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# Kvantummechanika gyakorlat
## 1. gyak

## Fotoelektromos effektus

1 foton adja át az energiáját 1 elektronnak

energiamegmaradás: $E_f = h \cdot \nu = W_k + \frac{1}{2} m v_{max}^2$

**1, Feladat** Káliumot világítanak meg 300 nm-es fénnyel, melynek hatására maximum 2.03 eV kinetikus energiájú elektronok lépnek ki a felületből

a,) Mekkora a beeső fotonok energiája? (eV-ben)

b,) Mekkora káliumra a kilépési munka?

c,) Mekkorának kell lennie ~~maximálisan~~ a fény hullámhosszának, hogy létrejöjjön a fotoelektromos effektus?

a,)

$$\left.\begin{array}{l} E_f = h \cdot \nu \\ \lambda \cdot \nu = c \Rightarrow \nu = \frac{c}{\lambda} \end{array}\right\} \Rightarrow E_f = h \cdot \frac{c}{\lambda} = 6.626 \cdot 10^{-34}\, J \cdot s \cdot \frac{2.998 \cdot 10^{8}\, \frac{m}{s}}{300 \cdot 10^{-9}\, m}$$

$$= 6.622 \cdot 10^{-19}\, J = \underline{\underline{4.133\, eV}}$$

$\| \; 1\, eV = 1.602 \cdot 10^{-19}\, J$

b,)

$$E_f = W_k + \frac{1}{2} m v_{max}^2 = W_k + E_{kin,max}$$

$$\Downarrow$$

$$W_k = E_f - E_{max}^{kin} = 4.133\, eV - 2.03\, eV = \underline{\underline{2.103\, eV}}$$

c,)

$$E_f = h \cdot \frac{c}{\lambda_{max}} = h\, \nu_{min} = E_{f\,min} \Rightarrow v_{max} = \emptyset$$

$$\Downarrow$$

$$h \cdot \frac{c}{\lambda_{max}} = W_k \Rightarrow \lambda_{max} = \frac{h \cdot c}{W_k}$$

$$\lambda_{max} = \frac{6.626 \cdot 10^{-34}\, J \cdot s \cdot 2.998 \cdot 10^{8}\, \frac{m}{s}}{2.103\, eV} = \frac{4.136 \cdot 10^{-15}\, eV \cdot 2.998 \cdot 10^{8}\, m}{2.103\, eV}$$

$\| \; h = 4.136 \cdot 10^{-15}\, eV \cdot s$

$$= \underline{\underline{589.62\, nm}}$$

# De Broglie hullámhossz

$\lambda = \frac{h}{p}$ minden anyagra $\quad p = \frac{E}{c}$ fotonra

$p = m v$ tömeges objektumra

tömeges anyag: $\nu = \frac{E}{h}$

**2. Feladat** Mekkora feszültséget kell alkalmazni elektronok gyorsítására, hogy azok de Broglie hullámhossza 5 nm, ill. 0.01 nm legyen.

$$\lambda = \frac{h}{p} = \frac{h}{m \cdot v} \Rightarrow v = \frac{h}{m \cdot \lambda}$$

energiamegmaradás: $\frac{1}{2} m v^2 = e \cdot U$

$$\Downarrow$$

$$U = \frac{1}{2e} m \cdot \left(\frac{h}{m \cdot \lambda}\right)^2 = \frac{h^2}{2e \cdot m} \cdot \frac{1}{\lambda^2}$$

ha $\lambda_1 = 5\ \text{nm}$: $$U_1 = \frac{(6.626 \cdot 10^{-34})^2\ J^2 s^2}{2 \cdot 1.602 \cdot 10^{-19}\ C \cdot 9.109 \cdot 10^{-31}\ kg} \cdot \frac{1}{(5 \cdot 10^{-9})^2\ m^2} =$$

$$\approx \underline{\underline{0.06\ V}}$$

ha $\lambda_2 = 0.01\ \text{nm}$: $$U_2 / U_1 = \left(\frac{1}{\lambda_2^2}\right) / \left(\frac{1}{\lambda_1^2}\right) = \left(\frac{\lambda_1}{\lambda_2}\right)^2$$

$$U_2 = \left(\frac{\lambda_1}{\lambda_2}\right)^2 U_1 = \left(\frac{5\ \text{nm}}{0.01\ \text{nm}}\right)^2 \cdot 0.06\ V = \underline{\underline{15\ kV}}$$

**3. Feladat** Mekkora az elektronmikroszkóp felbontóképessége?

$\overset{d_{min}}{\longleftrightarrow}$ $\quad d_{min} \approx \lambda_{min}$

$c$ határsebesség $\Rightarrow v < c \Rightarrow \frac{1}{c} < \frac{1}{v} \Rightarrow \frac{h}{m c} < \frac{h}{m v}$

$$\Rightarrow \lambda_{min} = \frac{h}{m c} \Rightarrow d_{min} = \frac{h}{m c}$$

$$d_{min} = \frac{6.626 \cdot 10^{-34}\ J s}{9.109 \cdot 10^{-31}\ kg \cdot 2.998 \cdot 10^{8}\ \frac{m}{s}} = \underline{\underline{2.426 \cdot 10^{-12}\ m}}$$

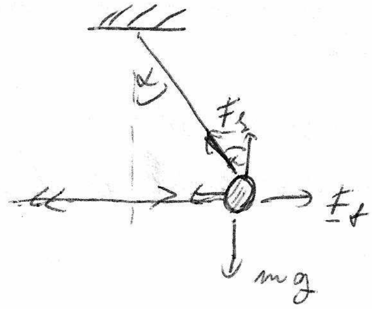
(atom mérete $\approx 10^{-10}$ m
atommag $\approx 10^{-15}$ m)

$\Rightarrow$ az atomok láthatóak.

## Fény-nyomás (fénynyomás)

**4. Feladat** Egy elhanyagolható tömegű szálon egy $m = 5\ \mu g$ tömegű piciny fémgömb függ. A gömböt vízszintesen 400 nm-es monokromatikus elhanyagolható keresztmetszetű fény-nyalábbal világítjuk meg. A fémgömb teljesen visszaveri a fényt és másodpercenként $10^{18}$ foton érkezik a gömbhöz. A függőlegessel mekkora szöget zár be az inga egyensúlyban? 2007. XI. 12.

$$n = 10^{18}\,\frac{1}{s}\ ;\quad m = 5\,\mu g\ ;\quad \lambda = 400\,nm$$




1 foton impulzusmegváltozása a visszaverődés során:

$$\Delta p = p_v - p_e = \frac{h}{\lambda} - \left(-\frac{h}{\lambda}\right) = 2\frac{h}{\lambda}$$

$\Delta\tau$ idő alatti imp. megvált.:

$$\Delta P = n \cdot \Delta\tau \cdot \Delta p$$

$$\Downarrow$$

$$F_f = \frac{\Delta P}{\Delta \tau} = n \cdot \Delta p = n \cdot 2\frac{h}{\lambda}$$

egyensúly: $\sum_i \underline{F} = 0 \Rightarrow$

$$\left.\begin{aligned} F_k \cdot \cos\alpha &= mg \\ F_k \cdot \sin\alpha &= F_f \end{aligned}\right\} \Rightarrow \operatorname{tg}\alpha = \frac{F_f}{mg}$$

$$\operatorname{tg}\alpha = \frac{2n \cdot h}{m g \lambda} = \frac{2 \cdot 10^{18}\frac{1}{s} \cdot 6.626 \cdot 10^{-34}\,kg\cdot\frac{m^2}{s^2}\cdot s}{5 \cdot 10^{-9}\,kg \cdot 9.81\frac{m}{s^2} \cdot 400 \cdot 10^{-9}\,m} =$$

$$= 0.068$$

$$\Rightarrow \underline{\underline{\alpha \approx 3.86^\circ}}$$

## Bohr-modell

Egy elektronnal rendelkező atomokra:

$$E = E_{kin} + E_{pot} = \frac{1}{2}mv^2 - \frac{kZe^2}{r} = -\frac{1}{2}\frac{kZe^2}{r}$$

$$\Big\|\ \frac{kZe^2}{r^2} = m\frac{v^2}{r}$$

imp. mom. kvantálás: $\left[ m v r = n\hbar \right]$ $\quad \hbar = 1.05 \cdot 10^{-34}\,J\cdot s$

$$\Downarrow$$

$\left[ r = n^2 \frac{a_0}{Z} \right]$ $\quad a_0 = \frac{\hbar^2}{m k e^2} = 0.0529\,nm$ Bohr-sugár

$\left[ E_n = -Z^2 \frac{E_0}{n^2} \right]$ $\quad E_0 = \frac{k^2 e^4 m}{2\hbar^2} \approx 13.6\,eV$

5.) Feladat

Egy hidrogén atom a 10. gerjesztett állapotában van.

a.) Mekkora az ehhez az állapothoz tartozó Bohr pálya sugara

b.) Mekkora az elektron impulzusmomentuma, kinetikus és potenciális energiája?

c.) Egy ilyen (teljes) energiájú elektronnak mekkora a de Broglie hullámhossza [redacted], és frekvenciája?

