{n=0}^{\infty} c_n t^n = 0 \qquad c_1 \stackrel{!}{=} 0$$

$(n+2)(n+1) c_{n+2} + (n+2) c_{n+2} + c_n = 0$

$(n+2)^2 c_{n+2} = -c_n \Rightarrow c_{n+2} = -\frac{c_n}{(n+2)^2} \Rightarrow c_3 = c_5 = \dots = c_{2n+1} = 0$

$c_2 = -\frac{c_0}{2}$

$c_4 = -\frac{c_2}{4!} = \frac{c_0}{2^2 \cdot 4^2}$

$c_{2k} = (-1)^k \frac{c_0}{2^2 \cdot 4^2 \cdots (2k)^2} =$

$= \frac{(-1)^k c_0}{2^{2k} (k!)^2}$

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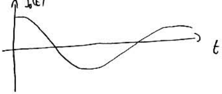
$Y(t)=C_0\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}(k!)^2}t^{2k}$ Bessel függvény



$Y(t)=: J_0$ nullindexű Bessel függvény

$R(\rho)=J_0(\lambda\rho)$

Peremfeltétel: $\rho=1$-re kielégíti $J_0(\lambda_n)=0$

$R(\rho)''+\frac{1}{\rho}R'+\lambda^2R=0 \qquad R(1)=0$

$\lambda_n \qquad J_0(\lambda_n)=0 \qquad \int_0^1 d\rho\cdot\rho\, J_0(\lambda_n\rho)\, J_0(\lambda_m\rho)=0 \quad m\neq n$

$$\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right)+\lambda^2\rho R=0$$

$$\int_0^1 d\rho\cdot\rho\cdot J_0(\lambda_n\rho)^2=N_n$$

$u_n=\operatorname{sh}[\lambda_n(h-z)]\,J_0(\lambda_n\rho)$

$u(\rho,z)=\sum_{n=1}^{\infty}a_n\operatorname{sh}[\lambda_n(h-z)]\,J_0(\lambda_n\rho) \qquad u(\rho,0)\stackrel{!}{=}\sum_{n=1}^{\infty}\underbrace{a_n\operatorname{sh}[\lambda_n\cdot h]}_{b_n}J_0(\lambda_n\rho)=f(\rho)$

$f(\rho)=\sum_{n=1}^{\infty}b_nJ_0(\lambda_n\rho)$

$b_n=\int_0^1 d\rho\cdot\rho\, f(\rho)\,J_0(\lambda_n\rho)\frac{1}{N_n}$

Bessel-egyenlet

$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-\nu^2)y=0 \iff \frac{d}{dx}\left(x\frac{dy}{dx}\right)+\left(x-\frac{\nu^2}{x}\right)y=0$

Megoldások $J_\nu(x)$ ha a megoldásra $\overset{\text{Taylor}}{T_1}(y)=c\cdot x^\lambda$

$\lambda^2x^{\lambda-1}+x^{\lambda+1}-\nu^2x^{\lambda-1}=0$

$y=x^\nu Y(x)$ $\Leftarrow$ $\lambda^2=\nu^2$

ez hatványsor

$\lambda=\pm\nu$

$x\frac{d^2Y}{dx^2}+(2\nu+1)\frac{dY}{dx}+xY=0$

$Y=\sum_{n=0}^{\infty}c_nx^n$

$\sum_{n=2}^{\infty}n(n-1)c_nx^{n-1}+(2\nu+1)\sum_{n=1}^{\infty}nc_nx^{n-1}+\sum_{n=0}^{\infty}c_nx^{n+1}=0$

$J_\nu(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k}$

$r(z) = z$ Wronsky

$$\{f(z), g(z)\} = \frac{c}{z}$$

$$\{J_\nu(z), J_{-\nu}(z)\} = -\frac{2\sin(\nu\pi)}{\pi z}$$ ha $\nu \in \mathbb{Z}$ $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$ vagyis lin. független ha $\nu \notin \mathbb{Z}$.

$$Y_\nu := \frac{J_\nu \cdot \cos\nu\pi - J_{-\nu}}{\sin\nu\pi}$$ másodfajú Bessel

$$Y_n = \lim_{\nu \to n} Y_n$$

$$\{J_\nu, Y_\nu\} = \frac{2}{\pi z}$$

$H^{(1)}_\nu = J_\nu + i Y_\nu$ Hankel fv

$$H^{(2)}_\nu = J_\nu - i Y_\nu$$

$C \in \{J, Y, H^{(1)}, H^{(2)}\}$

$$C_{\nu-1}(z) + C_{\nu+1}(z) = \frac{2\nu}{z} C_\nu(z)$$

$$C_{\nu-1}(z) - C_{\nu+1}(z) = 2 C'_\nu(z)$$