$Z = 1 \qquad n = 11$

a.) $$r = n^2 \frac{a_0}{Z} = 11^2 \cdot \frac{0.0529\,nm}{1} = \underline{\underline{6.4\,nm}}$$

b.) $$\underline{\underline{I}} = m v r = n \cdot \hbar = 11 \cdot 1.05 \cdot 10^{-34}\,Js = 1.155 \cdot 10^{-33}\,Js =$$
$$= \underline{\underline{7.2 \cdot 10^{-15}\,eVs}}$$

$$\underline{\underline{E_{kin}}} = \frac{1}{2} m v^2 = \frac{1}{2} \cancel{m} \cdot \frac{n^2 \hbar^2}{m^{\cancel{2}} r^2} = \frac{1}{2} \frac{n^2 \hbar^2}{m\, n^4 a_0^2} = \frac{1}{2} \frac{\hbar^2}{m a_0^2} \frac{1}{n^2} =$$

$$\Big\| \; v = \frac{n \hbar}{m r}$$

$$= \frac{1}{2} \frac{(1.05 \cdot 10^{-34}\,Js)^2}{9.109 \cdot 10^{-31}\,kg \cdot (0.0529 \cdot 10^{-9}\,m)^2} \cdot \frac{1}{11^2} = \qquad \Big\| \; \frac{J \cdot \cancel{kg} \frac{m^2}{\cancel{s^2}} \cdot \cancel{s^2}}{\cancel{kg}\, \cancel{m^2}} = J$$

$$= 1.787 \cdot 10^{-20}\,J \approx \underline{\underline{0.11\,eV}}$$

$$E = E_{kin} + E_{pot} = -\frac{E_0}{n^2} \Rightarrow E_{pot} = -\frac{E_0}{n^2} - E_{kin} = -\frac{13.6\,eV}{11^2} - 0.11\,eV$$

$$\underline{\underline{E_{pot} = -0.22\,eV}}$$

c.) $$|E| = h \cdot \nu_d \Rightarrow \nu_d = \frac{|E|}{h} = \frac{|E_{kin} + E_{pot}|}{h} = \frac{0.11\,\cancel{eV}}{4.136 \cdot 10^{-15}\,\cancel{eV}\,s} \Rightarrow$$

$$\Rightarrow \underline{\underline{\nu_d = 2.66 \cdot 10^{13}\,Hz}}$$

$$\lambda = \frac{h}{p} = \frac{h}{mv} = \frac{h}{\frac{I}{r}} = \frac{h \cdot r}{n \hbar} = \frac{h\, n^{\cancel{2}} a_0}{\cancel{n}\, \hbar} = \frac{\cancel{h}\, n\, a_0}{\cancel{h}} 2\pi = a_0 2\pi n$$

$$\underline{\lambda} = 0.0529\,nm \cdot 2\pi \cdot 11 = \underline{\underline{3.656\,nm}}$$

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# Kvantummech. gyak.
## 2.

**1. Feladat** Egy hullámcsomaggal leírható elektron $v = \frac{c}{3}$ sebességgel halad az x tengelyen. Mekkora a valószínűsége, hogy az elektront $t_1 = 1\,ns$ múlva a $(0, 1)$ intervallumban találjuk, ha a részecske Gauss-eloszlás szerint lokalizált? (A Gauss-eloszlás szórása: $\sigma = 10^{-1}$, eh.: $A := 1\sqrt{m}$, továbbá $\frac{d^2\omega}{dk^2}\Big|_{k_0} = \frac{\hbar}{m}$ és tegyük fel, hogy $\frac{d^3\omega}{dk^3}\Big|_{k_0}$ már elhanyagolható)

$$\Psi(x,t) = \int_{-\infty}^{\infty} A(k)\cdot e^{+i(kx - \omega(k)t)}\, dk =$$

$$A(k) = A\cdot e^{-\frac{(k-k_0)^2}{2\sigma^2}} \qquad [A] = \sqrt{m}$$

$$= A\cdot\int_{-\infty}^{\infty} e^{-\frac{(k-k_0)^2}{2\sigma^2}}\, e^{i(kx - \omega(k)t)}\, dk =$$

sorfejtés: $\omega(k) = \omega(k_0) + \overbrace{\frac{d\omega}{dk}\Big|_{k_0}}^{v}(k-k_0) + \frac{1}{2}\frac{d^2\omega}{dk^2}\Big|_{k_0}(k-k_0)^2 =$

$$z := k - k_0 \Rightarrow k = z + k_0$$

$$= \omega_0 + v\cdot z + \frac{1}{2} w\cdot z^2$$

$$= A\cdot\int_{-\infty}^{\infty} e^{-\frac{z^2}{2\sigma^2}}\, e^{i[(k_0+z)x - (\omega_0 + v\cdot z + \frac{1}{2} w z^2)t]}\, dz =$$

$k \to z \quad dk = dz$

$$= A\cdot e^{i(k_0 x - \omega_0 t)} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{z^2}{2\sigma^2}}\, e^{izx - ivtz - i\frac{1}{2}wtz^2}}\, dz = *$$

$$e^{-\frac{z^2}{2\sigma^2} - \frac{i}{2}wtz^2 + izx - ivtz} = e^{-z^2\underbrace{\left(\frac{1}{2\sigma^2} + \frac{iw}{2}t\right)}_{a^2} + iz(x-vt)}$$

a kitevő: $-z^2a^2 + iz(x-vt) = -a^2\left(z^2 - iz\frac{x-vt}{a^2}\right) =$

$$= -a^2\left[\left(z - \frac{i(x-vt)}{2a^2}\right)^2 - \left(\frac{i(x-vt)}{2a^2}\right)^2\right] =$$

$$= -a^2\left(z - \frac{i(x-vt)}{2a^2}\right)^2 + a^2\left(\frac{i(x-vt)}{2a^2}\right)^2 =$$

$$= -a^2\left(z - i\,\frac{x-vt}{2a^2}\right)^2 - \underbrace{\left(\frac{x-vt}{2a}\right)}_{b}^2 = -a^2\left(z - i\,\frac{b}{a}\right)^2 - ($$

$$\Downarrow$$

$$* = A\cdot e^{i(k_0 x - \omega_0 t)} \int_{-\infty}^{\infty} e^{-a^2\left(z - i\frac{b}{a}\right)^2 - b^2}\, dz =$$

$$= A\cdot e^{i(k_0 x - \omega_0 t)}\, e^{-b^2} \int_{-\infty}^{\infty} e^{-a^2\left(z - i\frac{b}{a}\right)^2}\, dz =$$

$$\tilde z := z - i\,\frac{b}{a}$$

$$\Downarrow$$

$$d\tilde z = dz$$

$$= A\cdot e^{i(k_0 x - \omega_0 t)}\, e^{-b^2} \int_{-\infty}^{\infty} e^{-a^2 \tilde z^2}\, d\tilde z = e^{i(k_0 x - \omega_0 t)}\, e^{-b^2}\, \frac{1}{a} \underbrace{\int_{-\infty} e}_{\sqrt{\pi}}$$

$$\xi := a\tilde z \Rightarrow d\xi = a\, d\tilde z$$

est e

ha ξ

$$= A\cdot e^{i(k_0 x - \omega_0 t)}\, e^{-b^2}\cdot \frac{\sqrt{\pi}}{a}$$

$$\Downarrow$$

$$|\psi(x,t)|^2 = A^2\left|e^{i(k_0 x - \omega_0 t)}\, e^{-b^2}\, \frac{\sqrt{\pi}}{a}\right|^2 = \frac{A^2\pi}{|a|^2}\cdot \underbrace{\left|e^{-b^2}\right|^2}_{\shortparallel} =$$

$$= \frac{A^2\pi}{|a|^2}\, e^{-b^2 - (b^*)^2}$$

$$e^{-b^2}\cdot\left(e^{-b^2}\right)^* = e^{-b^2 -}$$

$$b^2 + (b^*)^2 = \frac{(x-vt)^2}{4}\left(\frac{1}{a^2} + \frac{1}{a^{*2}}\right) = \frac{(x-vt)^2}{4}\,\frac{a^2 + a^{*2}}{|a|^4}$$

$$a^2 + a^{*2} = \frac{1}{2\sigma^2} + \frac{iw}{\cancel{2}}t + \frac{1}{2\sigma^2} - \frac{iw}{\cancel{2}}t = \frac{1}{\sigma^2}$$

$$|a|^2 = aa^* = \sqrt{\left(\frac{1}{2\sigma^2} + \frac{iw}{2}t\right)\left(\frac{1}{2\sigma^2} - \frac{iw}{2}t\right)} =$$

$$= \sqrt{\frac{1}{4\sigma^4} + \frac{w^2 t^2}{4}} = \frac{1}{2\sigma^2}\sqrt{1 + w^2 t^2 \sigma^4}$$

$$\Downarrow$$

$$b^2 + b^{*2} = \frac{(x-vt)^2}{\cancel{4}}\cdot\frac{\frac{1}{\sigma^2}}{\frac{1}{\cancel{4\sigma^4}_{\sigma^2}}\left(1 + w^2 t^2 \sigma^4\right)} = \frac{(x-vt)^2}{1 + w^2 t^2 \sigma^4}\cdot \sigma^2$$

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$$|\Psi(x,t)|^2 = \frac{2\pi\cdot\sigma^2\cdot A^2}{\sqrt{1+w^2t^2\sigma^4}}\, e^{-\sigma^2\frac{(x-vt)^2}{1+w^2t^2\sigma^4}}$$

haranggörbe: maximuma $v$ sebességgel halad, szélessége idővel nő

$x_1 := 1\,m$

a megtalálás valószínűsége a $(0, x_1)$ intervallumban a $t_1$ időpillanatban:

$$P([0,x_1]) = \int_0^{x_1} |\Psi(x,t_1)|^2 dx = \frac{2\pi\sigma^2\cdot A^2}{\sqrt{1+w^2t_1^2\sigma^4}} \int_0^{x_1} e^{-\sigma^2\frac{(x-vt_1)^2}{1+w^2t_1^2\sigma^4}} dx =$$

$$= \frac{2\pi\sigma A^2}{\alpha} \int_0^{x_1} e^{-\left(\frac{x-vt_1}{\alpha}\right)^2} dx =$$

$$\alpha := \frac{\sqrt{1+w^2t_1^2\sigma^4}}{\sigma}$$

$$\tilde{x} := \frac{x-vt_1}{\alpha} \Rightarrow d\tilde{x} = \frac{dx}{\alpha} \Rightarrow dx = \alpha\, d\tilde{x}$$

$$\tilde{x}_0 = -\frac{vt_1}{\alpha} \qquad \tilde{x}_1 = \frac{x_1 - vt_1}{\alpha}$$

$$= \frac{2\pi\sigma A^2}{\cancel{\alpha}}\cdot\cancel{\alpha}\int_{\tilde{x}_0}^{\tilde{x}_1} e^{-\tilde{x}^2} d\tilde{x} = 2\pi\sigma A^2\cdot\left(\int_{-|\tilde{x}_0|}^{0} + \int_0^{\tilde{x}_1}\right) e^{-\tilde{x}^2} d\tilde{x} =$$

$$\alpha = \frac{\sqrt{1+\left(\frac{6.626\cdot10^{-34}\,Js}{2\pi\cdot9.109\cdot10^{-31}\,kg}\cdot10^{-9}s\cdot10^{-2}\frac{1}{m^2}\right)^2}}{10^{-1}\,\frac{1}{m}} = 10\,m$$

$\sim 10^{-28}$

$$vt_1 = \frac{1}{3}\cdot2.998\cdot10^8\,\frac{m}{\cancel{s}}\cdot10^{-9}\cancel{s} \approx 0.1\,m$$

$$\sigma\cdot A^2 = 10^{-1}\cdot\frac{1}{\cancel{m}}\cdot1\,\cancel{m} = 10^{-1}\,; \quad \tilde{x}_0 = -\frac{0.1\,\cancel{m}}{10\,\cancel{m}} = -0.01$$

$$\tilde{x}_1 = \frac{1m-0.1m}{10m} = 0.09$$

$$= \frac{2\pi}{10}\cdot\left(\int_0^{0.09} + \int_0^{0.01}\right)e^{-x^2}dx = \underline{\underline{\frac{2\pi}{10}\left(\mathrm{Erf}(0.09) + \mathrm{Erf}(0.01)\right)}}$$

— o —

1 dim. Schrödinger egyenlet

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + V(x)\Psi(x)$$

$$\Psi(x,t) = \phi(x) \cdot e^{-i\omega t}$$

$$\Downarrow$$

$$-i\hbar\, i\omega\, \phi(x)\, \cancel{e^{-i\omega t}} = -\frac{\hbar^2}{2m}\phi''(x)\, \cancel{e^{-i\omega t}} + V(x)\phi(x)\, \cancel{e^{-i\omega t}}$$

$$\underbrace{\hbar\omega}_{E}\, \phi(x) - V(x)\phi(x) = -\frac{\hbar^2}{2m}\phi''(x)$$

$$\frac{\hbar^2}{2m}\phi''(x) = (V(x) - E)\,\phi(x)$$

$$\phi''(x) = \frac{2m}{\hbar^2}(V(x) - E)\,\phi(x)$$

integráljuk valamely $x_0$ körüli $2\varepsilon$ tartományra:

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \phi''(x)dx = \phi'(x_0+\varepsilon) - \phi'(x_0-\varepsilon) = \frac{2m}{\hbar^2}\int_{x_0-\varepsilon}^{x_0+\varepsilon} dx\, V(x)\,\phi(x) - \frac{2mE}{\hbar^2}\int_{x_0-}$$

$$\phi(x) \text{ korlátos} \Rightarrow \int_{x_0-\varepsilon}^{x_0+\varepsilon} \phi(x)\, dx = \phi(\xi)\cdot(x_0+\varepsilon - x_0+\varepsilon) = $$
$$= \phi(\xi)\cdot 2\varepsilon$$

$$\Downarrow$$

$$\lim_{\varepsilon\to 0}\left(\phi'(x_0+\varepsilon) - \phi'(x_0-\varepsilon)\right) = \frac{2m}{\hbar^2}\lim_{\varepsilon\to 0}\int_{x_0-\varepsilon}^{x_0+\varepsilon} V(x)\,\phi(x)\, dx =$$

$$= \begin{cases} 0 & \text{ha } V(x) \text{ korlátos } x_0\text{-} \\ \neq 0 & \text{ha } V(x) \text{ nem korlát} \end{cases}$$

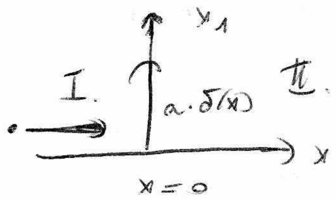
azonban $\phi(x)$ mindenképpen folytonos.

pl.: $V(x) = a\cdot\delta(x-x_0) \Rightarrow \int_{x_0-\varepsilon}^{x_0+\varepsilon} V(x)\,\phi(x)\, dx = \phi(x_0)$

Dirac-delta disztribúció: $\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$

ha $a \notin (x_1, x_2)$: $\int_{x_1}^{x_2} f(x)\,\delta(x-a)\,dx = 0$

<u>2. Feladat:</u> $t = ?$ $r = ?$ ha $V(x) = a\cdot\delta(x)$




elektron mozog $m = 511\ keV/c^2 \Rightarrow mc^2 = 5$

$E = 766.5\ keV$ és $a = \hbar\cdot c = 3.16\cdot 10^{-26}\ J\cdot m$

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mindkét tartományban $V(x)=0$

$$\Rightarrow \quad E = \hbar\omega = \frac{\hbar^2 k^2}{2m}$$

I. $\quad \phi_I = A e^{ikx} + B e^{-ikx}$

II. $\quad \phi_{II} = C e^{ikx}$

$\phi(x)$ folytonos $\Rightarrow \quad \phi_I(0) = \phi_{II}(0)$

$$\text{i.)} \quad A + B = C$$

azonban $\phi'(x)$ ugrik : $\quad \phi'(a^+) - \phi'(a^-) = \frac{2m}{\hbar^2} \lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} a\,\delta(x)\,\phi(x)\,dx =$

$$= \frac{2m}{\hbar^2} a \cdot \phi(0)$$

$$\Downarrow$$

$$\phi'_{II}(0) - \phi'_I(0) = \frac{2m}{\hbar^2} a\,\phi(0)$$

$$ikC - (ikA - ikB) = \frac{2m}{\hbar^2} a\,\phi(0)$$

$$\text{i.)} \quad ik(C - A + B) = \frac{2m}{\hbar^2} a\,\phi(0)$$

$$\Downarrow$$

$$\left.\begin{array}{l} \text{i.)} \quad A + B = C \\ \text{ii.)} \quad C - A + B = -\frac{i2ma}{\hbar^2 k}\,\underset{\substack{\uparrow \\ C}}{\phi(0)} \end{array}\right\} \Rightarrow \quad \cancel{A} + B - \cancel{A} + B = -2\frac{ima}{\hbar^2 k}(A+B)$$

$$\cancel{2}B = -\cancel{2}\frac{ima}{\hbar^2 k}A - \cancel{2}\frac{ima}{\hbar^2 k}B$$

$$B\left(1 + \frac{ima}{\hbar^2 k}\right) = -\frac{ima}{\hbar^2 k}A$$

$$\underline{\underline{r}} = \frac{j_B}{j_A} = \frac{\frac{\hbar k}{m}|B|^2}{\frac{\hbar k}{m}|A|^2} = \frac{|B|^2}{|A|^2} = \left|\frac{B}{A}\right|^2 = \left|\frac{-\frac{ima}{\hbar^2 k}}{1 + \frac{ima}{\hbar^2 k}}\right|^2 =$$

$$= \frac{\frac{m^2 a^2}{\hbar^4 k^2}}{1 + \frac{m^2 a^2}{\hbar^4 k^2}} = \frac{m^2 a^2}{\hbar^4 k^2 + m^2 a^2} = \frac{m^2 \cancel{\hbar^2} c^2}{\hbar^{\cancel{4}} k^2 + m^2 \cancel{\hbar^2} c^2} =$$

$$= \frac{m^2 c^2}{\hbar^2 k^2 + m^2 c^2} = \frac{m^{\cancel{2}} c^2}{2\cancel{m} E + m^{\cancel{2}} c^2} = \frac{mc^2}{mc^2 + 2E} = \frac{511\ \cancel{keV}}{511\ \cancel{keV} + 2 \cdot 766.5\ \cancel{keV}}$$

$$\frac{\hbar^2 k^2}{2m} = E \Rightarrow \hbar^2 k^2 = 2mE$$

$$= \frac{511}{511 + 2 \cdot 766.5} = \underline{\underline{\frac{1}{4}}}$$

$$t = \frac{j_C}{j_A} = \frac{\frac{\hbar k}{m}|C|^2}{\frac{\hbar k}{m}|A|^2} = \left|\frac{C}{A}\right|^2 = \left|\frac{A+B}{A}\right|^2 = \left|1+\frac{B}{A}\right|^2 =$$

$$= \left|1 + \frac{-i\frac{ma}{\hbar^2 k}}{1+\frac{ima}{\hbar^2 k}}\right|^2 = \left|\frac{1}{1+\frac{ima}{\hbar^2 k}}\right|^2 = \frac{1}{1+\frac{m^2a^2}{\hbar^4 k^2}} =$$

$$= \frac{\hbar^4 k^2}{\hbar^4 k^2 + m^2 a^2} = \frac{2Em}{2Em + m^2c^2} = \frac{2E}{2E+mc^2} =$$

$$= \frac{2\cdot 766.5\,keV}{2\cdot 766.5\,keV + 511\,keV} = \underline{\underline{\frac{3}{4}}}$$

Tehát $t + r = 1$ ahogy lennie kell.

# Kvantummech. gyak. 3.

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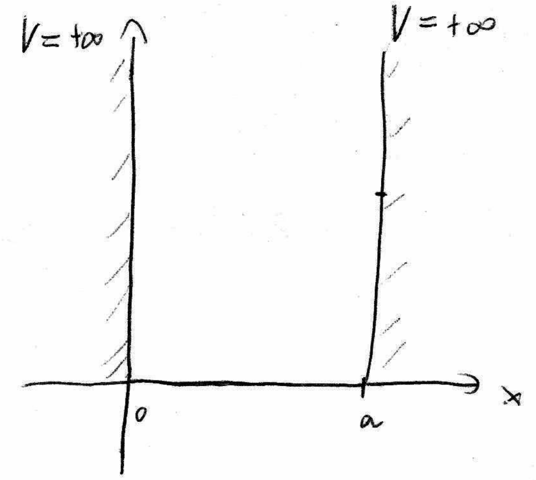
**1. Feladat:**

$\Psi(x) = ?$

$E = ?$




A megoldás a $V=+\infty$ régióban: $\quad \| \quad \phi''(x) = \frac{2m}{\hbar^2}(V-E)\phi(x)$

$$e^{\pm ikx} \Rightarrow -k^2 = \frac{2m}{\hbar^2}(V-E)$$

$$E<V \quad \Downarrow \quad k^2 = \frac{2m}{\hbar^2}(E-V)$$

$$k = \frac{\sqrt{2m}}{\hbar} i \sqrt{|E-V|}$$

$$\Rightarrow \text{mo.: } e^{\mp|k|x} \xrightarrow[V]{+\infty} \begin{cases} 0 \\ +\infty \to \text{ez értelmetlen} \end{cases}$$

$\Downarrow$

A falon el kell tűnjön a hullámfügv.

A két fal között:

$$\Psi(x) = A e^{ikx} + B e^{-ikx} \qquad k^2 = \frac{2mE}{\hbar^2}$$

A falaknál: $\Psi(0) = A + B = \emptyset \Rightarrow B = -A$

$$\Psi(x) = A(e^{ikx} - e^{-ikx}) = 2iA \sin kx$$

$$\Psi(a) = 2iA \sin ka = \emptyset \Rightarrow ka = n\pi$$

$$k = \frac{n\pi}{a}$$

$$\Downarrow$$

$$\frac{2mE}{\hbar^2} = \frac{n^2\pi^2}{a^2}$$

$$\boxed{E_n = \frac{n^2\pi^2\hbar^2}{2ma^2} \quad n \in \mathbb{N}}$$ diszkrét energiaszintek

$$\Psi_n(x) = 2iA \sin k_n x$$

normálás: $\int_0^a |\Psi_n(x)|^2 dx = 1$

$$\Downarrow$$

$$1 = \int_0^a 4|A|^2 \sin^2(k_n x)\,dx = 4|A|^2 \int_0^a \frac{1}{2}(1-\cos 2k_n x)\,dx =$$

$$\Big\| \sin^2\alpha = \frac{1}{2}(1-\cos 2\alpha)$$

$$= 2|A|^2\left[x - \frac{\sin 2k_n x}{2}\right]_0^a = 2|A|^2\Big[a - \underbrace{\frac{\sin 2k_n a}{2}}_{\frac{1}{2}\sin 2n\pi = \emptyset}\Big] = 2|A|^2 a$$

$$\Downarrow$$

$$|A|^2 = \frac{1}{2a}$$

$$\Downarrow$$

$$A = \frac{1}{\sqrt{2a}}e^{i\varphi}$$

$$\Downarrow$$

$$\psi_n(x) = 2e^{i\frac{\pi}{2}}\cdot\frac{e^{i\varphi}}{\sqrt{2a}}\sin k_n x = \sqrt{\frac{2}{a}}\,e^{i(\varphi+\frac{\pi}{2})}\sin\left(n\pi\frac{x}{a}\right)$$

$$\boxed{\psi_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}\cdot e^{i\varphi'} \qquad \varphi'\in\mathbb{R}}$$

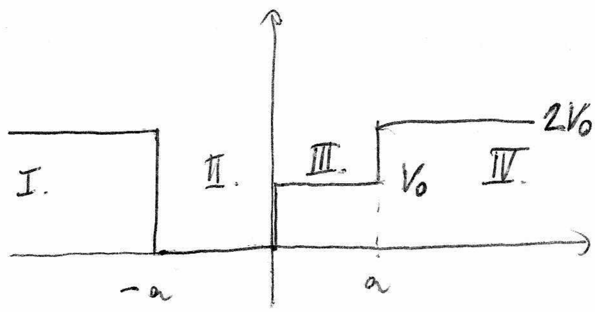
$\psi_n(x)$ csak egy fázisfaktor erejéig meghatározott

## 2. feladat

$$a := \frac{\hbar}{\sqrt{2mV_0}}$$

$V_0 < E < 2V_0$

$E = ?$




I. ; IV.

$$-k^2 = \frac{2m}{\hbar^2}(2V_0 - E) > 0 \qquad i\varkappa := k$$

$$-(i\varkappa)^2 = \frac{2m}{\hbar^2}(2V_0 - E)$$

$$\varkappa = \sqrt{2m(2V_0 - E)}/\hbar$$

I.) $\psi_I(x) = Ae^{\varkappa x} + Be^{-\varkappa x}$

$$\lim_{x\to-\infty}\psi_I(x) = \emptyset \Rightarrow B = \emptyset$$

$$\Downarrow$$

$$\psi_I(x) = Ae^{\varkappa\cdot x}$$

IV.) $\psi_{IV}(x) = A'e^{\varkappa x} + B'e^{-\varkappa x} \qquad \lim_{x\to\infty}\psi_{IV}(x) = \emptyset \Rightarrow A' =$

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$$\Downarrow$$

$$\psi_{IV}(x) = B e^{-\varkappa x}$$

II.

$$-k^2 = -\frac{2m}{\hbar^2} E$$

$$k = \sqrt{2mE}/\hbar$$

$$\psi_{II}(x) = C e^{ikx} + D e^{-ikx}$$

III.

$$-q^2 = -\frac{2m}{\hbar^2}(V_0 - E)$$

$$q^2 = \frac{2m}{\hbar^2}(E - V_0) > 0$$

$$q = \sqrt{2m(E-V_0)}/\hbar$$

$$\psi_{III}(x) = F e^{iqx} + G e^{-iqx}$$

folytonossági feltételek:

<u>$x=-a$:</u>

i.) $\psi_I(-a) = \psi_{II}(-a)$

ii.) $\psi'_I(-a) = \psi'_{II}(-a)$

i.) $A e^{-\varkappa a} = C e^{-ika} + D e^{ika}$

ii.) $A\varkappa e^{-\varkappa a} = C ik e^{-ika} - D ik e^{ika}$

<u>$x=0$:</u>

iii.) $\psi_{II}(0) = \psi_{III}(0)$

iv.) $\psi'_{II}(0) = \psi'_{III}(0)$

iii.) $C + D = F + G$

iv.) $ik(C-D) = iq(F-G)$

<u>$x=a$:</u>

v.) $\psi_{III}(a) = \psi_{IV}(a)$

vi.) $\psi'_{III}(a) = \psi'_{IV}(a)$

v.) $F e^{iqa} + G e^{-iqa} = B e^{-\varkappa a}$

vi.) $iqF e^{iqa} - iqG e^{-iqa} = -B\varkappa e^{-\varkappa a}$

— o —

i.) $\Rightarrow$ $D = A e^{-\varkappa a} e^{-ika} - C e^{-2ika}$

ii.) $A\varkappa e^{-\varkappa a} = C ik e^{-ika} - ik(A e^{-\varkappa a} - C e^{-ika}) =$

$$= 2Ci\xi\, e^{-i\xi a} - Ai\xi\, e^{-\kappa a}$$

$$\Downarrow$$

$$A e^{-\kappa a}(\kappa + i\xi) = 2Ci\xi\, e^{-i\xi a}$$

$$\boxed{C = A\,\frac{\kappa + i\xi}{2i\xi}\, e^{-\kappa a}\, e^{+i\xi a}}$$

$$\text{v.)} \Rightarrow \boxed{D = A\left(e^{-\kappa a} e^{-i\xi a} - \frac{\kappa+i\xi}{2i\xi}\, e^{-\kappa a} e^{-i\xi a}\right) =}$$

$$= A\underbrace{\left(1 - \frac{\kappa+i\xi}{2i\xi}\right)}_{\frac{2i\xi - \kappa - i\xi}{2i\xi} = \frac{i\xi-\kappa}{2i\xi}} e^{-\kappa a} e^{-i\xi a} = \boxed{A\,\frac{i\xi-\kappa}{2i\xi}\, e^{-\kappa a} e^{-i\xi a}}$$

q · iii.) ± iv.) $\Rightarrow$ $q(C+D) + \xi(C-D) = 2qF \Rightarrow$

$$\Rightarrow \boxed{F = C\,\frac{\xi+q}{2q} - D\,\frac{\xi-q}{2q}} \qquad \| \ \xi > q$$

" $q(C+D) - \xi(C-D) = 2qG \Rightarrow$

$$\Rightarrow \boxed{G = -C\,\frac{\xi-q}{2q} + D\,\frac{\xi+q}{2q}}$$

C, D-t beírva F és G-be:

$$\underline{\underline{F}} = A\, e^{-\kappa a} \left\{ \frac{\xi+q}{2q}\,\frac{\kappa+i\xi}{2i\xi}\, e^{i\xi a} - \frac{\xi-q}{2q}\,\frac{i\xi-\kappa}{2i\xi}\, e^{-i\xi a} \right\} =$$

$$= A\,\frac{e^{-\kappa a}}{4i\xi q}\left\{(\xi+q)(\kappa+i\xi)\, e^{i\xi a} + (\xi-q)(\kappa-i\xi)\, e^{-i\xi a}\right\}$$

$$\underline{\underline{G}} = A e^{-\kappa a}\left\{ -\frac{\xi-q}{2q}\,\frac{\kappa+i\xi}{2i\xi}\, e^{i\xi a} + \frac{\xi+q}{2q}\,\frac{i\xi-\kappa}{2i\xi}\, e^{-i\xi a}\right\} =$$

$$= -A\,\frac{e^{-\kappa a}}{4i\xi q}\left\{(\xi-q)(\kappa+i\xi)\, e^{i\xi a} + (\xi+q)(\kappa-i\xi)\, e^{-i\xi a}\right\}$$

$$\Downarrow$$

v.) és vi.) egyenletekbe beírva:

v.):

$$A\,\frac{\cancel{e^{-\kappa a}}}{4i\xi q}\Big[\overbrace{(\xi+q)(\kappa+i\xi)\, e^{i(\xi+q)a} + (\xi-q)(\kappa-i\xi)\, e^{-i(\xi-q)a} - (\xi-q)(\kappa+i\xi)\, e^{i(\xi-q)a} - (\xi+q)(\kappa-i\xi)\, e^{-i(\xi+q)a}}^{f(\xi,q,\kappa)}\Big] = B\,\cancel{e^{-\kappa a}}$$

vi.):

$$A\,\frac{\cancel{e^{-\kappa a}}}{4\xi}\Big[\overbrace{(\xi+q)(\kappa+i\xi)\, e^{i(\xi+q)a} + (\xi-q)(\kappa-i\xi)\, e^{-i(\xi-q)a} + (\xi-q)(\kappa+i\xi)\, e^{i(\xi-q)a} + (\xi+q)(\kappa-i\xi)\, e^{-i(\xi+q)a}}^{g(\xi,q,\kappa)}\Big] = -B\kappa\,\cancel{e^{-\kappa a}}$$

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$$\text{v.)}\quad A\cdot f(\xi,q,\varkappa) - B\cdot 4i\xi q = 0$$
$$\text{vi.)}\quad A\cdot g(\xi,q,\varkappa) + B\, 4\xi\varkappa = 0$$

$$\begin{pmatrix} f & -4i\xi q \\ g & 4\xi\varkappa \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

nem triv. mo.: $\begin{vmatrix} f & -4i\xi q \\ g & 4\xi\varkappa \end{vmatrix} = 0$

$$\Rightarrow\quad f\cdot \cancel{4}\cancel{\xi}\varkappa + g\,\cancel{4}\, i\cancel{\xi} q = 0$$
$$f\cdot\varkappa + g\cdot iq = 0$$

$$\varkappa(\xi+q)(\varkappa+i\xi)e^{i(\xi+q)a} + \varkappa(\xi-q)(\varkappa-i\xi)e^{-i(\xi-q)a} -$$
$$- \varkappa(\xi-q)(\varkappa+i\xi)e^{i(\xi-q)a} - \varkappa(\xi+q)(\varkappa-i\xi)e^{-i(\xi+q)a} +$$
$$+ iq(\xi+q)(\varkappa+i\xi)e^{i(\xi+q)a} + iq(\xi-q)(\varkappa-i\xi)e^{-i(\xi-q)a} +$$
$$+ iq(\xi-q)(\varkappa+i\xi)e^{i(\xi-q)a} + iq(\xi+q)(\varkappa-i\xi)e^{-i(\xi+q)a} = 0$$

$$(\xi+q)(\varkappa+i\xi)(\varkappa+iq)e^{i(\xi+q)a} + (\xi+q)(\varkappa-i\xi)(iq-\varkappa)e^{-i(\xi+q)a} +$$
$$+ (\xi-q)(\varkappa+i\xi)(iq-\varkappa)e^{i(\xi-q)a} + (\xi-q)(\varkappa-i\xi)(\varkappa+iq)e^{-i(\xi-q)a} = 0$$

$$(\xi+q)\left[(\varkappa^2+i(\xi+q)\varkappa-\xi q)e^{i(\xi+q)a} + (-\varkappa^2+i(\xi+q)\varkappa+\xi q)e^{-i(\xi+q)a}\right] +$$
$$+ (\xi-q)\left[(-\varkappa^2-i(\xi-q)\varkappa-\xi q)e^{i(\xi-q)a} + (\varkappa^2-i(\xi-q)\varkappa+\xi q)e^{-i(\xi-q)a}\right] = 0$$

$$(\xi+q)\left[(\varkappa^2-\xi q)\cancel{2i}\sin(\xi+q)a + \cancel{i}\varkappa(\xi+q)\cancel{2}\cos(\xi+q)a\right] +$$
$$+ (\xi-q)\left[-(\varkappa^2+\xi q)\cancel{2i}\sin(\xi-q)a - \cancel{i}\varkappa(\xi-q)\cancel{2}\cos(\xi-q)a\right] = 0$$

$$(\xi\varkappa^2-\xi^2 q+q\varkappa^2-\xi q^2)\sin(\xi+q)a + (-\xi\varkappa^2-\xi^2 q+q\varkappa^2+\xi q^2)\sin(\xi-q)a +$$
$$+ \varkappa(\xi^2+2\xi q+q^2)\cos(\xi+q)a + \varkappa(-\xi^2+2\xi q-q^2)\cos(\xi-q)a = 0$$

$$\xi(\varkappa^2-q^2)\underbrace{(\sin(\xi+q)a-\sin(\xi-q)a)}_{\cancel{2}\cos\xi a\sin qa} + q(\varkappa^2-\xi^2)\underbrace{(\sin(\xi+q)a+\sin(\xi-q)a)}_{\cancel{2}\sin\xi a\cos qa} +$$
$$+ \varkappa(\xi^2+q^2)\underbrace{(\cos(\xi+q)a-\cos(\xi-q)a)}_{-\cancel{2}\sin\xi a\sin qa} + 2\varkappa\xi q\underbrace{(\cos(\xi+q)a+\cos(\xi-q)a)}_{\cancel{2}\cos\xi a\cos qa} = 0$$

$/:\ \varkappa\xi q\cos\xi a\cos qa$

$$\frac{\varkappa^2-q^2}{\varkappa q}\,\mathrm{tg}\,qa + \frac{\varkappa^2-\xi^2}{\varkappa\xi}\,\mathrm{tg}\,\xi a - \frac{\xi^2+q^2}{\xi q}\,\mathrm{tg}\,qa\cdot\mathrm{tg}\,\xi a + 2 = 0$$

$$\frac{\varkappa^2 - q^2}{\varkappa q} = \frac{\cancel{\frac{2m}{\hbar^2}}(2V_0 - E - E + V_0)}{\cancel{\frac{2m}{\hbar^2}}\sqrt{(2V_0 - E)(E - V_0)}} = \frac{3V_0 - 2E}{\sqrt{(2V_0-E)(E-V_0)}} = \frac{3-2x}{\sqrt{(2-x)(x-1)}}$$

$x := \frac{E}{V_0}$ [redacted] felt.: $1 < x < 2$

$$\frac{\varkappa^2 - k^2}{\varkappa k} = \frac{\cancel{\frac{2m}{\hbar^2}}(2V_0 - E - E)}{\cancel{\frac{2m}{\hbar^2}}\sqrt{(2V_0-E)(E)}} = \frac{2(V_0-E)}{\sqrt{(2V_0-E)E}} = \frac{2(1-x)}{\sqrt{(2-x)\cdot x}} = -2\frac{x-1}{\sqrt{x(2-x)}}$$

$$\frac{k^2 + q^2}{kq} = \frac{\cancel{\frac{2m}{\hbar^2}}(E + E - V_0)}{\cancel{\frac{2m}{\hbar^2}}\sqrt{E(E-V_0)}} = \frac{2E - V_0}{\sqrt{E(E-V_0)}} = \frac{2x-1}{\sqrt{x(x-1)}}$$

$$ka = \sqrt{\frac{2m}{\hbar^2}E}\,a = \underbrace{\frac{\sqrt{2m}\,a}{\hbar}\sqrt{V_0}}_{\alpha}\sqrt{\frac{E}{V_0}} = \alpha\cdot\sqrt{x}$$

$$qa = \frac{\sqrt{2m}\,a}{\hbar}\sqrt{V_0}\sqrt{\frac{E}{V_0} - 1} = \alpha\sqrt{x-1}$$

$$\frac{3-2x}{\sqrt{(2-x)(x-1)}}\,\mathrm{tg}(\alpha\sqrt{x-1}) - \frac{2(x-1)}{\sqrt{x(2-x)}}\,\mathrm{tg}(\alpha\sqrt{x}) - \frac{2x-1}{\sqrt{x(x-1)}}\,\mathrm{tg}(\alpha\sqrt{x-1})\,\mathrm{tg}(\alpha\sqrt{x}) + 2 = 0$$

$a = \frac{\hbar}{\sqrt{2m}\sqrt{V_0}}$ esetén $\alpha = 1$; $\uparrow$ diszkrét energiasávok

$$F(x) := \frac{2(x-1)}{\sqrt{x(2-x)}}\,\mathrm{tg}\sqrt{x} + \frac{2x-1}{\sqrt{x(x-1)}}\,\mathrm{tg}\sqrt{x-1}\,\mathrm{tg}\sqrt{x} - \frac{3-2x}{\sqrt{(2-x)(x-1)}}\,\mathrm{tg}\sqrt{x-1} = 2$$

$1 < x < 2$

Van-e megoldás?

$$\lim_{x\to 1^+} F(x) = 0 + \underbrace{\lim_{y\to 0^+}\frac{\mathrm{tg}\, y}{y}}\cdot \mathrm{tg}\,1 - \lim_{y\to 0^+}\frac{\mathrm{tg}\, y}{y} = \mathrm{tg}\,1 - 1 \approx 0.56$$

$$\lim_{y\to 0^+}\frac{\sin y}{y}\cdot\underbrace{\frac{1}{\cos y}}_{1} = 1$$

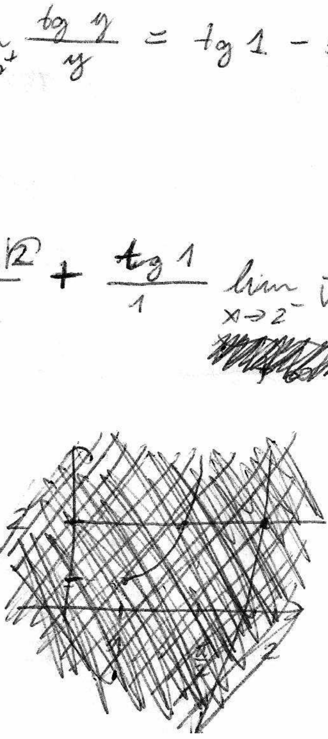
$$\lim_{x\to 2^-} F(x) = \frac{2\,\mathrm{tg}\sqrt{2}}{\sqrt{2}}\lim_{x\to 2^-}\frac{1}{\sqrt{2-x}} + \frac{3\,\mathrm{tg}\,1\cdot\mathrm{tg}\sqrt{2}}{\sqrt{2}} + \frac{\mathrm{tg}\,1}{1}\lim_{x\to 2^-}\frac{1}{\sqrt{2-x}} =$$

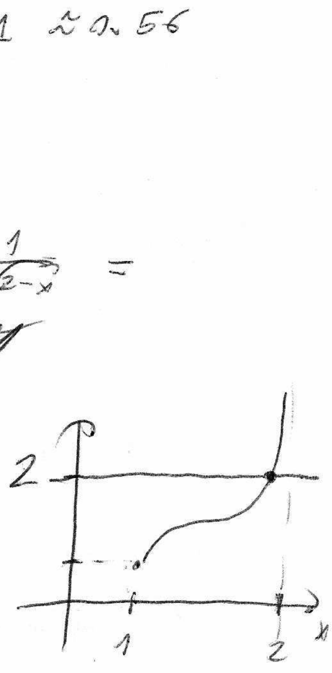
[redacted] [redacted]

$$= \underbrace{\left(\frac{2\,\mathrm{tg}\sqrt{2}}{\sqrt{2}} + \mathrm{tg}\,1\right)}_{>0}\lim_{x\to 2^-}\frac{1}{\sqrt{2-x}} = +\infty$$

$\sqrt{x} = \frac{\pi}{2}$-nél $\mathrm{tg}\sqrt{x}$ divergál

$\Rightarrow x = \frac{\pi^2}{4} > 2$





# Kvantummechanika gyak.

## 4.

operátorok: $\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$ $\qquad \hat{x} = x.$

Hogy lesz az operátorokból mérhető fiz. mennyiség?

Általában adott rendszeren többször mérve az impulzust más-más értéket kapunk → fiz. mennyiségek értékeinek valószínűségi eloszlása lesz.

## Várható érték, szórás

$\hat{p}$ várható értéke $\Psi$ állapotban:

$$\bar{p}_\Psi = \int dx\, \Psi^*(x,t)\, \hat{p}\, \Psi(x,t) = \int dx\, \Psi^*(x,t) \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t) \rightarrow \text{szám}$$

jelölés: $\bar{p}_\Psi = \langle \Psi | \hat{p} | \Psi \rangle$

$\hat{x}$ várható értéke $\Psi$ állapotban:

$$\bar{x}_\Psi = \langle \Psi | \hat{x} | \Psi \rangle = \int dx\, \Psi^*(x,t)\, x\, \Psi(x,t) \rightarrow \text{szám}$$

Szórásnégyzet (várható értéktől való átlagos eltérés)

$$\overline{(\Delta p)^2} \equiv \overline{(\hat{p}-\bar{p})^2} = \langle \Psi | (\hat{p} - \underbrace{\bar{p}}_{\bar{p}\equiv\bar{p}\cdot 1})^2 | \Psi \rangle = \langle \Psi | \hat{p}^2 - 2\bar{p}\hat{p} + \bar{p}^2 | \Psi \rangle =$$

$$= \langle \Psi | \hat{p}^2 | \Psi \rangle - 2\bar{p} \underbrace{\langle \Psi | \hat{p} | \Psi \rangle}_{\bar{p}} + \bar{p}^2 \underbrace{\langle \Psi | \Psi \rangle}_{1} =$$

$$= \langle \Psi | \hat{p}^2 | \Psi \rangle - 2\bar{p}^2 + \bar{p}^2 = \langle \Psi | \hat{p}^2 | \Psi \rangle - \bar{p}^2 =$$

$$= \int dx\, \Psi^*(x,t)\, \hat{p}^2\, \Psi(x,t) - \bar{p}^2 =$$

$$\Big\| \quad \hat{p}^2 = \hat{p}\hat{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}\frac{\hbar}{i}\frac{\partial}{\partial x} = \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial x^2}$$

$$= \int dx\, \Psi^*(x,t) \left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial x^2} \Psi(x,t) - \bar{p}^2$$

$$\overline{(\Delta x)^2} \equiv \overline{(\hat{x}-\bar{x})^2} = \langle \Psi | \hat{x}^2 | \Psi \rangle - \bar{x}^2 = \int dx\, \Psi^*(x,t)\, x^2\, \Psi(x,t) - \bar{x}^2$$

$$\Big\| \quad \hat{x}^2 = \hat{x}\hat{x} = x^2.$$

## Sajátállapot

Van-e olyan állapot, amikor a szórás 0?

Ha $\exists$ olyan $p \in \mathbb{R}$, hogy: $\hat{p}|\Psi_p\rangle = p|\Psi_p\rangle$ akkor:

$$\bar{p} = \langle \psi_p | \hat{p} | \psi_p \rangle = \langle \psi_p | p | \psi_p \rangle = p \underbrace{\langle \psi_p | \psi_p \rangle}_{1} = p$$

$p$ a $\hat{p}$ operátor sajátértéke $\psi_p$ a sajátállapota (sajátfüggvénye

$\Downarrow$

a várható érték megegyezik a sajátértékkel

Mik a $\hat{p}$ sajátállapotai: $\frac{\hbar}{i} \frac{\partial}{\partial x} \psi_p(x,t) = p\, \psi_p(x,t)$

$$\frac{\partial \psi_p}{\partial x} = \frac{1}{\hbar} i p\, \psi_p \Rightarrow \frac{d\psi_p}{\psi_p} = \frac{1}{\hbar} i p\, dx$$

$$\ln \psi_p = \frac{1}{\hbar} i p x + c$$

$$\psi_p = c\, e^{\frac{1}{\hbar} i p x} \qquad \psi_p^* \psi_p = |c|^2 = 1 \Rightarrow c =$$

választ

$\hat{p}$ szórása: $\overline{(\Delta p^2)}$ [redacted] $= \langle \psi_p | \hat{p}^2 | \psi_p \rangle - \bar{p}^2 = p^2 \underbrace{\langle \psi_p | \psi_p \rangle}_{1} - \bar{p}^2 = \emptyset$

$\Downarrow$

$\psi_p \to$ határozott [redacted] impulzusú állapot

$\psi_x \to$ határozott helysajátállapot $\Delta x = \emptyset$

Heisenberg határozatlansági reláció $\Rightarrow$ $\nexists$ olyan állapot, amely egyszerre hely- és impulzus- sajátállapot

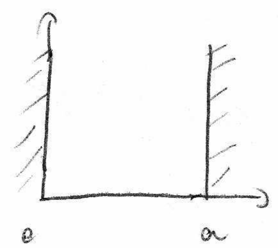
$$[\hat{x}, \hat{p}] = i\hbar \overset{\text{belátható}}{\Longrightarrow} \Delta\hat{x}\, \Delta\hat{p} \geq \frac{\hbar}{2}$$

megj.: Schrödinger-egyenlet:

$$\left(-\frac{\hbar^2}{2m} \Delta + V(x)\right) \psi(x) = \underbrace{\left(\frac{\hat{p}^2}{2m} + V(\hat{x})\right)}_{\hat{H}} \psi(x) = E\psi$$

$\Rightarrow \hat{H} \psi(x) = E \psi(x)$ Hamilton-operátor sajátérték-egyenlete

<u>Feladat</u>:




$$\psi_n(x) = \sqrt{\frac{2}{a}} \sin \frac{n\pi}{a} x \; e^{i\varphi} \qquad x \in [0, a]$$

$n = 1, 2, \ldots$ || ugyanis a fázisból lehet (-1)

$\bar{p}, \overline{(\Delta p)^2}, \bar{x}, \overline{(\Delta x)^2} = ?$

$$\bar{p} = \langle \psi_n | \hat{p} | \psi_n \rangle = \int_0^a dx\, \psi_n^*(x) \frac{\hbar}{i} \frac{\partial}{\partial x} \psi_n(x) \overset{e^{i\varphi} \text{ kiesik}}{=}$$

$$= \int_0^a dx \frac{2}{a} \sin\left(\frac{n\pi x}{a}\right) \frac{\hbar}{i} \frac{\partial}{\partial x} \sin\left(\frac{n\pi x}{a}\right) = \frac{2\hbar}{ia} \int_0^a \sin\left(\frac{n\pi x}{a}\right) \left(\frac{n\pi}{a}\right) \cdot \cos\left(\frac{n\pi x}{a}\right)$$

$$= \frac{\hbar n\pi}{ia^2} \int_0^a dx\, 2 \sin\left(\frac{n\pi x}{a}\right) \cos\left(\frac{n\pi x}{a}\right) = \frac{\hbar n\pi}{ia^2} \int_0^a dx \sin\left(\frac{2n\pi x}{a}\right) =$$

$$= \frac{\hbar n\pi}{i a^2}\left[-\frac{a}{2n\pi}\cos\left(\frac{2n\pi}{a}x\right)\right]_0^a = -\frac{\hbar}{2ia}\left[\cos(2n\pi) - \cos 0\right] = \emptyset$$

$\Rightarrow \quad \bar{p} = 0$ (ugyanakkora valószínűséggel megy balra mint jobbra)

$$\bar{x} = \langle \psi_n | \hat{x} | \psi_n \rangle = \int_0^a dx\, \psi_n^*(x)\, x\, \psi_n(x) = \int_0^a dx\, \frac{2}{a}\, x \sin^2\left(\frac{n\pi x}{a}\right) =$$

$$= \frac{2}{a}\left(\frac{a}{n\pi}\right)^2 \int_0^{n\pi} y \sin^2 y \, dy = \frac{2a}{(n\pi)^2}\left(I(n\pi) - I(0)\right)$$

$y := \frac{n\pi x}{a} \Rightarrow dy = \frac{n\pi}{a} dx \Leftrightarrow dx = \frac{a}{n\pi} dy$

$x = \frac{a}{n\pi} y$

$x = 0 \Rightarrow y = 0$

$x = a \Rightarrow y = n\pi$

ahol

$$I(y) := \int dy\, y \underbrace{\sin^2 y}_{\frac{1}{2}(1-\cos 2y)} = \frac{1}{2}\int (y - y\cos 2y)\, dy =$$

$$= \frac{1}{2}\left\{\frac{y^2}{2} - \int y \cos 2y\, dy\right\} = \frac{y^2}{4} - \frac{1}{2}\int \underbrace{y}_{g} \underbrace{\cos 2y}_{f'}\, dy \overset{\text{parc. int.}}{=}$$

$$= \frac{y^2}{4} - \frac{1}{2}\left\{y \cdot \frac{1}{2}\sin 2y - \int \frac{1}{2}\sin 2y\, dy\right\} =$$

$$= \frac{y^2}{4} - \frac{y}{4}\sin 2y + \frac{1}{4}\left(-\frac{\cos 2y}{2}\right) + c = \frac{y^2}{4} - \frac{y}{4}\sin 2y - \frac{1}{8}\cos 2y + c$$

$$\Downarrow$$

$$I(y) = \frac{y^2}{4} - \frac{y}{4}\sin 2y - \frac{1}{8}\cos 2y + c$$

$$I(n\pi) = \frac{(n\pi)^2}{4} - \frac{n\pi}{4}\underbrace{\sin 2n\pi}_{\emptyset} - \frac{1}{8}\underbrace{\cos 2n\pi}_{1} + c = \frac{(n\pi)^2}{4} - \frac{1}{8} + c$$

$$I(0) = -\frac{1}{8} + c$$

$$\Downarrow$$

$$\bar{x} = \frac{2a}{(n\pi)^2}\left(\frac{(n\pi)^2}{4} - \cancel{\frac{1}{8}} + \cancel{c} + \cancel{\frac{1}{8}} - \cancel{c}\right) = \frac{a}{2}$$

$\frac{\partial^2}{\partial x^2}\sin\left(\frac{n\pi}{a}x\right) = -\left(\frac{n\pi}{a}\right)^2 \sin$

$$\overline{(\Delta p)^2} = \int_0^a dx\, \psi_n^*(x)\left(\frac{\hbar}{i}\right)^2 \frac{\partial^2}{\partial x^2}\psi_n(x) - \underbrace{\bar{p}^2}_{\emptyset} = -\hbar^2 \int_0^a dx\, \frac{2}{a}\sin\left(\frac{n\pi x}{a}\right)\left(-\left(\frac{n\pi}{a}\right)^2\right)\sin\frac{n\pi x}{a}$$

$$= \left(\frac{\hbar n\pi}{a}\right)^2 \cdot \frac{2}{a}\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx = \left(\frac{\hbar n\pi}{a}\right)^2 \frac{1}{a}\int_0^a \left(1 - \cos\frac{2n\pi x}{a}\right) dx =$$

$$= \frac{(\hbar n\pi)^2}{a^3}\left\{\left[x - \underbrace{\frac{\sin\frac{2n\pi}{a}x}{\frac{2n\pi}{a}}}_{\to\,\emptyset}\right]_0^a\right\} = \left(\frac{\hbar n\pi}{a}\right)^2$$

$$\Rightarrow \boxed{\Delta p \equiv \sqrt{\overline{(\Delta p)^2}} = \frac{\hbar n\pi}{a}}$$

$$\overline{(\Delta x)^2} = \int_0^a dx\ \psi_n^*(x)\ x^2 \psi_n(x) - \underbrace{\bar{x}^2}_{\frac{a^2}{4}} =$$

$$= \frac{2}{a}\int_0^a dx \left(x^2 \sin^2\Big(\underbrace{\frac{n\pi x}{a}}_{y}\Big)\right) - \frac{a^2}{4} = \frac{2}{a}\left(\frac{a}{n\pi}\right)^3 \int_0^{n\pi} dy\, y^2 \sin^2 y - \frac{a^2}{4} = *$$

$$dx = \frac{a}{n\pi}\, dx$$

$$x = \frac{a}{n\pi}\, y$$

$$\int dy\, y^2 \sin^2 y = \int dy\, y \cdot y \sin^2 y = y\, I(y) - \int dy\, I(y) =$$

$$= \frac{y^3}{4} - \frac{y^2}{4}\sin 2y - \frac{y}{8}\cos 2y - \int dy \left(\frac{y^2}{4} - \frac{1}{4} y \sin 2y - \frac{1}{8}\cos 2y\right) =$$

$$= \frac{y^3}{4} - \frac{y^2}{4}\sin 2y - \frac{y}{8}\cos 2y - \frac{y^3}{12} + \frac{1}{4}\left(-\frac{y}{2}\cos 2y + \frac{1}{4}\sin 2y\right) + \frac{1}{16}\sin 2y + c =$$

$$\int dy\, y \sin 2y = -y\frac{\cos 2y}{2} + \frac{1}{2}\int dy \cos 2y = -\frac{y}{2}\cos 2y + \frac{1}{4}\sin$$

$$= \frac{y^3}{6} + \sin 2y\left(\frac{1}{8} - \frac{y^2}{4}\right) - \frac{y}{4}\cos 2y + c$$

$$* = \frac{2a^2}{(n\pi)^3}\left\{\underbrace{\left[\frac{y^3}{6}\right]_0^{n\pi}}_{\frac{(n\pi)^3}{6}} + \underbrace{\left[\sin 2y\left(\frac{1}{8} - \frac{y^2}{4}\right)\right]_0^{n\pi}}_{\emptyset} - \underbrace{\left[\frac{y}{4}\cos 2y\right]_0^{n\pi}}_{\frac{n\pi}{4}}\right\} - \frac{a^2}{4} =$$

$$= \frac{2a^2}{(n\pi)^3}\left(\frac{(n\pi)^3}{6} - \frac{n\pi}{4}\right) - \frac{a^2}{4} = a^2\left[\frac{1}{3} - \frac{1}{4} - \frac{1}{2(n\pi)^2}\right] = \frac{a^2}{12}\left(1 - \frac{6}{(n\pi)^2}\right)$$

Tehát:

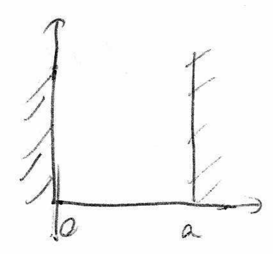
$$\overline{(\Delta x)^2} = \frac{a^2}{12}\left(1 - \frac{6}{(n\pi)^2}\right)$$

Más operátorok várható értékét és szórását is hasonlóan számoljuk.

pl.: $\bar{E} = \int dx\ \Psi^*(x,t)\ \hat{H}\ \Psi(x,t) =$

$$= \int dx\ \Psi^*(x,t)\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t)$$

Feladat:




$\bar{E} = ?$

Legyen a részecske a $\Psi(x) = A\, x\,(a-x)$ hullámfgv.-vel leírt állapotban ha $x \in [0,a]$ és $\Psi(x) \equiv 0$ $x \notin [0,a]$

2007.12.6

$$\bar{E} = \int_0^\infty dx\; \psi^*(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x) =$$

$\|$ $x \in [0,a] \Rightarrow V(x) = 0$
$\phantom{\|}$ $x \notin [0,a] \Rightarrow \psi(x) \equiv 0$

$$= \int_0^a dx\; A^*(x(a-x)) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}\right) A(x(a-x)) =$$

$$= |A|^2 \int_0^a dx\; x(a-x)\left(-\frac{\hbar^2}{2m}\right)(-2) = \frac{|A|^2\hbar^2}{m}\int_0^a dx\; x(a-x) =$$

$\|$ $\frac{\partial^2}{\partial x^2}(x(a-x)) = \frac{\partial}{\partial x}(a-x-x) = -2$

$$= \frac{|A|^2\hbar^2}{m}\int_0^a (ax - x^2)\,dx = \frac{|A|^2\hbar^2}{m}\left[a\frac{x^2}{2} - \frac{x^3}{3}\right]_0^a = \frac{|A|^2\hbar^2}{m}\left(\frac{a^3}{2} - \frac{a^3}{3}\right) =$$

$$= \frac{|A|^2\hbar^2}{6m}$$

## oszcillátor algebra

megj.: $\langle\varphi|\hat p|\psi\rangle = \int dx\; \varphi^*(x)\frac{\hbar}{i}\frac{\partial}{\partial x}\psi(x) = \frac{\hbar}{i}\int dx\left\{\frac{\partial}{\partial x}(\varphi^*(x)\psi(x)) - \right.$

$\left. - \left(\frac{\partial}{\partial x}\varphi^*(x)\right)\psi(x)\right\} = \frac{\hbar}{i}\underbrace{\left[\varphi^*(x)\psi(x)\right]_0^\infty}_{0} - \int dx\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\varphi^*(x)\right)\psi(x) =$

$= \int dx\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\varphi(x)\right)^*\psi(x) = \int dx\,(\hat p\,\varphi(x))^*\psi(x) = \langle\hat p\varphi|\psi\rangle$

azaz $\langle\varphi|\hat p\psi\rangle = \langle\hat p\varphi|\psi\rangle$ hasonlóan: $\langle\varphi|\hat x\psi\rangle = \langle\hat x\varphi|\psi$

továbbá ha $\alpha \in \mathbb{C}$ $\langle\varphi|\alpha\cdot\psi\rangle = \int dx\,\varphi^*(x)\,\alpha\cdot\psi(x) = \int dx\,(\alpha^*\varphi(x))^*\psi$

$= \langle\alpha^*\varphi|\psi\rangle = \alpha\langle\varphi|\psi\rangle$

tehát $\langle\varphi|\alpha\psi\rangle = \alpha\langle\varphi|\psi\rangle$

$\langle\alpha\varphi|\psi\rangle = \alpha^*\langle\varphi|\psi\rangle$

— o —

léptető operátorok: $a = \frac{1}{\sqrt{2}}(\hat X + i\cdot\hat P)$ $\qquad \hat X = \sqrt{\frac{m\omega}{\hbar}}\,\hat x$

$a^+ = \frac{1}{\sqrt{2}}(\hat X - i\cdot\hat P)$ $\qquad \hat P = \frac{1}{\sqrt{m\hbar\omega}}\,\hat p$

$[a, a^+] = 1$ $\qquad \hat N := a^+a$ $\qquad \hat N|N\rangle = N|N\rangle \rightarrow$ diszkrét állapotok

$a^+|N\rangle = \alpha_N|N+1\rangle$

$a|N\rangle = \beta_N|N-1\rangle$

Feladat: Határozzuk meg $\alpha_N$-t és $\beta_N$-t ha $\langle N|M\rangle =$
$\forall N, M \in \mathbb{N}$-re.

Ehhez nézzük a köv várhatóértékeket:

$$\langle N | a a^+ | N \rangle = ?$$

kitérő:

$$\langle N|a|M\rangle = \langle N|\left(\frac{1}{\sqrt2}(\hat X + i\hat P)\right)|M\rangle = \langle N|\frac{1}{\sqrt2}\hat X|M\rangle$$
$$+ \langle N|\frac{1}{\sqrt2}i\hat P|M\rangle = \langle N|\frac{1}{\sqrt2}\hat X M\rangle + \langle N|\frac{1}{\sqrt2}i\hat P M\rangle$$
$$= \langle \frac{1}{\sqrt2}\hat X N|M\rangle + \langle\left(\frac{1}{\sqrt2}i\right)^*\hat P N|M\rangle = \langle\left(\frac{1}{\sqrt2}\hat X N - \frac{1}{\sqrt2}i P N\right)|M\rangle$$
$$= \langle\Big(\underbrace{\frac{1}{\sqrt2}\hat X - \frac{1}{\sqrt2}i\hat P}_{a^+}\Big)N|M\rangle = \langle a^+ N|M\rangle$$

tehát: $\langle N|a|M\rangle = \langle N|aM\rangle = \langle a^+N|M\rangle$

hasonlóan: $\langle N|a^+|M\rangle = \langle N|a^+M\rangle = \langle a N$

$\Downarrow$

$$\langle N|aa^+|N\rangle = \langle N|aa^+N\rangle = \langle \underbrace{a^+N}_{} | \underbrace{a^+N}_{}\rangle = \langle \alpha_N(N+1)|\alpha_N($$
$$\langle\alpha_N(N+1)|\alpha_N(N+1)\rangle$$
$$= \alpha_N^*\alpha_N\langle N+1|N+1\rangle = |\alpha_N|^2 \underbrace{\langle N+1|N+1\rangle}_{1} = |\alpha_N|$$

másrészt:

$$\langle N|aa^+|N\rangle = \langle N|\mathbb{1} + \underbrace{a^+a}_{\hat N}|N\rangle = \langle N|(|N\rangle + \underbrace{\hat N|N}_{N\cdot|N}$$
$$= \langle N|(\underbrace{|N\rangle + N\cdot|N\rangle}_{(1+N)|N\rangle}) = \langle N|(1+N)|N\rangle = (1+N)\underbrace{\langle N|}_{1}$$
$$= 1+N$$

$\Downarrow$ legyen $\alpha_N$ valós

$$|\alpha_N|^2 = 1+N \Rightarrow \alpha_N = \sqrt{1+N}$$

$$\langle N|a^+a|N\rangle = \langle aN|aN\rangle = \langle\beta_N(N-1)|\beta_N(N-1)\rangle =$$
$$= |\beta_N|^2\langle N-1|N-1\rangle = |\beta_N|^2$$
$$\langle N|a^+a|N\rangle = \langle N|\hat N|N\rangle = N\underbrace{\langle N|N\rangle}_{1} = N$$

$\Rightarrow |\beta_N|^2 = N \Rightarrow$ $\beta_N$ legyen valós $\beta_N = \sqrt N$

$\Downarrow$

$$a^+|N\rangle = \sqrt{1+N}\,|N+1\rangle$$
$$a|N\rangle = \sqrt{N}\,|N-1\rangle$$

2007.12.03

**Feladat:** Hat. meg $p^2$ várható értékét oszcillátor sajátállapotok esetén!

$$a = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \; ; \; a^+ = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p})$$

$$\Downarrow$$

$$a - a^+ = \frac{1}{\sqrt{2}} 2i\hat{P} = i\sqrt{2}\frac{1}{\sqrt{m\hbar\omega}}\hat{p} \Rightarrow \hat{p} = \frac{i}{\sqrt{2}}\sqrt{m\hbar\omega}\,(a^+ - a)$$

$$\Downarrow$$

$$\underline{\underline{\langle N|\hat{p}^2|N\rangle}} = -\frac{m\hbar\omega}{2}\langle N|\underbrace{(a^+ - a)(a^+ - a)}|N\rangle =$$

kommutátort beírva

$$a^+a^+ - a^+a - aa^+ + aa = a^+a^+ + aa - a^+a - (1 + a^+a) = a^+a^+ + aa - 2a^+a - 1$$

$$= -\frac{m\hbar\omega}{2}\langle N|(a^+a^+ + aa - 2a^+a - 1)|N\rangle = -\frac{m\hbar\omega}{2}\Big\{\langle N|a^+\sqrt{N+1}|N+1\rangle + \langle N|a\sqrt{N}|N-1\rangle - 2\langle N|\underbrace{\hat{N}|N\rangle}_{N\cdot|N\rangle} - \underbrace{\langle N|N\rangle}_{1}\Big\} =$$

$$= -\frac{m\hbar\omega}{2}\Big\{\underbrace{\langle N|N+2\rangle}_{\emptyset}\sqrt{N+1}\sqrt{N+2} + \underbrace{\langle N|N-2\rangle}_{\emptyset}\sqrt{N}\sqrt{N-1} - 2N - 1\Big\} =$$

$$= \underline{\underline{m\hbar\omega\left(N + \frac{1}{2}\right)}}$$

# Kvantummechanika gyak

## 5.

A 4. gyakorlat vége +

### Spin

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$\underline{n}$ irányú spin operátor:

$$\hat{\underline{S}}\,\underline{n} = \frac{1}{2}\left(\sigma_x \sin\theta\cos\varphi + \sigma_y \sin\theta\sin\varphi + \sigma_z\cos\theta\right) =$$

$$= \frac{1}{2}\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

sajátvektorok: $|+\rangle_{\underline{n}} = \begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\varphi}{2}} \end{pmatrix}$ $\qquad |-\rangle_{\underline{n}} = \begin{pmatrix} -\sin\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2}\, e^{i\frac{\varphi}{2}} \end{pmatrix}$

**1. Feladat:** Határozzuk meg egy általánosan polarizált nyaláb átlagos ~~spinjét~~ x irányú spinjét

$$\hat{S}_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\langle \hat{S}_x \rangle = {}_{\underline{n}}\langle +|\, \hat{S}_x\, |+\rangle_{\underline{n}} = \frac{1}{2}\begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}\, e^{+i\frac{\varphi}{2}} \end{pmatrix}^{+}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}\, e^{+i\frac{\varphi}{2}} \end{pmatrix} =$$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2} e^{i\frac{\varphi}{2}},\ \sin\frac{\theta}{2} e^{-i\frac{\varphi}{2}}\right)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} \\ \sin\frac{\theta}{2}\, e^{i\frac{\varphi}{2}} \end{pmatrix} =$$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2} e^{i\frac{\varphi}{2}},\ \sin\frac{\theta}{2} e^{-i\frac{\varphi}{2}}\right)\begin{pmatrix} \sin\frac{\theta}{2}\, e^{i\frac{\varphi}{2}} \\ \cos\frac{\theta}{2}\, e^{-i\frac{\varphi}{2}} \end{pmatrix} =$$

$$= \frac{1}{2}\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{i\varphi} + \sin\frac{\theta}{2}\cos\frac{\theta}{2}\, e^{-i\varphi}\right) = \frac{1}{4}\sin\theta\left(e^{i\varphi} + e^{-i\varphi}\right) =$$

$$= \frac{1}{2}\sin\theta\cos\varphi$$

**2. Feladat:** Egy $\frac{1}{2}$ spinű részecske hullámfüggvénye: $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \begin{pmatrix} e^{i\alpha} \cos \ldots \\ e^{i\beta} \sin \ldots \end{pmatrix}$

Milyen eredményt kapunk átlagosan ebben az állapotban a spin tetszőleges $\underline{n}$ irányú vetületére?

$$\langle \hat{S}_{\underline{n}} \rangle = (\psi_1^*, \psi_2^*) \cdot \hat{S}_{\underline{n}} \cdot \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} =$$

$$= \frac{1}{2} (e^{-i\alpha} \cos\delta, e^{-i\beta} \sin\delta) \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix} \begin{pmatrix} e^{i\alpha} \cos\delta \\ e^{i\beta} \sin\delta \end{pmatrix} =$$

$$= \frac{1}{2} (e^{-i\alpha} \cos\delta, e^{-i\beta} \sin\delta) \begin{pmatrix} \cos\delta \cos\theta\, e^{i\alpha} + \sin\delta \sin\theta\, e^{i\beta} e^{-i\varphi} \\ \sin\theta \cos\delta\, e^{i\alpha} e^{i\varphi} - \cos\theta \sin\delta\, e^{i\beta} \end{pmatrix} =$$

$$= \frac{1}{2} \Big[ \cos^2\delta \cos\theta + \sin\delta \cos\delta \sin\theta\, e^{-i(\alpha-\beta+\varphi)} + \sin\delta \cos\delta \sin\theta\, e^{i(\alpha}$$

$$- \sin^2\delta \cos\theta \Big] = \frac{1}{2} \Big[ (\cos^2\delta - \sin^2\delta) \cos\theta + 2 \sin\delta \cos\delta \sin\theta \cos(\alpha$$

$$= \frac{1}{2} \left[ \cos 2\delta \cos\theta + \sin 2\delta \sin\theta \cos(\alpha - \beta + \varphi) \right]$$